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ON LIE ALGEBRAS OF VECTORFIELDS, LIE ALGEBRAS OF DIFFERENTIAL OPERATORS AND (NONLINEAR) FILTERING.

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Dedicated to my teacher and friend Nico Kuiper on the occasion of his 60^{th} birthday with gratitude for the attitude to mathematics that he taught me by example and instruction.

1. Introduction and/or abstract.

The (nonlinear) optimal (recursive) filtering problem which gives rise to the mathematical problems to be described and discussed below is the following. Suppose the state x_t of a stochastic system evolves according to the Ito stochastic differential equation $dx_t = f(x_t)dt + G(x_t)dw_t$ where f and G are vector and matrix valued functions of the appropriate dimensions and w_t is a Wiener (noise) process. The state x_t is not directly observable. What can be measured are noise corrupted outputs y_t depending on x_t according to $dy_t = h(x_t)dt + dv_t$, where v_t is another Wiener noise process. We now want to find the best estimate $\hat{x}_t$ of x_t given $y^t = \{y_s : 0 \leq s \leq t\}$ and more precisely we would like to construct a finite dimensional "machine" for calculating $\hat{x}_t$ recursively. (What this means is explained below). It now turns out that there is a natural Lie algebra (of differential operators) associated to this problem and that the structure of this Lie algebra and questions of representability (by vectorfields) concerning this algebra are intimately connected to the existence of optimal finite dimensional recursive filters. Much what follows below reports on joint work with Steve Marcus of the University of Texas at Austin.

2. Representation questions.

Let me start with (a simplified version of) a representation problem of Lie algebras which has acquired considerable importance in the theory of optimal filtering and then proceed (in the next section) to discuss how this representation problem arises. The question is :

(2.1.) Problem. When can a Lie algebra (usually infinite dimensional) over $\mathbb{R}$ be realized (represented) as a Lie algebra of smooth vectorfields on a smooth finite dimensional manifold, and when can this be done analytically (algebraically) on a real analytic (algebraic) manifold.

Below, there are two general results concerning this. Very little else is known. First a bit of notation. Let M be a smooth finite dimensional manifold. Then $V(M)$ denotes the Lie algebra of smooth vectorfields on M (usually viewed as first order differential operators, i.e. as derivations of $F(M)$ the ring of smooth functions on M, cf. [19,Ch.I, §2]) ; and if M is real analytic then $V_{an}(M)$ denotes the Lie algebra of analytic vectorfields.

(2.2.) The finite dimensional case. Let L be a finite dimensional Lie algebra over $\mathbb{R}$. Then, by Ado's theorem [2, §7], L has a finite dimensional faithful representation, i.e. for some $n \in \mathbb{N}$ there is an injective homomorphism $L \to gl_n(\mathbb{R})$ where $gl_n(\mathbb{R})$ denotes the Lie algebra of real $n \times n$ matrices. Now

$$(2.3.) \quad (a_{ij}) \mapsto \sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j}$$

defines an injective homomorphism of Lie algebras $gl_n(\mathbb{R}) \to V(\mathbb{R}^n)$. Combining this with Ado's theorem it follows that every finite dimensional Lie algebra can be represented as a Lie algebra of vectorfields on $\mathbb{R}^n$ for some n.

For the applications to be discussed below it is also most important to find, if possible, low dimensional M such that L can be imbedded in $V(M)$. This leads to

(2.4.) Problem. Given a Lie algebra L (finite or infinite dimensional). What is the smallest natural number m such that L can be imbedded in $V(M)$, where M

is a smooth manifold of dimension m .

I know nothing about the question of whether the topological type of M will play a role here or whether the question is essentially local. Thus for instance one would like to know whether the extra requirement "M is compact" or "M is $\mathbb{R}^n$ for some n " would make a difference in problems (2.1.) and (2.4.).

Of course, problem (2.4.) can also be asked concerning imbeddings $L \to V_{an}(N)$, N an analytic manifold. The answers can certainly be different. Thus an abelian Lie algebra of countable dimension can be imbedded in $V(\mathbb{R})$ but an n-dimensional abelian Lie algebra can not be imbedded in $V_{an}(\mathbb{R})$ if $n \geq 2$, but can be imbedded in $V_{an}(\mathbb{R}^2)$ (for all n including $n = \infty$) .

(2.5.) Example : The oscillator algebra. The Lie algebras L arising from filtering problems as described in section 1. above are all Lie algebras of (higher order) differential operators. A nice simple example for the linear case is the so-called oscillator algebra $\underline{h}$ which has a basis $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2$, x, $\frac{d}{dx}$, 1 . (Here a function $f(x)$ is considered as the multiplication operator $p(x) \mapsto f(x)p(x)$, and the bracket of two (differential) operators D_1, D_2 is defined as $[D_1, D_2] = D_1D_2 - D_2D_1$). Writing $A = \frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2$ one checks that $[A,x] = \frac{d}{dx}$, $[A,\frac{d}{dx}] = x$, $[\frac{d}{dx},x] = 1$, $[A,1] = [x,1] = [\frac{d}{dx},1] = 0$ so that we have a four dimensional Lie algebra with a one dimensional center $\mathbb{R}.1$. Let $\underline{k}$ be the Lie algebra $\underline{h}/\mathbb{R}.1$. This algebra admits the following representation in $V_{an}(\mathbb{R}^2)$ (which comes from the so-called Kalman filter ; c.f. below in section (3.3.)).

$$(2.6.) \quad \frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2 \mapsto (1-y^2)\frac{\partial}{\partial y} - yz\frac{\partial}{\partial z} , \quad \frac{d}{dx} \mapsto \frac{\partial}{\partial z} , \quad x \mapsto y\frac{\partial}{\partial z}$$

(2.7.) The Heisenberg-Weyl algebras. The Lie algebras of vectorfields $V(M)$ are quite large and contain most of the better known Lie algebras. For instance the simple infinite dimensional filtered Lie algebras (of Lie and Cartan) , [9, 29] are defined as subalgebras of $V(M)$ and as another example the free Lie algebra on 2 generators can be imbedded in $V(\mathbb{R})$.

Now consider the Heisenberg-Weyl algebras $W_n = \mathbb{R}\langle x_1,\dots,x_n,\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\rangle$ of all differential operators (of any order) in $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ with polynomial coefficients. (A basis for W_n as vector space is formed by the monomials $x^\alpha \frac{\partial^\beta}{\partial x^\beta}$ (where α and β are multiindices)). Two elementary facts concerning the Weyl algebras W_n are

(2.8.) Proposition. W_n has a one dimensional centre $\mathbb{R}.1$, and $W_n/\mathbb{R}.1$ is simple.

And concerning its relations with the Lie algebras $V(M)$ we have

(2.9.) Theorem. ([16]). Let M be a finite dimensional smooth manifold, $n \in \mathbb{N} = \{1,2,\dots\}$. Then there are no nonzero homomorphisms $W_n \to V(M)$, $W_n/\mathbb{R}.1 \to V(M)$.

The present proof [16] of this theorem first reduces to the case of Lie algebra homomorphisms into the Lie algebra of formal power series vectorfields $\hat{V}_m = \{\sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} : f_i(x)$ formal power series in $x_1,\dots,x_m\}$, by killing of the ideal of all germs of vectorfields near a point whose coefficients are flat functions. Now $\hat{V}_m$ has a natural filtration $\hat{V}_m = L_{-1} \supset L_o \supset L_1 \supset \dots$ where L_j consists of all expressions $\Sigma a_{\alpha,i}x^\alpha \frac{\partial}{\partial x_i}$ for which $a_{\alpha,i} = 0$ if $|\alpha| = \alpha_1 + \dots + \alpha_n \leq i$. Further $\hat{V}_m/L_j$ is a finite dimensional vector space for all j, $[L_i,L_j] \subset L_{i+j}$ and $\bigcap_j L_j = \{0\}$. Thus if there existed a nonzero $W_n \to \hat{V}_m, W_n/\mathbb{R}.1 \to \hat{V}_m$, then W_m or $W_m/\mathbb{R}.1$ would inherit a similar filtration.

One now proves that the Lie algebras W_m and $W_m/\mathbb{R}.1$ do not admit such filtration. This part of the proof is long and computationally and combinatorially involved. It would be nice to have also a more conceptual proof perhaps involving the relation of W_n with classifying spaces for foliations and/or using Gęlfand-Fuks cohomology, [28] and [9, especially the last section].

An obvious question to ask concerning the W_n is

(2.10.) Problem. Characterize W_n in terms of some of its properties. And, particularly in view of the matters to be discussed below in section 3, it would be nice to have criteria to decide when a given subset of elements of W_n generates all of W_n (as a Lie algebra).

Let me also state the natural extension problem (such extensions arise e.g. when treating linear identification or adaptive control problems as nonlinear filtering problems) :

(2.11.) Problem. Let $0 \to \underline{a} \to \underline{g} \xrightarrow{\pi} \underline{h} \to 0$ be an exact sequence of Lie algebras with abelian kernel $\underline{a}$. Suppose that we have given an imbedding $\alpha : \underline{h} \to V(M)$. Does there exist an imbedding $\tilde{\alpha} : \underline{g} \to V(M')$ which lifts the present one in the sense that there exists a morphism of smooth manifolds $\phi : M' \to M$ which takes the vectorfield $\tilde{\alpha}(X)$ into $\alpha(\pi(X))$ for all $X \in \underline{g}$. (Because $\underline{a}$ inbeds in $V(\mathbb{R})$ and $V_{an}(\mathbb{R}^2)$ one naturally thinks in terms of M' as a product space $M' = M \times \mathbb{R}^s$ or possibly a vectorbundle over M).

(2.12.) Representations in $\hat{V}_n$. To conclude this section let us consider briefly the formal part of problem (2.1.), i.e. the question of when a Lie algebra L can be represented as a subalgebra of $\hat{V}_n$. Then L inherits a filtration $L = L_{-1} \supset L_o \supset L_1 \supset \dots$ such that $[L_i, L_j] \supset L_{i+j}$, $\cap L_i = \{0\}$ and L/L_i is finite dimensional. Moreover there is a growth condition on $\dim L/L_i$ which says that this number grows slower than ci^n as $i \to \infty$ for a suitable constant c. This leads to

(2.13.) Problem. Let L be a filtered Lie algebra, $L = L_{-1} \supset L_o \supset \dots$, $[L_i, L_j] \subset L_{i+j}$, $\dim L/L_i < \infty$, $\cap L_i = \{0\}$. Suppose moreover that for some n there is a constant c such that $\dim L/L_i \leq ci^n$ for all i. Does there then exist an imbedding of Lie algebras $L \to \hat{V}_m$ for some $m \in \mathbb{N}$.

Obviously, the answer is yes if one adds the primitivity and transitivity requirements which make L one of the simple infinite Lie algebras of Lie and Cartan [9]. Thus the answer seems to be yes for the basic building blocks for this class of algebras. I should add that large classes of the Lie algebras of nonlinear filtering theory are of this filtered type [16].

3. Nonlinear filtering and Lie algebras.

Now let us see what the Lie algebra representation problems of section 2 above have to do with optimal recursive filtering.

Consider a stochastic dynamical system

(3.1.) $dx_t = f(x_t)dt + G(x_t)dw_t,\ dy_t = h(x_t)dt + dv_t,\ x_0 = x(0), y_0 = 0$

Here f, G, h are vector and matrix valued functions of the appropriate dimensions, w_t and v_t are unit variance Wiener noise processes. The processes w_t and v_t are assumed independent of each other and of the initial state $x(0)$. The problem is to find recursive methods to calculate $\hat{x}_t = E[x_t|y^t]$ the (least squares) best estimate of the state x_t given the observations up to time t, $y^t = \{y_s : 0 \leq s \leq t\}$. More generally we are interested in the best estimates $\hat{\phi}(x_t)$ of functions $\phi(x_t)$ of the state given y^t. Here by definition a finite dimensional recursive estimator for $\hat{\phi}(x_t)$ is a system on a finite dimensional manifold of the form

(3.2.) $d\eta_t = \alpha(\eta_t)dt + \beta(\eta_t)dy_t,\ \hat{\phi}(x_t) = \gamma(\eta_t)\ ,\ \eta_0 = \eta(0)$

where α, β are vectorfields on a finite dimensional manifold M. Such a machine permits the calculation of $\hat{\phi}(x_t)$ by a simple updating procedure for η_t after which $\hat{\phi}(x_t)$ is obtained by applying γ. Obviously such a procedure has advantages in "on line" situations and it is also not totally unreasonable to ask for such calculating devices because the Kalman filter of considerable fame and enormous applicability is precisely such a machine.

(3.3.) Example : The Kalman-Bucy filter. Suppose we are dealing with a linear stochastic system, i.e. a system of the form

(3.4.) $dx_t = Ax_t dt + Bdw_t,\ dy_t = Cx_t dt + \sqrt{R}dv_t$

where A, B, C, are matrices of the appropriate dimensions and R is a positive definite symmetric (covariance) matrix. All may depend on t. Then an optimal recursive filter for the conditional state $\hat{x}_t$ is given by the equations

(3.5.) $d\hat{x}_t = A\hat{x}_t dt + P_t C^T R^{-1}(dy_t - C\hat{x}_t dt)$, $\hat{x}_0 = \hat{x}(0)$

(3.6.) $dP_t = (AP_t + P_t A^T + BB^T - P_t C^T R^{-1} C P_t)dt$, $P_0 = P(0)$

where the upper T denotes transposes. Here (3.6.) is an equation for the square matrix P . (Matrix Riccati equation). This is precisely a machine of the type (3.2.) for $\hat{x}_t$, with $\eta_t = (\hat{x}_t, P_t)$ and γ the projection on the first coordinate.

(3.7.) <u>The Duncan-Mortensen-Zakai equation.</u> For simplicity (of notation mainly) we shall from now on assume that $h(x_t)$ is scalar valued. Suppose that the system (3.1.) is sufficiently regular so that $\hat{x}_t$ admits a probability density $p(x,t)$, the conditional probability of x_t given y^t . A certain unnormalized version $\rho(x,t)$ of $p(x,t)$ then satisfies the so-called Duncan-Mortensen-Zakai equation

(3.8.) $d\rho(t,x) = L\rho(t,x)dt + h(x)\rho(x,t)dy_t$

where L is the Fokker-Plank operator defined by

(3.9.) $L(.) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}((GG^T)_{ij}.) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f_i.)$

where f_i is the i-th component of $f(x)$ and $(GG^T)_{ij}$ the (i,j)-th component of $G(x)G^T(x)$. Cf. e.g. [8] for a derivation of equation (3.8.).

Equation (3.8.) is an infinite dimensional version of a so-called bilinear system, that is a system of equations of the form $\dot{x} = Ax + Bxu$ where A and B are matrices. And for such systems it is known that the Lie algebra generated by the matrices A and B plays an important role in studying such systems (Wei-Norman theory ; cf. e.g. [6]) . This analogy was first noticed by Brockett and the idea to analyze the Lie algebra generated by the two operators L and $h(x)$ to study the optimal recursive filtering properties of nonlinear systems (3.1.) seems to be due independently to Brockett and Mitter. [3,4,5,6,25,26,27] .

Equation (3.8.) is an Ito stochastic differential equation. In order to be able to calculate the brackets of the differential operators involved in it in the normal way it is necessary to bring it in its Fisk-Stratonovic form

(3.10.) $d\rho(t,x) = (L - \frac{1}{2}h^2(x))\rho(t,x)dt + h(x)\rho(t,x)dy_t$

The Lie algebra generated by the two operators $L - \frac{1}{2}h^2(x)$, $h(x)$ is called the <u>estimation Lie algebra</u> of the system (3.1.).

(3.11.) <u>Example : Linear noise linearly observed</u> [3] . The simplest nontrivial linear system (3.4.) is undoubtedly the one-dimensional system

(3.12.) $dx_t = dw_t,\ dy_t = x_t dt + dv_t$

In this case the two-operators occurring in the (Fisk-Stratonovic form of the) Duncan-Mortensen-Zakai equation (3.10.) are $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2, x$. Thus the Lie algebra generated by them is the four dimensional oscillator algebra of example (2.5.) above. The Kalman filter for $\hat{x}$ in this case is given by the equations

(3.13.) $d\hat{x}_t = P_t(dy_t - \hat{x}_t dt),\ dP_t = (1-P_t^2)dt$

so that the two vectorfields involvedin this calculating machine of type (3.2.) are $a = (1-P^2)\frac{\partial}{\partial P} - Px\frac{\partial}{\partial x}$, $b = P\frac{\partial}{\partial x}$. The Lie algebra generated by these two vectorfields is closely related to the oscillator algebra. It is in fact the quotient by its center of the oscillator algebra, cf. (2.5.) above. This relationship between the estimation Lie algebra of (3.12.) and the Lie algebra of the recursive filter (3.13.) is no accident [3] . It is also striking that the Lie algebra is precisely the Lie algebra of the Euclidean harmonic oscillator. It turns out that there are indeed deep analogies between the problems of nonlinear filtering and those of quantum field theory [25,27] .

(3.14.) <u>The estimation algebra and representation questions</u>. Now suppose that there is a machine of the type (3.2.) for estimating a certain statistic $\hat{\phi}(x_t)$. (Equations (3.2.) are supposed to be in Fisk-Stratonovic form, which, anyway, is necessary for stochastic equations on general manifolds [7]) . Then there are two ways to process the data y_s , $0 \leq s \leq t$ to obtain $\hat{\phi}(x_t)$. The first way is to run the y_s through the conditional density equation (3.10.) to obtain $\rho(t,x)$ (from a given starting density $\rho(0,x)$) ; from $\rho(t,x)$ calculate $p(t,x)$ by normalising and then obtain $\hat{\phi}(x_t)$ by integrating $\phi(x)$ against $p(t,x)$. The second way is to run our data through the machine (3.2.), which we can assume to be of minimal dimension. Thus we have two machines processing inputs with the same results. If both were finite dimensional this would imply [30] that there is a morphism from the part of the first machine reachable from the starting point to the second machine, which in turn implies that there is a homomorphism from the Lie algebra generated by the vectorfields of the first machine into the Lie algebra generated by the vectorfields of the second machine. Conjecturally this theorem extends (under suitable assumptions) to the case that the first machine is infinite dimensional. Thus if a finite dimensional machine (3.2.) for calculating $\hat{\phi}(x_t)$ exists there should be a corresponding homomorphism of Lie algebras from the estimation Lie algebra of the system to the Lie algebra generated by the vectorfields of the filter. This is precisely what happened in the case of the example (3.11.) above.

(3.15.) Homomorphisms between Lie algebras and morphisms between systems. There is a partial converse to the result discussed above [21] . It goes, roughly, as follows. Consider a system $\dot{x} = \alpha_1(x) + \beta_1(x)u$ on a manifold M_1 and a second system $\dot{x} = \alpha_2(x) + \beta_2(x)u$ on a manifold M_2 . Let L_i , $i = 1, 2$, be the Lie algebra of vectorfields generated by α_i, β_i . Let $x_i \in M_i$, $i = 1, 2$, and suppose that $\alpha_1 \mapsto \alpha_2$, $\beta_1 \mapsto \beta_2$ induces a homomorphism of Lie algebras $L_1 \to L_2$ which takes the isotropy subalgebra of L_1 at x_1 into the isotropy subalgebra of L_2 at x_2 . Then there is a morphism of manifolds from a neighbourhood of x_1 to a neighbourhood of x_2 which takes the trajectories of the first system into the trajectories of the second system.

Hopefully this result also extends to the case where the first manifold M_1 is infinite dimensional. Given a homomorphism of the estimation algebra into some $V(M)$ this is almost the same as exponentiating the resulting action of the (usually infinite dimensional) estimation algebra on M to an action of a semigroup of operators. In this connection I am curious to know whether similar phenomena can occur as in the case of actions of Banach Lie groups on finite dimensional manifolds. In that case one has the result [12] that under certain irreducibility and transitivity assumptions the Banach Lie group is necessarily finite dimensional.

Thus we have two more problems involving both system theoretic and representation theoretic ideas.

(3.16.) Two problems. Extend the "minimal realization results" of [30] and the "existence of morphisms of systems" of [21] to the infinite dimensional case.

(3.17.) Representing Lie algebras together with a module. Suppose that we have a morphism of manifolds $\phi : M \to N$ which induces a homomorphism of Lie algebras from a certain subalgebra $L \subset V(M)$ into $V(N)$. This is precisely the situation of (3.14.) and (3.15.) above. Let $\alpha : L \to V(N)$ be this homomorphism of Lie algebras. The map ϕ induces a homomorphism of the rings of functions $\phi^* : F(N) \to F(M)$ and because α is compatible with ϕ we have that ϕ^* is a homomorphism of L-modules where $F(N)$ acquires its L-module structure via α.

Of course ϕ is recoverable from ϕ^* by looking at the real ideals of $F(M)$ and $F(N)$, cf. [10].

Thus the representability problem of Lie algebras coming from filtering theory is not just a question of representing Lie algebras by vectorfields but a question of representing a Lie algebra together with a given representation by means of vectorfields.

(3.18.) Problem. Let L be a Lie algebra together with an L-module P. When does there exist a homomorphism of Lie algebras $L \to V(M)$, where M is finite dimensional smooth manifold, such that there exists also a morphism of L-modules $F(M) \to P$.

(3.19.) The case that P is finite dimensional. It is perhaps worth remarking that if P is a finite dimensional vectorspace (3.18.) is easy. Choose coordinates in P. To a function f on $P = \mathbb{R}^n$ associate the vector $\frac{\partial f}{\partial x_1}(0),\ldots,\frac{\partial f}{\partial x_n}(0)$ and define $L \to V(P)$, by $g \mapsto \Sigma\, a_{ij} x_i \frac{\partial}{\partial x_j}$ if (a_{ij}) is the matrix by which g acts on P. Then $F(P) \to P$ is indeed a homomorphism of L-modules.

This case is in fact relevant in the setting discussed above because it may happen that there are submodules of finite codimension in the space of functions on which L acts. This happens e.g. when the functions f and G in (3.1.) are both zero in 0, cf. [16].

4. Estimation Lie algebras.

Given a stochastic system (3.1.) (with scalar observations) we have discussed a certain Lie algebra associated to it generated by the two operators $L - \frac{1}{2} h^2(x)$, $h(x)$ occurring in equation (3.10.), and we have seen that this Lie algebra has much to say about the existence or nonexistence of finite dimensional recursive filters for various statistics of the conditional state. This algebra is called the estimation Lie algebra of the system (3.1.), and it is an almost totally open question which algebras can arise in this way and, how to decide when the algebra will be infinite or finite dimensional. Let us start with some examples.

(4.1.) Example : The cubic sensor. Consider the system

(4.2.) $dx_t = dw_t \ , \ dy_t = x_t^3 dt + v_t$

In this case the estimation algebra is generated by $\frac{d^2}{dx^2} - \frac{1}{2} x^6, \ x^3$, and we have

(4.3.) Theorem ([16]). The estimation algebra of the cubic sensor is $W_1 = \mathbb{R} < x, \frac{d}{dx} >$. For this particular system the conjectural statement of (3.14.) above has been proved [31] , [18] so that combined with theorem (2.9.) this result implies.

(4.4.) Theorem ([31]). There exist no finite dimensional exact filters (3.2.) for any statistic $\hat{\varphi}(x_t)$, φ nonconstant, of the cubic sensor (4.2.).

(4.5.) Example ([23]). Consider the two dimensional system

(4.6.) $dx_{1t} = dw_t, \ dx_{2t} = x_{1t}^2 dt \ ; \ dy_t = x_t dt + dv_t$

The estimation Lie algebra of this example has as basis the operators a , b_i, c_i, d_i, $i \in \mathbb{N} \cup \{0\}$ given by

$$a = -x_1^2 \frac{\partial}{\partial x_2} + \frac{1}{2} \frac{\partial^2}{\partial x_1^2} - \frac{1}{2} x_1^2, \ b_i = x_1 \frac{\partial^i}{\partial x_2^i} \ , \ c_i = \frac{\partial}{\partial x_1} \frac{\partial^i}{\partial x_2^i} \ , \ d_i = \frac{\partial^i}{\partial x_2^i}$$

with the bracket relations $[a,b_i] = c_i$, $[a,c_i] = b_i + 2b_{i+1}$,

$[a,d_j] = [b_i,d_j] = [c_i,d_j] = 0$

$[b_i,c_j] = -d_{i+j}$, $[b_i,b_j] = [c_i,c_j] = 0$. This estimation algebra has many ideals and these do indeed correspond to exact filters for various statistics [23].

(4.7.) Example ([22]). Consider the linear system with partially unknown parameters

$$dx_t = ax_t dt,\ da = 0,\ dc = 0\ ;\ dy_t = cx_t dt + dv_t\ ;\ x_0 = x(0),\ y_0 = 0$$

In this case the estimation algebra has a basis $b_o = a + ax\frac{\partial}{\partial x} + \frac{1}{2}c^2x^2$, $b_i = a^i cx$, $i = 1,2,\ldots$ with the bracket relations $[b_i,b_j] = 0$, $i,j \geq 1$, $[b_o,b_i] = b_{i+1}$. It is perfectly easy to represent this Lie algebra by means of vectorfields on $\mathbb{R}^2$, e.g. by assigning to b_o the vectorfield $e^y\frac{\partial}{\partial y}$ and to b_i the vectorfield $(i-1)!e^{iy}\frac{\partial}{\partial x}$. This would give a 2-dim. calculating machine and it seems most unlikely that this can give information for all three (independent) unknowns a,c,x. The four dimensional representation $b_o \mapsto a\frac{\partial}{\partial y} + ax\frac{\partial}{\partial x} + (\frac{1}{2}c^2x^2)\frac{\partial}{\partial y}$, $b_i \mapsto a^i cx\frac{\partial}{\partial y}$ will do a better job. Thus as is also clear from (3.15.) above not all representations of the estimation Lie algebra will be relevant for filters.

(4.8.) Example ([13]). Consider the stochastic system

(4.9.) $dx_t = dw_t,\ dy_t = (x_t + \varepsilon x_t^3)dt + dv_t$

where ε is a (small) fixed parameter. In this case one finds that the estimation algebra is equal to W_1 for all $\varepsilon \neq 0$.

(4.10.) Example ([14]). Consider the stochastic system

(4.11.) $dx_t = dw_{1t} + \varepsilon x_t dw_{2t},\ dy_t = x_t dt + dv_t$

where again ε is a (small) fixed parameter. In this case also one finds that the estimation algebra is equal to W_1 for all $\varepsilon \neq 0$.

These examples and several more suggest that estimation algebras have a strong tendency to be equal to a Heisenberg-Weyl algebra suggesting the question (conjecture really).

(4.12.) Question. Let f,g and h in the stochastic system (3.1.) be polynomial in $x_1, x_2, \ldots, x_n$. Is it true that generically the estimation algebra of (3.1.) is equal to W_n ?

More ambitiously one would like to know all subalgebras of W_n which can arise as an estimation algebra and in particular

(4.13.) Question. Are there (up to isomorphism) other finite dimensional estimation algebras in W_n then the ones coming from linear systems ?

One result in this direction can be found in [25]:

(4.14.) Theorem ([25]). Consider a one dimensional nonlinear system of the form $dx_t = f(x_t)dt + dw_t$, $dy_t = h(x_t)dt + dv_t$. Then the estimation algebra is finite dimensional only in the case $h(x) = \alpha x + \beta$, $f_x + f^2 = ax^2 + bx + c, \alpha, \beta, a, b, c \in \mathbb{R}$.

For polynomial f this means that f is of the form $f(x) = dx + e$. For more general f and h the resulting class of filtering problems is one which was discovered by Benes ([1]) and this class is equivalent in a certain precise way to the filtering problem of example (3.11.), ([25]).

(4.15.) Problem. Which subalgebras of W_n can arise as estimation algebras ? Similar questions have come up in quantum physics [20], suggesting additional evidence concerning the deep relations between the problems of nonlinear filtering and those of quantum physics. One striking result from [20] is the following.

(4.16.) Theorem. Let L be a semisimple Lie algebra over $\mathbb{C}$ of rank r . Then L cannot be realized in $W_n \otimes \mathbb{C} = \mathbb{C} < x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} >$ if $r > n$.

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