# A SHORT TUTORIAL ON LIE ALGEBRAS

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This tutorial does not correspond to an actual oral lecture during the conference at Les Arcs in June, 1980. However, to improve accessibility and understandability of the material in this volume it seemed wise to include a small section on the basic facts and definitions concerning Lie algebras which play a role in control and nonlinear filtering theory. This is what these few pages attempt to do.

1. DEFINITION OF LIE ALGEBRAS. EXAMPLES. Let k be a field and V a vectorspace over k. (For the purpose of this volume it suffices to take $k = \mathbb{R}$ or (rarely) $k = \mathbb{C}$; the vectorspace V over k need not be finite dimensional). A Lie algebra structure on V is then a bilinear map (called brackett multiplication)

(1.1) $$[\ ,\ ]: V \times V \to V$$

such that the two following conditions hold

(1.2) $$[u,u] = 0 \text{ for all } u \in V$$



M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 95–108.

(1.3) $[u,[v,w]] + [v,[w,u]] + [w,[u,v] = 0$ for all $u,v,w, \in V$.

The last identity is called the Jacobi identity. Of course the bilinearity of (1.1) means that $[au+bv,w] = a[u,w] + b[v,w]$, $[u,bv+cw] = b[u,v] + c[u,w]$. From (1.2) it follows that

(1.4) $$[u,v] = -[v,u]$$

by considering $[u+v,u+v] = 0$ and using bilinearity.

1.5. Example. The Lie algebra associated to an associative algebra.

Let A be an associative algebra over k. Now define a new multiplication (brackett) on A by the formula

(1.6) $$[v,w] = vw - wv, \quad w,v \in A$$

Then A with this new multiplication is a Lie algebra. (Exercise: check the Jacobi identity (1.3)).

1.9. Remark. In a certain precise sense all Lie algebras arise in this way. That is for every Lie algebra L there is an associative algebra A containing L such that $[u,v] = uv - vu$. I.e. every Lie algebra arises as a subspace of an associative algebra A which happens to be closed under the operation $(u,v) \to uv - vu$. Though this "universal enveloping algebra" construction is quite important it will play no role in the following and the remark is intended to make Lie algebras easier to understand for the reader.

1.7. Example. Let $M_n(k)$ be the associative algebra of all n x n matrices with coefficients in k. The associated Lie algebra is written $g\ell_n(k)$; i.e. $g\ell_n(k)$ is the $n^2$-dimensional vectorspace of all n x n matrices with the brackett multiplication $[A,B] = AB - BA$.

1.8. Example. Let $s\ell_n(k)$ denote the subspace of all n x n matrices of trace zero. Because $Tr(AB-BA) = 0$ for all n x n matrices A,B, we see that $[A,B] \in s\ell_n(k)$ if $A,B \in s\ell_n(k)$ giving us an $(n^2-1)$-dimensional sub Lie algebra of $g\ell_n(k)$.

1.10. Example. The Lie algebra of first order differential operators with $C^\infty$-coefficients.

Let $V_n$ be the space of all differential operators (on the space $F(\mathbb{R}^n)$ of $C^\infty$-functions (i.e. arbitrarily often differentiable functions in $x_1,\ldots,x_n$)) of the form

$$X = \sum_{i=1}^{n} f_i(x_1,\ldots,x_n) \frac{\partial}{\partial x_i} \tag{1.11}$$

where the $f_i$, $i = 1, \ldots, n$ are $C^\infty$-functions. Thus $X: F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ is the operator $X(\phi) = \sum_{i=1}^{n} f_i \frac{\partial \phi}{\partial x_i}$. Now define a brackett operation on $V_n$ by the formula

$$[X,Y] = \sum_{i,j} (f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}) \tag{1.12}$$

if $X = \Sigma f_i \frac{\partial}{\partial x_i}$, $Y = \Sigma g_j \frac{\partial}{\partial x_j}$. This makes $V_n$ a Lie algebra. Check that $[X,Y](\phi) = X(Y(\phi)) - Y(X(\phi))$ for all $\phi \in F(\mathbb{R}^n)$.

1.13. Example. Derivations. Let A be any algebra (i.e. A is a vectorspace together with any bilinear map (multiplication) $A \times A \to A$: in particular A need not be associative). A derivation on A is a linear map $D: A \to A$ such that

(1.14) $$D(uv) = (Du)v + u(Dv)$$

For example let $A = \mathbb{R}[x]$ and $D$ the operator $\frac{d}{dx}$. The $D$ is a derivation. The operators (1.11) of the example above are derivations on $F(\mathbb{R}^n)$.
Let $\mathrm{Der}(A)$ be the vectorspace of all derivations. Define $[D_1,D_2] = D_1D_2 - D_2D_1$. Then $[D_1,D_2]$ is again a derivation and this brackett multiplication makes $\mathrm{Der}(A)$ a Lie algebra over $k$.

1.15. <u>Example.</u> <u>The Weyl algebra</u> $W_1$. Let $W_1$ be the vectorspace of all (any order) differential operators in one variable with polynomial coefficients. I.e. $W_1$ is the vectorspace with basis $x^i \frac{d^j}{dx^j}$, $i,j \in \mathbb{N} \cup \{0\}$. ($x^i$ is considered as the operator $f(x) \to x^i f(x)$). Consider $W_1$ as a space of operators acting, say, on $k[x]$. Composition of operators makes $W_1$ an associative algebra and hence gives $W_1$ also the structure of a Lie algebra. For example one has

$$[x \frac{d^2}{dx^2}, x^2 \frac{d}{dx}] = 3x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx}, \quad [x \frac{d}{dx}, x^i \frac{d^j}{dx^j}] = (i-j)x^i \frac{d^j}{dx^j}$$

1.16. <u>Example.</u> <u>The oscillator algebra.</u> Consider the four dimensional subspace of $W_1$ spanned by the four operators $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2$, $x$, $\frac{d}{dx}$, $1$. One easily checks that (under the brackett multiplication of $W_1$)

(1.17)
$$[\tfrac{1}{2}\frac{d^2}{dx^2} - \tfrac{1}{2}x^2, x] = \frac{d}{dx}, \quad [\tfrac{1}{2}\frac{d^2}{dx^2} - \tfrac{1}{2}x^2, \frac{d}{dx}] = x, \quad [\frac{d}{dx}, x] = 1$$
$$[\tfrac{1}{2}\frac{d^2}{dx^2} - \tfrac{1}{2}x^2, 1] = [x,1] = [\frac{d}{dx}, 1] = 0$$

Thus this four dimensional subspace is a sub-Lie-algebra of $W_1$. It is called the oscillator Lie algebra (being intimately associated to the harmonic oscillator).

2. HOMOMORPHISMS, ISOMORPHISMS, SUBALGEBRAS AND IDEALS.

2.1. Sub-Lie-algebras. Let L be a Lie algebra over k and V a subvectorspace of L. If $[u,v] \in V$ for all $u,v \in L$. Then V is a sub-Lie-algebra of L. We have already seen a number of examples of this, e.g. the oscillator algebra of example 1.16 as a sub-Lie-algebra of the Weyl algebra $W_1$ and the Lie-algebra $s\ell_n(k)$ as a sub-Lie-algebra of $g\ell_n(k)$. Some more examples follow.

2.2. The Lie-algebra $so_n(k)$. Let $so_n(k)$ be the subspace of $g\ell_n(k)$ consisting of all matrices A such that $A + A^T = 0$ (where the upper $^T$ denotes transposes). Then if $A,B \in so_n(k)$

$[A,B] + [A,B]^T = AB - BA + (AB-BA)^T = A(B+B^T) - B(A+A^T) +$

$+ (B^T+B)A^T - (A^T+A)B^T = 0$ so that $[A,B] \in so_n(k)$. Thus $so_n(k)$ is a sub-Lie-algebra of $g\ell_n(k)$.

2.3. The Lie-algebra $t_n(k)$. Let $t_n(k)$ be the subspace of $g\ell_n(k)$ consisting of all upper triangular matrices. Because product and sum of upper triangular matrices are again upper triangular $t_n(k)$ is a sub-Lie-algebra of $g\ell_n(k)$.

2.4. The Lie-algebra $sp_n(k)$. Let Q be the 2n x 2n matrix

$Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Now let $sp_n(k)$ be the subspace of all 2n x 2n

matrices A such that $AQ + QA^T = 0$. Then as above in example 2.2 one sees that $A,B \in sp_n(k) \Rightarrow [A,B] \in sp_n(k)$ so that $sp_n(k)$ is a sub-Lie-algebra of $g\ell_{2n}(k)$.

2.5. Ideals. Let L be a Lie-algebra over k. A subvectorspace $I \subset L$ with the property that for all $u \in I$ and all $v \in L$ we have $[u,v] \in I$ is called an ideal of L. An example is $s\ell_n(k) \subset g\ell_n(k)$, cf. example 1.8 above. Another example follows.

2.6. Example. The Heisenberg Lie-algebra. Consider the 3-dimensional subspace of $W_1$ spanned by the operators x, $\frac{d}{dx}$, 1. The formulas (1.17) show that this subspace is an ideal in the oscillator algebra.

2.7. Example. The centre of a Lie algebra. Let L be a Lie algebra. The centre of L is defined as the subset $Z(L) = \{z \in L | [u,z] = 0$ for all $u \in L\}$. Then Z(L) is a subvector space of L and in fact an ideal of L. As an example it is easy to check that the centre of $g\ell_n(k)$ consists of scalar multiples of the unit matrix $I_n$.

2.8. Homomorphisms and isomorphisms. Let $L_1$ and $L_2$ be two Lie algebras over k. A morphism of $\alpha: L_1 \to L_2$ vectorspaces (i.e. a k-linear map) is a homomorphism of Lie algebras if $\alpha[u,v] = \alpha(u),\alpha(v)$ for all $u,v \in L_1$. The homomorphism α is called an isomorphism if it is also an isomorphism of vectorspaces.

2.9. Example. Consider the following three first-order differential operators in two variables x,P

$$a = (1-p^2)\frac{\partial}{\partial p} - Px\frac{\partial}{\partial x}, \quad b = P\frac{\partial}{\partial x}, \quad c = \frac{\partial}{\partial x}$$

Then one easily calculates (cf. (1.9)) $[a,b] = c$, $[a,c] = b$, $[b,c] = 0$. Now define α from the oscillator algebra of example 1.16 to this 3-dimensional Lie algebra as the linear map $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2 \to a$, $x \to b$, $\frac{d}{dx} \to c$, $1 \to 0$. Then the formulas above and (1.17) show that α is a homomorphism of Lie algebras.

2.10. Kernel of a homomorphism. Let $\alpha: L_1 \to L_2$ be a homomorphism of Lie algebras. Let $Ker(\alpha) = \{u \in L_1 | \alpha(u) = 0\}$. Then Ker(α) is an ideal in $L_1$.

2.11. Quotient Lie algebras. Let L be a Lie algebra and I an ideal in L. Consider the quotient vector space L/I and the

quotient morphisms of vector spaces $L \overset{\alpha}{\to} L/I$. For all $\bar{u},\bar{v} \in L/I$ choose $u,v \in L$ such that $\alpha(u) = \bar{u}$, $\alpha(v) = \bar{v}$. Now define $[\bar{u},\bar{v}] = \alpha[u,v]$. Check that this does not depend on the choice of u,v.
This then defines a Lie-algebra structure on L/I and $\alpha: L \to L/I$ becomes a homomorphism of Lie-algebras.

2.12. Image of a homomorphism. Let $\alpha: L_1 \to L_2$ be a homomorphism of Lie algebras. Let $\mathrm{Im}(\alpha) = \alpha(L_1) = \{u \in L_2 \mid \exists v \in L_1, \alpha(v) = u\}$. Then $\mathrm{Im}\alpha$ is a sub-Lie-algebra of $L_2$ and $\alpha$ induces an isomorphism $L_1/\mathrm{Ker}(\alpha) \simeq \mathrm{Im}(\alpha)$.

2.13. Exercise. Consider the 3-dimensional vector space of all real upper triangular 3 x 3 matrices with zero's on the diagonal. Show that this a sub-Lie-algebra of $g\ell_3(\mathbb{R})$, and show that it is isomorphic to the 3-dimensional Heisenberg-Lie-algebra of example 2.6 but that it is not isomorphic to the 3-dimensional Lie-algebra $s\ell_2(\mathbb{R})$ of example 1.8.

2.14. Exercise. Show that the four operators $x^2$, $\frac{d^2}{dx^2}$, $x\frac{d}{dx}$, 1 span a 4-dimensional subalgebra of $W_1$, and show that this 4-dimensional Lie algebra contains a three dimensional Lie algebra which is isomorphic to $s\ell_2(\mathbb{R})$.

2.15. Exercise. Show that the six operators $x^2$, $\frac{d^2}{dx^2}$, x, $\frac{d}{dx}$, $x\frac{d}{dx}$, 1 space a six dimensional sub-Lie-algebra of $W_1$. Show that x, $\frac{d}{dx}$, 1 space a 3-dimensional ideal in this Lie-algebra and show that the corresponding quotient algebra is $s\ell_2(\mathbb{R})$.

## 3. LIE ALGEBRAS OF VECTORFIELDS.

Let M be a $C^\infty$-manifold (cf. the tutorial on manifolds and vectorfields in this volume). Intuitively a vectorfield on M specifies a tangent vector t(m) at every point $m \in M$. Then given a $C^\infty$-function f on M we can for each $m \in M$ take the derivation of f at m in the direction t(m), giving us a new function g on

M. This can be made precise in varying ways; e.g. as follows.

3.1. The Lie algebra of vectorfields on a manifold M. Let M be a $C^\infty$-manifold, and let F(M) be the $\mathbb{R}$-algebra (pointwise addition and multiplications) of all smooth (= $C^\infty$) functions f: $M \to \mathbb{R}$. By definition a $C^\infty$-vectorfield on M is a derivation X: F(M) $\to$ F(M). The Lie algebra of derivations of F(M) cf. example 1.13, i.e. the Lie-algebra of smooth vectorfields on M, is denoted V(M).

3.2. Derivations and vectorfields. Now let $M = \mathbb{R}^n$ so that F(M) is simply the $\mathbb{R}$-algebra of $C^\infty$-functions in $x_1, \ldots, x_n$. Then it is not difficult to show that every derivation X: $F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ is necessarily of the form

$$X = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \tag{3.3}$$

with $g_i \in F(\mathbb{R}^n)$. For a proof cf. [4, Ch.I,§2]. The corresponding vectorfield on $\mathbb{R}^n$ now assigns to $x \in \mathbb{R}^n$ the tangent vector $(g_1(x), \ldots, g_n(x))^T$.

On an arbitrary manifold we have representations (3.3) locally around every point and these expressions turn out to be compatible in precisely the x way needed to define a vectorfield as described in the tutorial on manifolds and vectorfields in this volume [3].

3.4. Homomorphisms of Lie algebras of vectorfields. Let M and N be $C^\infty$-manifolds and let $\alpha$: L $\to$ V(N) be a homomorphism of Lie algebras where L is a sub-Lie-algebra of V(M). Let $\phi$: M $\to$ N be a smooth map. Then $\alpha$ and $\phi$ are said to be compatible if

$$\phi^*(\alpha(X)f) = X(\phi^*(f)) \quad \text{for all } f \in F(N) \tag{3.5}$$

where $\phi^*$ is the homomorphism of algebras $F(N) \to F(M)$, $f \to \phi^*(f) = f \circ \phi$.

In terms of the Jacobian of $\phi$ (cf.[3]), this means that

$$J(\phi)(X_m) = \alpha(X)_{\phi(m)} \tag{3.6}$$

where $X_m$ is the tangent vector at m of the vectorfield X.

If $\phi : M \to N$ is an isomorphism of $C^\infty$-manifolds there is always precisely one homomorphism of Lie-algebras $\alpha : V(M) \to V(N)$ compatible with $\phi$ (which is then an isomorphism). It is defined (via formula (3.5)) by

$$\alpha(X)(f) = (\phi^*)^{-1}X(\phi^*f), \quad f \in F(N) \tag{3.7}$$

3.8. Isotropy subalgebras. Let L be a sub-Lie-algebra of V(M) and let m M. The isotropy subalgebra $L_m$ of L at m consists of all vectorfields in L whose tangent vector in m is zero, or, equivalently

$$L_m = \{X \in L | Xf(m) = 0 \quad \text{all } f \in F(M)\} \tag{3.9}$$

Now suppose that $\alpha: L \to V(N)$ and $\phi: M \to N$ are compatible in the sense of 3.4 above. Then it follows easily from (3.5) that

$$\alpha(L_m) \subset V(N)_{\phi(m)} \tag{3.10}$$

i.e. $\alpha$ isotropy subalgebras into isotropy subalgebras. Inversely if we restrict our attention to analytic vectorfields then condition (3.10) on $\alpha$ at m implies that locally there exists a $\phi$ which is compatible with $\alpha$ [7].

## 4. SIMPLE, NILPOTENT AND SOLVABLE LIE ALGEBRAS.

4.1. Nilpotent Lie algebras let L be a Lie-algebra over k.

The descending central seems of L is defined inductively by

$$C^1L = L,\ C^{i+1}L = [L, C^iL],\ i \geq 1 \tag{4.2}$$

It is easy to check that the $C^iL$ are ideals. The Lie algebra L is called nilpotent if $C^nL = \{0\}$ for n big enough.

For each $x \in L$ we have the endomorphism adx: $L \to L$ defined by $y \to [x,j]$. It is now a theorem that if L is finite dimensional then L is nilpotent iff the endomorphisms adx are nilpotent for all $x \in L$. Whence the terminology.

4.3. Solvable Lie algebras. The derived seems of Lie algebras of a Lie algebra L is defined inductively by

$$D^1L = L,\ D^{i+1}L = [D^iL, D^iL]\ ,\ i \geq 1 \tag{4.4}$$

It is again easy to check that the $D^iL$ are ideals. The Lie algebra L is called solvable if $D^nL = \{0\}$ for n large enough.

4.5. Examples. The Heisenberg Lie algebra of example 2.6 is nilpotent. The Oscillator algebra of example 1.16 is solvable but not nilpotent. The sub-Lie-algebra of $W_1$ with vector-space basis $x^2, \frac{d^2}{dx^2}, x, \frac{d}{dx}, 1, x\frac{d}{dx}$ is neither nilpotent, nor solvable. The Lie-algebra $t_n(k)$ of example 2.3 is solvable and in a way is typical of finite dimensional solvable Lie algebras in the sense that if k is algebraically closed (e.g. k = C), then every finite dimensional solvable Lie algebra over k is isomorphic to a sub-Lie-algebra of some $t_n(k)$.

4.6. Exercise. Show that sub-Lie-algebras and quotient-Lie-algebras of solvable Lie algebras (resp. nilpotent Lie algebras) are solvable (resp. nilpotent).

4.7. Abelian Lie-algebras. A lie algebra L is called abelian if $[L,L] = \{0\}$, i.e. if every brackett product is zero.

4.8. Simple Lie-algebras. A Lie algebra L is called simple if it is not abelian and if it has no other ideals than 0 and L. (Given the second condition the first one only rules out the zero- and one-dimensional Lie algebras). These simple-Lie-algebras and the abelian ones are in a very precise sense the basic building blocks of all Lie algebras.

The finite dimensional simple Lie algebras over $\mathbb{C}$ have been classified. They are the Lie algebras $sl_n(\mathbb{C})$, $sp_n(\mathbb{C})$, $so_n(\mathbb{C})$ of examples 1.8, 2.4 and 2.2 above and five additional exceptional Lie algebras. For infinite dimensional Lie algebras things are more complicated. The socalled filtered, primitive, transitive simple Lie algebras have also been classified (cf. e.g. [2]). One of these is the Lie-algebra $\hat{V}_n$ of all formal vector fields $\Sigma f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$, where the $f_i(x)$ are (possibly non converging) formal power series in $x_1,\ldots, x_n$. This class of infinite dimensional simple Lie algebras by no means exhausts all possibilities. E.g. the quotient-Lie-algebras $W_n/\mathbb{R}.1$ are simple and non-isomorphic to any of those just mentioned.

4.9. Exercise. Let $V_{alg}(\mathbb{R}^n)$ be the Lie algebra of all differential operators (vector fields) of the form $\Sigma f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$ with $f_i(x_1,\ldots,x_n)$ polynomial. Prove that $V_{alg}(\mathbb{R}^n)$ is simple.

## 5. REPRESENTATIONS.

Let L be a Lie algebra over k and M a vectorspace over k. A representation of L in M is a homomorphism of Lie algebras.

$$\rho : L \to \mathrm{End}_k(M) \tag{5.1}$$

where $\mathrm{End}_k(M)$ is the vectorspace of all k-linear maps $M \to M$ which is of course given the Lie algebra structure

$[A,B] = AB - BA$. Equivalently a representation of L in M consists of a k-bilinear map

(5.2) $\sigma : L \times M \to M$

such that, writing xm for $\sigma(x,m)$, we have $x,y\ m = x(ym) - y(xm)$ for all $x,y \in L$, $m \in M$. The relation between the two definitions is of course $\sigma(x,m) = \rho(x)(m)$.

Instead of speaking of a representation of L in M we also speak (equivalently) of the L-module M.

5.3. Example. The Lie algebra $gl_n(k)$ of all $n \times n$ matrices naturally acts on $k^n$ by $(A,v) \to Av \in k^n$ and this defines a representation $gl_n(k) \times k^n \to k^n$. The Lie algebra V(M) of vectorfields on a manifold M acts (by its definition) on F(M) and this is a representation of V(M). A quite important theorem concerning the existence of representations is

5.4. Ado's theorem. Cf. e.g. [1,§7]. If k is a field of characteristic zero, e.g. $k = \mathbb{R}$ or C and L is finite dimensional then there is a faithful representation $\rho: L \to \mathrm{End}(k^n)$ for some n. (Here faithful means that $\rho$ is injective).

Thus every finite dimensional Lie algebra L over k (of characteristic zero) can be viewed as a subalgebra of some $gl_n(k)$, and this subalgebra can then be viewed as a more concrete matrix "representation" of the "abstract" Lie algebra L.

5.5. Realizing Lie-algebras in V(M). A question of some importance for filtering theory is when a LIe algebra L can be realized as a sub-Lie-algebra of V(M), i.e. when L can be represented in F(M) by means of derivations of several papers in this volume for a discussion of the relevance of this problem. For finite dimensional Lie algebras Ado's theorem gives the answer because $(a_{ij}) \to \Sigma a_{ij} x_i \frac{\partial}{\partial x_j}$ defines an injective homomorphism of Lie-algebras $gl_n(\mathbb{R}) \to V(\mathbb{R}^n)$ (Exercise: check this)

## 6. LIE ALGEBRAS AND LIE GROUPS.

6.1. Lie groups. A (finite dimensional) Lie group is a finite dimensional smooth manifold G together with smooth maps $G \times G \to G$, $(x,y) \to xy$, $G \to G$, $x \to x^{-1}$ and a distinguished element $e \in G$ which make G a group. An example is the open subset of $\mathbb{R}^{n^2}$ consisting of all invertible $n \times n$ matrices with the usual matrix multiplication.

6.2. Left invariant vectorfields and the Lie algebra of a Lie group.

Let G be a Lie group. Let for all $g \in G$, $L_g : G \to G$ be the smooth map $x \to gx$. A vectorfield $X \in V(G)$ is called left invariant if $X(L_g^*f) = XL_g^*(Xf)$ for all functions f on G. Or, equivalently, if $J(L_g)X_x = X_{gx}$ for all $x \in G$, cf. section 3.4 above. Especially from the last condition it is easy to see that $X \to X_e$ defines an isomorphism between the vectorspace of left invariant vectorfields on G and the tangent space of G at e. Now the brackett product of two left invariant vectorfields is easily seen to be left invariant again so the tangent space of G at e (which is $\mathbb{R}^n$ if G is n-dimensional) inherits a Lie algebra structure. This is the Lie algebra Lie(G) of the Lie group G. It reflects so to speak the infinitesimal structure of G. A main reason for the importance of Lie algebras in many parts of mathematics and its applications is that this construction is reversible to a great extent making it possible to study Lie groups by means of their Lie algebras.

6.3. Exercise. Show that the Lie algebra of the Lie group $GL_n(\mathbb{R})$ of invertible real $n \times n$ matrices is the Lie algebra $gl_n(\mathbb{R})$.

## 7. POSTSCRIPT.

The above is a very rudimentary introduction to Lie algebras. Especially the topic "Lie algebras and Lie groups" also called "Lie theory" has been given very little space, in spite of the fact that it is likely to become of some importance in filtering (integration of a representation of a Lie algebra to a representation of a LIe (semi)group). The books [1, 4, 5, 6, 8] are all recommended for further material. My personal favourite (but by no means the easiest) is [4]; [6] is a classic and in its present incarnation very good value indeed.

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