ECONOMETRIC INSTITUTE

SOME RESULTS AND SPECULATIONS ON THE ROLE OF LIE ALGEBRAS IN FILTERING

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REPRINT SERIES no. 298

This article appeared in "Stochastic Systems: The Mathematics of Filtering and Identification and Applications", D. Reidel, Publ. Comp., Dordrecht, Holland (1981).

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SOME RESULTS AND SPECULATIONS ON THE ROLE OF LIE ALGEBRAS IN FILTERING.

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1. INTRODUCTION. SETTING THE STAGE.

Consider a stochastic dynamical system of the type

$$dx_t = f(x_t)dt + G(x_t)dw_t, \quad dy_t = h(x_t)dt + dv_t \tag{1.1}$$

where f,G,h are (sufficiently regular) vector and matrix valued functions, and w and v are unit variance Wiener processes independent of the initial state x(0) and independent of each other. We are interested in ways of calculating the conditional expectation $\hat{\phi}(x_t)$ (best least squares estimates) of functions $\phi(x_t)$ given the observations $y^t = \{y_s: 0 \leq s \leq t\}$ through time t. In particular we are interested in finite dimensional recursive filters for $\hat{\phi}(x_t)$. By definition this means a machine driven by the observations:

$$d\eta_t = \alpha(\eta_t)dt + \beta(\eta_t)dy_t \tag{1.2}$$

defined on a finite dimensional manifold M (so that $\eta_t \in M$ and

M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 591–604.

$\alpha(\eta_t)$, $\beta(\eta_t)$ are vectorfields on M), such that for a suitable output function

$$\gamma(\eta_t) = \hat{\phi}(x_t) \tag{1.3}$$

(Equations (1.2), (1.3) together form a finite dimensional recursive filter for the statistic $\hat{\phi}(x_t)$.

Now a certain unnormalized version $\rho(x,t)$ of the conditional density for x_t given y^t satisfies the Duncan-Mortensen-Zakai equation. Written in Fisk-Stratonovic form this equation is

$$d\rho(x,t) = (\mathcal{L} - \tfrac{1}{2}\sum_{i=1}^{p} h^i(x)^2)\rho(t,x)dt + \sum_{i=1}^{p} h^i(x)\rho(t,x)dy_t^i \tag{1.4}$$

where $\mathcal{L}$ is the Fokker-Planck operator

$$\mathcal{L}(\cdot) = \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}((GG^T)^{i,j}\cdot) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f^i\cdot) \tag{1.5}$$

(Here f^i, h^i is the i-th component of f, h and $(GG^T)^{i,j}$ is the (i,j)-th entry of the product of the matrix G with its transpose); cf. [7] for a derivation of the Duncan-Mortensen-Zakai equation. The Lie algebra of differential operators generated by $\mathcal{L} - \frac{1}{2}\sum h^i(x)^2$ and $h^1(x), \ldots, h^p(x)$ is called the estimation Lie algebra. (Here $h^j(x)$ is the multiplication operator $\varphi(x) \mapsto h^i(x)\varphi(x)$). We refer to the two appendices on "manifolds and vectorfields" and on "Lie algebras" in this volume for basic background information on these topics.

Both Brockett and Mitter have independently proposed the study of this estimation Lie algebra as an approach to the filtering properties of (1.1). This idea has been quite remarkably successful. Some evidence for this lies in the following. First equation (1.4) is bilinear (albeit infinite dimensional) and the Lie algebra generated by the matrices A,B in a control system $\dot{x} = Ax + (Bx)u$ is known to be influential ([5]). Second in the case of a linear system

$$(1.6) \quad dx_t = Axdt + Bdw_t, \; dy_t = Cx_t dt + dv_t$$

the Lie algebra of equation (1.5) and the Lie algebra of the Kalman filter of (1.6) are closely related [2]. The third point requires more explanation. Suppose that a finite dimensional filter (1.2), (1.3) existed. The equations are supposed to be in Fisk-Stratonovic form so that they make sense on a manifold [6]). Then we have two ways for calculating $\hat{\phi}(x_t)$: once via (1.2), (1.3) and once via (1.4) followed by normalization and integration. We can assume (1.2), (1.3) to be minimal and by a conjectured generalization of Sussmann's minimal realization result [20] we would have a homomorphism of the estimation Lie algebra onto the Lie algebra generated by the vectorfields $a(\eta_t)$ and $b(\eta_t)$ in (1.2). This is precisely what happens in the case of linear systems [2]. And inversely given such a homomorphism of Lie algebras satisfying an additional isotropy subalgebra condition a suitable generalization of the results of [13] or [23] would give a filter. Thus we would have a correspondence between statistics which are finite dimensionally recursively computable and certain homomorphisms of Lie algebras of the estimation algebra into Lie algebras of vectorfields on manifolds. Most of what follows makes little sense unless this is more or less true. There is, fortunately, a fair amount of positive evidence (linear case [2,4], finite state case [4,5] certain bilinear systems [15,26], cubic sensor [21,11]).

There are still more reasons for the importance of the estimation algebra involving representation theory, functional integration and deep analogies with quantum physics [17,18,19].

2. EXAMPLES OF ESTIMATION ALGEBRAS.

2.1. The simplest nonzero linear system, [2]. The stochastic dynamical system is $dx_t = dw_t$, with observations $dy_t = x_t dt + dv_t$. The estimation algebra is four dimensional with basis $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2, x, \frac{d}{dx}, 1$. It is a well-known Lie algebra (especially in physics). It is called the oscillator algebra.

2.2. Heisenberg-Weyl algebras. Let W_n denote the associative algebra $\mathbb{R}<x_1,\dots,x_n; \frac{d}{dx_1},\dots,\frac{d}{dx_n}>$ of all (partial) differential operators in $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ (of any order) with polynomial coefficients. As an associative algebra it is generated by the symbols $x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ subject to the relations suggested by the notations used, i.e. $x_i \frac{\partial}{\partial x_i} x_i - x_i \frac{\partial}{\partial x_i} = 1$, $x_i x_j = x_j x_i$, $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}$, and $x_i \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} x_i$ if $i \neq j$. A basis for W_n (as a vectorspace over $\mathbb{R}$) consists of the monomials $x^\alpha \frac{\partial^\beta}{\partial x^\beta} = x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \dots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$, $\alpha_i, \beta_j \in \mathbb{N} \cup \{0\}$. In this paper W_n is always considered as a Lie algebra (with the bracket operation $[D,D'] = DD'-D'D$). The Lie algebra W_n has a one dimensional centre $\mathbb{R}.1$ (consisting of scalar multiples of the identity operator) and $W_n/\mathbb{R}.1$ is simple.

2.3. The cubic sensor. The system is $dx_t = dw_t$ with observations, $dy_t = x_t^3 dt + dv_t$. In this case the estimation algebra is equal to all of W_1. For a proof cf. [10].

2.4. Quadratic observations. Now consider $dx_t = dw_t$, $dy_t = x_t^2 dt + dv_t$. Then the estimation algebra is $W_1^{(2)}$ which is the subalgebra of W_1 spanned by all monomials of the form $x^i \frac{d^j}{dx^j}$ with $i - j$ even.

2.5. Example of mixed linear bilinear type. The system is $dx_{1t} = dw_{1t}$, $dx_{2t} = x_{1t}dt + dx_{1t}dw_{2t}$ with observations $dy_t = x_{2t}dt + dv_t$. Here the estimation algebra turns out to be equal to W_2,[10].

2.6. Example. The system is $dx_{1t} = dw_t$, $dx_{2t} = x_{1t}^2dt$ with observations $dy_{1t} = x_{1t}dt + dv_{1t}$, $dy_{2t} = x_{2t}dt + dv_{2t}$. Here again the estimation algebra is W_2, [10].

2.7. Example,[15]. The system is $dx_{1t} = dw_t$, $dx_{2t} = x_{1t}^2dt$ with observations $dy_{1t} = x_{1t}dt + dv_t$. In this case the estimation Lie algebra has as a basis the operators

$$A = -x_1^2\frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} - \frac{1}{2}x_1^2,\quad B_i = x_1\frac{\partial^i}{\partial x_2^i},\quad C_i = \frac{\partial}{\partial x_1}\frac{\partial^i}{\partial x_2^i},$$

$$D_i = \frac{\partial^i}{\partial x_2^i}\quad i = 0,1,2,\ \ldots$$ with the bracket relations

$[A,B_i] = C_i$, $[A,C_i] = B_i + 2B_{i+1}$, $[B_i,C_j] = -D_{i+j}$ and all other brackets between basis elements equal to zero.

2.8. Example. The system is $dx_t = dw_t$ with observations $dy_t = (x_t+\varepsilon x_t^3)dt + dv_t$. Here ε is a (small) parameter. In this case one finds that the estimation algebra is equal to W_1 for all $\varepsilon \neq 0$ (and of course equal to the oscillator algebra if $\varepsilon=0$).

2.9. Example. The system is $dx_t = dw_{1t} + \varepsilon x_t dw_{2t}$ with observations $dy_t = x_t dt + dv_t$. In this also one finds that the estimation algebra is equal to W_1 for all $\varepsilon \neq 0$.

2.10. Degree increasing estimation algebras. Consider systems of the form $dx_t = f(x_t)dt + G(x_t)dw_t$, $dy_t = h(x_t)dt + dv_t$ and assume that f, G and h are smooth and that all components of f and G are zero for x = 0. Consider the Lie algebra of all differential operators of the form $\Sigma f_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}$, α a multiindex, $f_\alpha(x)$ smooth (finite sums). This algebra acts on the space $F(\mathbb{R}^n)$ of all smooth functions in $x_1, \ldots, x_n$. Let $F_i(\mathbb{R}^n)$ denote the subspace of all functions $\phi \in F(\mathbb{R})$ such that

$\frac{\partial^\alpha \phi}{\partial x^\alpha}(0) = 0$ for all α with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq i$. Then $F(\mathbb{R}^n)/F_i(\mathbb{R}^n)$ is a finite dimensional vectorspace (isomorphic to the vectorspace of all polynomials in $x_1, \ldots, x_n$ of total degree $\leq i$). Now under the assumptions on f and G stated, the Fokker-Planck operator $\mathcal{L}$ maps $F_i(\mathbb{R}^n)$ into itself and multiplication with h(x) always does so. Hence for these systems the estimation algebra L maps $F_i(\mathbb{R}^n)$ into itself. Let $L_i = \{D \in L \mid DF(\mathbb{R}^n) \subset F_i(\mathbb{R}^n)\} = \mathrm{Ker}(L \to \mathrm{End}(F(\mathbb{R}^n)/F_i(\mathbb{R}^n)))$. Then L_i is an ideal of L, L/L_i is finite dimensional, $L \supset L_1 \supset \ldots$ and if f, G, h are all three analytic then $\cap L_i = \{0\}$.

2.11. Pro-finite dimensional algebras. An infinite dimensional Lie algebra L will be called profinite dimensional if there exists a sequence of ideals $L_1 \supset L_2 \supset \ldots$ such that L/L_i is finite dimensional for all i and $\cap L_i = \{0\}$. Thus the degree increasing estimation algebras of 2.10 above are examples if f, G, h are analytic (or at least not flat at 0). Another example of a profinite dimensional Lie algebra is 2.7. The relevance of this property for the existence of (approximate) filters will be discussed in 6.1 below.

2.12. Identification of linear systems with noise corrupted coefficients.

The system is $dx_t = a_t x_t dt + dw_{1t}$, $da_t = dw_{2t}$ with observations $dy_t = x_t dt + dv_t$. The estimation algebra is again W_2.

3. WEYL ALGEBRAS.

As we saw in section 2 above the (Heisenberg-)Weyl algebras W_n often occur as estimation algebras. Thus, according to the introduction, it becomes important to study the homomorphisms of W_n into the Lie algebras V(M) of vectorfields on finite dimensional manifolds M.

3.1. Nonimbedding theorem. Let M be a finite dimensional smooth manifold. Then for all $n \geq 1$ there are no nonzero homomorphisms of Lie algebras $W_n \to V(M)$ or $W_n/\mathbb{R}.1 \to V(M)$.

3.2. The cubic sensor. For the cubic sensor the conjectured generalization of Sussmann's minimal realization result has been proved (during this conference in fact) [21,11] and as a consequence of this and 3.1, 2.3 we have

3.3. Theorem. For the cubic sensor 2.3 there exist no nonzero statistics which can be computed by finite dimensional filters (1.2) - (1.3).

Of course this theorem says nothing about approximate methods. The reader is also invited in this connection to look at the contribution by M. Zakai in this volume [22].

It seems most likely that the proof of theorem 3.1 can be adapted easily to yield a similar result for $W_1^{(2)}$ which would give an analogue of theorem 3.3 for example 2.4.

4. A NUMBER OF OPEN PROBLEMS.

The results of sections 2 and 3 above suggest a large number of open problems.

4.1. Problem. First and foremost there is the question of the appropriate generalizations of the results of Krener and Sussmann discussed in section 1.

4.2. Problem. Determine (up to isomorphism) all finite dimensional Lie subalgebras of W_1 and more generally W_n. An obvious example is Q_n which as a vector space is spanned by the monomials $x^\alpha \frac{d^\beta}{dx^\beta}$ with $|\alpha| + |\beta| \leq 2$. Thus Q_1 is 6 dimensional. Another example is the subalgebra spanned as a vector space by $\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}$, $1, x, \ldots, x^m$ for some m. Conjecturally all finite dimensional subalgebras of W_1 are isomorphic to subalgebras of one of these. Thus the algebra spanned by $x\frac{\partial}{\partial x}$, $x^2\frac{\partial}{\partial x}$, $\frac{\partial}{\partial x}$, 1 which is isomorphic to $gl_2(\mathbb{R})$ is also isomorphic to the subalgebra of Q_1 spanned by $\frac{d^2}{dx^2}$, x^2, 1, $x\frac{d}{dx}$. Another example of a finite subalgebra of W_1 is the linear span of

$1, x, \frac{d}{dx}, x^2, x\frac{d}{dx} + x^3, \frac{d^2}{dx^2} + 2x^2\frac{d}{dx} + x^4$ which is isomorphic to Q_1.

4.3. Problem. Are there finite dimensional estimation algebras (in W_n) which are not isomorphic to the estimation algebra of a linear system? In particular can the classical finite dimensional Lie algebras arise as estimation Lie algebras?

4.4. Problem. Consider the Lie algebra of all expressions $\Sigma f_i(x) \frac{\partial}{\partial x_i} + g(x)$, $f_i(x)$, $g(x)$ smooth functions on $\mathbb{R}^n$. Can this Lie algebra arise as an estimation Lie algebra (up to isomorphism)?

4.5. Problem. The classical simple infinite dimensional (filtered) Lie algebras of Lie and Cartan are all subalgebras of the algebra $\hat{V}_n$ of formal vectorfields in n-variables. Can one of these algebras arise as an estimation Lie algebra? There are many infinite dimensional Lie algebras contained in the V(M). One example of an infinite dimensional estimation algebra which can be embedded in a V(M) occurs in [14]. More are needed.

4.6. Problem. If there is no noise in the state equations the Fokker-Planck operator degenerates to a first order differential operator and the resulting estimation algebra is always naturally an algebra of vectorfields. What does this imply for filtering, and what happens if the noise term in the state equations is given a coefficient ε and we let ε go to zero?

4.7. Problem. Develop tests for the finite dimensionality of the Lie algebra generated by a finite set of elements of W_n.

5. MORPHISMS BETWEEN SYSTEMS, COMPATIBLE REPRESENTATIONS AND ISOTROPY SUBALGEBRAS.

5.1. <u>Isotropy subalgebras</u> Let $L \subset V(M)$ be a Lie algebra of vectorfields on M. Let $x \in M$. Then the isotropy subalgebra L_x of L at x consists of all $X \in L$ such that the tangent vector at x of X is zero. Equivalently if X is seen as a derivation on the algebra F(M) of smooth functions on M, cf. [12] on the appendix on manifolds and vectorfields in this volume, then $X \in L_x$ iff $(Xf)(x) = 0$ for all $f \in F(M)$. Now let $\phi : M \to N$ be a morphism of smooth manifolds and suppose that ϕ is compatible with a homomorphism of Lie algebras $\alpha : L \to V(N)$. This means that $(\alpha X)_{\phi(m)} = d\phi(X_m)$ for all $m \in M$. In terms of derivations it means that

$$X(\phi^*(g)) = \phi^*(\alpha(X)(g)), \quad g \in F(N) \tag{5.2}$$

where $\phi^*(g)$ is the function on M defined by $\phi^*(g)(m) = g(\phi(m))$. Another way of stating (5.2) is that ϕ^* is a homomorphism of L-modules where V(N) acquires its L-module structure via α. It immediately follows from (5.2) that if $\phi : M \to N$ and $\alpha : L \to V(N)$ are compatible, then for all $x \in M$, $\alpha(L_x) \subset V(N)_{\phi(x)}$. This is the extra condition on homomorphisms of Lie algebras involved in Krener's theorem [13]; cf. also Sussmann's paper [23].

5.3. <u>Estimation algebras with representations</u>. Thus to construct finite dimensional filters we need not just any homomorphism of Lie algebras from the estimation Lie algebra into a V(M), we need one which is compatible with the natural representation of the estimation algebra acting on (unnormalized) densities $\rho(x)$ and V(M) acting on F(M). That is we need a homomorphism of Lie algebras $\alpha: L \to V(M)$ together with a linear map ψ: {functions on densities} $\to$ F(M) which is a homomorphism of L-modules (where V(M) acquires its L-module structure via α). It is easy to find homomorphisms of Lie algebras $L \to V(M)$ which do not satisfy this extra condition. Thus for example in [14] there occurs an estimation Lie algebra with basis a, b_1, b_2,... and bracketts $[a,b_i] = b_{i+1}$, $[b_i,b_j] = 0$. An ad hoc representation of this Lie algebra by means of vectorfields is $a \mapsto e^y \frac{\partial}{\partial y}$, $b_i \mapsto (i-1)!e^{iy} \frac{\partial}{\partial x}$, and this realization of L does not correspond to a filter for the conditional density.

6. APPROXIMATE AND SUBOPTIMAL FILTERS.

6.1. Power series expansions. Let us consider again the case of the degree increasing estimation algebras of section 2.10 above. In this case we had a homomorphism of Lie algebras $L \to L/L_i \to \mathrm{End}(F/F_i)$ (where F is the space of smooth functions on $\mathbb{R}^n$). Now F/F_i is a finite dimensional vectorspace, say $F/F_i \simeq \mathbb{R}^r$. Choose coordinates $\eta_1, \ldots, \eta_r$ in $\mathbb{R}^r$ and map $A \in \mathrm{End}(\mathbb{R}^r)$ to the vectorfield $\Sigma a_{ij}\eta_i \frac{\partial}{\partial \eta}$. This gives us a homomorphism of Lie algebras $L \to V(\mathbb{R}^r)$ and this homomorphism comes together with a natural map {space of smooth densities} $\to \mathbb{R}^r$, viz. $\rho \mapsto (\frac{\partial \rho}{\partial x^\alpha}(0))_\alpha$ where α runs through all multiindices such that $|\alpha| \leq i$, and, virtually by the definition of the various maps, $L \to V(\mathbb{R}^r)$ is compatible with {space of smooth densities} $\to \mathbb{R}^r$. Thus the isotropy subalgebra condition is automatically fulfilled in this case. So that (modulo the appropriate generalizations of [13], [23]) we should obtain a sequence of filters for various statistics $\psi_1, \psi_2, \psi_3, \ldots$. The fact that $\cap L_i = \{0\}$ if f,G,h are analytic should correspond to a statement that the statistics $\psi_1, \psi_2, \ldots$ determine $\rho(x,t)$ uniquely.

In fact if $\rho(x,t)$ admits a power series expansion $\rho(x,t) = \Sigma x^\alpha \rho_\alpha(t)$, then these various statistics ought to be the $\sum_{|\alpha|\leq i} x^\alpha \rho_\alpha(t)$. Quite possibly these filters exist even when $\rho(x,t)$ cannot be shown to admit a power series expansion and then converge to $\rho(x,t)$ in some singular way. More generally one may hope for generalized power series expansions when the estimation algebra is profinite dimensional (in an isotropy subalgebra respecting way).

6.2. Perturbation and deformation techniques. As we have seen the estimation Lie algebras of examples 2.8 and 2.9 are both equal to W_1 for all $\varepsilon \neq 0$. Yet the associated "Lie algebras mod ε^n" are finite dimensional for all n [8]. There should be approximate filters corresponding to these Lie algebras corresponding (more or less) to the calculation of the first n terms in a power series development (if it exists) of $\rho(t,x)$ in powers of ε, $\rho(t,x) = \rho_0(t,x) + \varepsilon\rho_1(t,x) + \varepsilon^2\rho_2(t,x) + \ldots$ Similar ideas seem to be involved in [1].

6.3. Suboptimal filters. If one throws away the second observation in example 2.6 one finds example 2.7 which has an estimation algebra of profinite dimensional type. Moreover for this particular example the various ideals do correspond to filters for various moments [15]. These are suboptimal filters in the case of the original system. The question arises whether quite generally a quotient of a sub-Lie-algebra of the estimation algebra corresponds (under suitable compatibility, i.e. isotropy subalgebra, conditions) to a suboptimal filter for some statistic. We are also curious to know whether there exists an estimation Lie algebra L which is not itself realizable in a V(M) but which is a union of subalgebras $L = \sum_{i=1}^{\infty} L_i, L_1 \subset L_2 \subset \ldots$ such that each L_i is realizable in some V(M).

6.4. Changes in output structure. Quite generally the following question seems to merit investigation: What happens to the estimation algebra when the output structure is changed, e.g. when an output is added, when the output is processed through another system before being observed, when a component of the state is made observable, ... etc.

ACKNOWLEDGEMENT.

The work of S.I. Marcus was supported in part by the U.S. National Science Foundation under grant ENG 76-11106.

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