A SHORT ELEMENTARY PROOF OF GROTHENDIECK'S THEOREM ON ALGEBRAIC VECTORBUNDLES OVER THE PROJECTIVE LINE*

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Let $E$ be an algebraic (or holomorphic) vectorbundle over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Then Grothendieck proved that $E$ splits into a sum of line bundles $E = \bigoplus L_i$ and that the isomorphism classes of the $L_i$ are (up to order) uniquely determined by $E$. The $L_i$ in turn are classified by an integer (their Chern numbers) so that $m$-dimensional vectorbundles over $\mathbb{P}^1(\mathbb{C})$ are classified by an $m$-tuple of integers

$$\kappa(E) = (\kappa_1(E), \ldots, \kappa_m(E)), \quad \kappa_1(E) \geq \kappa_2(E) \geq \cdots \geq \kappa_m(E), \quad \kappa_i \in \mathbb{Z}.$$ 

In this short note we present a completely elementary proof of these facts which, as it turns out, works over any field $k$.

1. Introduction

Let $E$ be a holomorphic (or algebraic) vectorbundle over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. (By [2] holomorphic and algebraic vectorbundles over $\mathbb{P}^1(\mathbb{C})$ amount to the same thing). In [1] Grothendieck proved that $E$ splits into a sum of line bundles $E = \bigoplus L_i$ and that the isomorphism classes of the $L_i$ are (up to order) uniquely determined by $E$. The line bundles $L_i$ in turn are classified by an integer (their first Chern number) so that $m$-dimensional vectorbundles over $\mathbb{P}^1(\mathbb{C})$ are classified by an $m$-tuple of integers

$$\kappa(E) = (\kappa_1(E), \ldots, \kappa_m(E)), \quad \kappa_1(E) \geq \cdots \geq \kappa_m(E), \quad \kappa_i \in \mathbb{Z}.$$ 

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Below we give a completely elementary proof of these facts, which, as it turns out, works over any field \( k \). Of course 'completely elementary' means that such concepts as 'degree of a line bundle' or 'first Chern number' or 'cohomology' or 'intersection number' are not needed or mentioned below. All we use is some linear algebra (or matrix manipulation).

2. Vectorbundles over \( \mathbb{P}^1_k \)

Let \( k \) be any field. The projective line \( \mathbb{P}^1_k \) over \( k \) can be obtained as follows. Let \( U_1 = \text{Spec}(k[s]), U_2 = \text{Spec}(k[t]), U_{12} = \text{Spec}(k[s,s^{-1}]) = U_1 \setminus \{0\}, U_{21} = \text{Spec}(k[t,t^{-1}]) = U_2 \setminus \{0\} \). Now glue \( U_1 \) and \( U_2 \) together by identifying \( U_{12} \) and \( U_{21} \) by means of the isomorphism

\[
k[s,s^{-1}] \cong k[t,t^{-1}], \quad s \mapsto t^{-1}.
\]

Now let \( E \) be an \( m \)-dimensional vectorbundle over \( \mathbb{P}^1_k \) defined over \( k_1 \) and let \( \mathbb{A}^m = \text{Spec}(k[X_1, \ldots, X_m]) \). Then \( E|_{U_i}, i = 1,2 \), is trivial, i.e. \( E|_{U_i} \cong U_i \times \mathbb{A}^m \), so that \( E \) can be viewed (up to isomorphism) as obtained by glueing together \( U_1 \times \mathbb{A}^m \) and \( U_2 \times \mathbb{A}^m \) by identifying \( U_1 \setminus \{0\} \times \mathbb{A}^m \) and \( U_2 \setminus \{0\} \times \mathbb{A}^m \) by means of an isomorphism of the form

\[
(s,v) \mapsto (s^{-1}, A(s,s^{-1})v)
\]

where \( A(s,s^{-1}) \) is a matrix with coefficients in \( k[s,s^{-1}] \) which has nonzero determinant for all \( s \neq 0, s^{-1} \neq 0 \). This last fact means that

\[
\det(A(s,s^{-1})) = s^n, \quad n \in \mathbb{Z}.
\]

A vectorbundle automorphism of \( U_1 \times \mathbb{A}^m \) is necessarily of the form \( (s,v) \mapsto (s, U(s)v) \) where \( U(s) \) is a matrix with coefficients in \( k[s] \) with \( \det U(s) \in k \setminus \{0\} \) and similarly an automorphism of \( U_2 \times \mathbb{A}^m \) is given by a matrix \( V(s^{-1}) \) with coefficients in \( k[s^{-1}] \) with determinant in \( k \setminus \{0\} \). Different trivializations of \( E|_{U_i} \) differ by an automorphism of \( U_i \times \mathbb{A}^m \). It follows that

**Proposition 2.3.** Isomorphism classes of \( m \)-dimensional algebraic vectorbundles over \( \mathbb{P}^1_k \) correspond bijectively to equivalence classes of polynomial \( m \times m \) matrices \( A(s,s^{-1}) \) over \( k[s,s^{-1}] \) such that \( \det A(s,s^{-1}) = s^n, \quad n \in \mathbb{Z} \) where the equivalence relation is the following: \( A(s,s^{-1}) \sim A'(s,s^{-1}) \) iff there exist polynomial invertible \( m \times m \) matrices \( U(s), V(s^{-1}) \) over \( k[s] \) and \( k[s^{-1}] \) respectively with constant determinant such that

\[
A'(s,s^{-1}) = V(s^{-1})A(s,s^{-1})U(s).
\]
3. A canonical form for matrices over $k[s, s^{-1}]$

Now let us study canonical forms for $m \times m$ matrices over $k[s, s^{-1}]$ under the equivalence relation defined in Proposition 2.3 above. The result is

**Proposition 3.1.** Let $A(s, s^{-1})$ be an $m \times m$ matrix over $k[s, s^{-1}]$ with determinant equal to $s^n$ for some $n \in \mathbb{Z}$. Then there exist polynomial $m \times m$ matrices $V(s^{-1})$ and $U(s)$ with constant nonzero determinant such that

$$V(s^{-1})A(s, s^{-1})U(s) = \begin{pmatrix}
s^{r_1} & 0 \\
0 & s^{r_2} \\
& \ddots \\
0 & s^{r_m}
\end{pmatrix}$$

with $r_1 \geq r_2 \geq \cdots \geq r_m$, $r_i \in \mathbb{Z}$. The $r_i$ are uniquely determined by $A(s, s^{-1})$. Moreover if $A(s, s^{-1})$ is polynomial in $s$ then $r_i \geq 0$, $i = 1, \ldots, m$, and if $A(s, s^{-1})$ is polynomial in $s^{-1}$ then $r_i \leq 0$, $i = 1, \ldots, m$.

**Proof.** Let's prove uniqueness first. Write $D(r_1, \ldots, r_m)$ for the matrix on the right in (3.2). Suppose there were two such matrices equivalent to $A(s, s^{-1})$. Then there would be polynomial matrices with constant nonzero determinant $U(s), V(s^{-1})$ such that

$$V(s^{-1})D(r_1, \ldots, r_m) = D(r'_1, \ldots, r'_m)U(s).$$

If $A$ is a matrix let

$$A_{j_1, \ldots, j_k}^{i_1, \ldots, i_k}$$

denote the minor of $A$ obtained by taking the determinant of the submatrix of $A$ obtained by removing all rows with index in $\{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\}$ and all columns with index in $\{1, \ldots, m\} \setminus \{j_1, \ldots, j_k\}$. Then of course

$$(AB)^{i_1, \ldots, i_k}_{j_1, \ldots, j_k}^{l_1, \ldots, l_k} = \sum_{r_1 < \cdots < r_k} A^{i_1, \ldots, i_k}_{l_1, \ldots, l_k} B^{r_1, \ldots, r_k}_{j_1, \ldots, j_k}.$$

Using this on the equality $V(s^{-1})D(r_1, \ldots, r_m) = D(r'_1, \ldots, r'_m)U(s)$ one finds that

$$V_{i_1, \ldots, i_k}^{l_1, \ldots, l_k}(s^{-1})s^{r_1 + \cdots + r_k} = s^{r'_1 + \cdots + r'_k}U_{l_1, \ldots, l_k}^{i_1, \ldots, i_k}(s)$$

for all $i_1 < \cdots < i_k$. Now for some $i_1, \ldots, i_k$,

$$U_{l_1, \ldots, l_k}^{i_1, \ldots, i_k}(s) \neq 0.$$

Hence $r'_1 + \cdots + r'_k \leq r_1 + \cdots + r_k$ for some $i_1 < \cdots < i_k$, and hence certainly $r'_1 + \cdots + r'_k \leq r_1 + \cdots + r_k$ for all $k$. Multiplying with $V(s^{-1})^{-1}$ on the left and $U(s)^{-1}$ on the right in $V(s^{-1})D(r_1, \ldots, r_m) = D(r'_1, \ldots, r'_m)U(s)$ and repeating the argument gives $r_1 + \cdots + r_k \leq r'_1 + \cdots + r'_k$ for all $k$ and hence $r_i = r'_i$, $i = 1, \ldots, m$. 


It remains to prove existence. First multiply \( A(s, s^{-1}) \) with a suitable power \( s^n \in \mathbb{N} \cup \{0\} \) to obtain a polynomial matrix \( B(s) \). Then by post multiplication with suitable \( U(s) \) (column operations) we can find a \( B'(s) \) with \( b'_{i1} \neq 0 \) and \( b'_{ij} = 0 \) for \( i = 2, \ldots, m \) (\( b'_{11} \) is the greatest common divisor of \( b_{11}, \ldots, b_{1m} \)). Of course \( b'_{11} = s^{k_1} \) for some \( k_1 \in \mathbb{N} \cup \{0\} \) because \( \det B(s) \) is a power of \( s \). Let \( B_2 \) be the lower-right \((m-1) \times (m-1)\) submatrix of \( B \). By induction we can assume that the proposition holds for \((m-1) \times (m-1)\) matrices. (The case \( m = 1 \) is trivial). So there are \( U_2(s), V_2(s^{-1}) \) such that \( V_2(s^{-1})B_2U_2(s) \) is of the form of the right hand side of (3.2).

Then

\[
C(s) = \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} s^{k_1} & 0 & \cdots & 0 \\ c_2 & s^{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_m & 0 & \cdots & s^{k_m} \end{pmatrix}
\]

(3.6)

for certain \( k_1, k_2, \ldots, k_m \in \mathbb{N} \cup \{0\} \) (same \( k_1 \) as before) and \( c_i \in k[s, s^{-1}], i = 2, \ldots, m \). Subtracting suitable \( k[s^{-1}] \) multiples of the first row from rows \( 2, \ldots, m \) (which premultiplication with a \( V(s^{-1}) \)) we can moreover see to it that \( c_i \in k[s] \).

Now consider all polynomial matrices of the form (3.4) which are equivalent to \( B(s) \). Choose one for which \( k_1 \) is maximal. Such a one exist because \( k_1 \leq \deg (\det B(s)) \) because \( k_2, \ldots, k_m \geq 0 \). We claim that then \( k_1 \geq k_i, i = 2, \ldots, m \). Indeed, suppose that \( k_1 < k_i \). Subtracting a suitable \( k[s^{-1}] \) multiple of the first row from the \( i \)-th row we find a matrix (3.4) with \( c_i = s^{k_1+1}c_i(s) \). Now interchange the first and the \( i \)-th row to find a polynomial matrix \( B'(s) \) such that the greatest common divisor of its first row elements is \( s^{k_1} \) with \( k_1 \geq k_1 + 1 \). Now apply to \( B'(s) \) the same procedure as above to \( B(s) \). This would give a \( C'(s) \) of the form (3.4) with \( k'_1 > k_1 \), a contradiction. We can therefore assume that in (3.4) \( k_1 \geq k_i, c_i \in k[s], i = 2, \ldots, m \). Subtracting suitable \( k[s]\)-multiples of the 2-nd, \ldots, \( m \)-th columns from the first one we find a matrix (3.4) with degree \( (c_i) \leq k_i \). But then \( \deg(c_i) < k_1 \) so that a suitable \( k[s^{-1}] \) multiple of \( s^{k_1} \) is equal to \( c_i \) so that a further premultiplication with a \( V(s^{-1}) \) gives a matrix (3.4) with \( c_2 = \cdots = c_m = 0 \). This proves the first half of the last part of the statement of the proposition and shows that there are \( k_1, \ldots, k_n \in \mathbb{N} \cup \{0\} \) \( k_1 \geq \cdots \geq k_m \) (by permuting columns and rows if necessary) and \( U(s), V(s^{-1}) \) constant nonzero determinant such that

\[
V(s^{-1})s^nA(s, s^{-1})U(s) = V(s^{-1})B(s)U(s) = D(k_1, \ldots, k_m).
\]

Multiplying with \( s^{-n} \) gives

\[
V(s^{-1})A(s, s^{-1})U(s) = D(r_1, \ldots, r_m) \]

with \( r_i = k_i - n \). The second half of the last statement of the proposition is proved as the first half starting with a matrix \( B(s^{-1}) \) and using row (resp. column) operations everywhere where we used column (resp. row) operations above. This concludes the proof of Proposition 3.1.
4. Classification of vectorbundles over $\mathbb{P}^1_k$

Let $O(n)$, $n \in \mathbb{Z}$ be the line bundle over $\mathbb{P}^1_k$ defined by the glueing matrix $A(s,s^{-1})=s^{-n}$. Obviously then the bundle defined by the glueing matrix $A(s,s^{-1})=D(r_1,\ldots,r_m)$ is equal to the direct sum $O(-r_1)\oplus\cdots\oplus O(-r_m)$.

**Theorem 4.1.** Let $E$ be an algebraic $m$-dimensional vectorbundle over $\mathbb{P}^1_k$ which is defined over $k$. Then $E$ is isomorphic over $k$ to a direct sum of line bundles

$$E=O(\kappa_1)\oplus\cdots\oplus O(\kappa_m), \quad \kappa_1 \geq \cdots \geq \kappa_m, \quad \kappa_i \in \mathbb{Z}, \quad i=1,\ldots,m,$$

and the $\kappa_i$ are uniquely determined by the isomorphism class of $E$.

**Remarks 4.2.** It is perhaps worth remarking that $E$ is positive (meaning that all the $\kappa_i(E)\geq 0$) if the glueing matrix $A(s,s^{-1})$ is polynomial in $s^{-1}$ and that $E$ is negative (i.e. $\kappa_i(E)\leq 0$ all $i$) if $A(s,s^{-1})$ is polynomial in $s$. This follows from the last statement of Proposition 3.1. Also $E$ contains a summand $O(n)$ with $n>0$ if $\deg(\det A(S,s^{-1}))<0$. Finally it follows that vectorbundles over $\mathbb{P}^1_k$ have no forms, i.e. if $E$ and $E'$ are two vectorbundles over $k$ which become isomorphic over the algebraic closure $\bar{k}$ of $k$ then $E$ and $E'$ are also isomorphic over $k$. This can of course also be seen by other, more sophisticated, means (e.g. Galois cohomology).

**References**
