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REPRESENTATION OF S_n AND THE GEOMETRY OF LINEAR SYSTEMS

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1. Introduction

Let $K = (K_1,\ldots,K_m)$, $K_1 \geq \ldots \geq K_m$, $K_i \in \mathbb{N} \cup \{0\}$, $\Sigma K_i = n$ be a partition of n. We define a partial order on the set of all m-part partitions of n as follows

$$K > K' \Leftrightarrow \sum_{i=1}^{r} K_i \leq \sum_{i=1}^{r} K'_i, \quad r=1,\ldots,m \qquad (1.1)$$

We shall say that K *specializes to* K' or that K *is more general than* K' if (1.1) holds. The reverse ordering has been called the dominance order [1]. This order occurs in many different parts of pure and applied mathematics and we now proceed to discuss some of these.

1.2. The Snapper Conjecture

Let $K = (K_1,\ldots,K_m)$ be a partition of n. Let S_K be the subgroup $S_{K_1} \times S_{K_2} \times \ldots \times S_{K_m}$ of S_n, the symmetric group on n letters. For example $S_{(2,2,1)} \subset S_5$ is the subgroup consisting of the permutations (1), (12), (34), (12)(34). Let $\rho(K)$ be the representation of S_n obtained by taking the trivial representation of the subgroup S_K and inducing it up to S_n. Then the Snapper conjecture says that $\rho(K)$ is a direct summand of $\rho(K')$ if $K < K'$. Proofs of this statement can be found in [2] and [3].

1.3. The Gale-Ryser Theorem [5],[6]

Let μ and ν be two partitions of n. Then there is a matrix of zeros and ones whose columns sum to μ and whose rows sum to ν iff $\nu \geq \mu^*$. There μ^* is the dual partition of μ defined by $\mu^*_i = \#\{j \mid \mu_j \geq i\}$. For example $(2,2,1)^* = (3,2)$. As a rule we shall not distinguish between two partitions if one of them is obtained from the other by adding some zeros.

1.4. Double Stochastic Matrices ([5])

A matrix $M = (m_{ij})$ is called double stochastic if $m_{ij} \geq 0$ for all i,j and $\sum_i m_{ij} = 1$ for all j and $\sum_j m_{ij} = 1$ for all i. Let μ and ν be two partitions of n. Then there is a double stochastic matrix M such that $\mu = M\nu$ (so that μ is an average of ν) if $\mu > \nu$.

1.5. Completely Reachable Systems

Let $L^{cr}_{m,n}$ denote the space of all completely reachable control systems $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. That is, $L^{cr}_{m,n}$ is the space of all pairs (A,B) consisting of a real $n \times n$ matrix A and a real $n \times m$ matrix B such that the $n \times (n+1)m$ matrix $R(A,B) = (B;AB;\ldots;A^nB)$ has rank n. The transformations: $(A,B) \mapsto (A+BK,B)$, K a real $m \times n$ matrix (feedback), $(A,B) \mapsto (SAS^{-1},SB)$, S an invertible real $n \times n$ matrix (basis change in state space) and $(A,B) \mapsto (A,BT)$, T an invertible real $m \times m$ matrix (basis change in input space) define an action of the Lie group of all block triangular matrices

$$\begin{pmatrix} S & 0 \\ K & T \end{pmatrix} \in GL_{n+m}(\mathbb{R})$$

on $L^{cr}_{m,n}$. This group is called the feedback group.

For each $(A,B) \in L^{cr}_{m,n}$ let $K(A,B)$ be the set of Kronecker indices of (A,B) (ordered in descending order). For each m-part partition K of n let $O_K = \{(A,B) \mid K(A,B) = K\}$. Then

1.6. Theorem ([15])

The orbits of the feedback F acting on $L^{cr}_{m,n}$ are precisely the O_K.

It follows that the topological closure $\bar{O}_K$, i.e. the set of systems which can arise as limits (degenerations) of a family of systems with Kronecker indices K is necessarily a union of O_K and some other orbits (possibly none). Concerning this, several people (Byrnes, Hazewinkel, Kalman, Martin . . .) have noticed that

1.7. Theorem

$\bar{O}_K \supset O_{K'}$, iff $K > K'$.

1.8. Gerstenhaber-Hesselink Theorem

Let N be the space of all nilpotent $n \times n$ matrices, i.e. $N_n = \{A \in \mathbb{R}^{n\times n} \mid A^n = 0\}$. Let $SL_n(\mathbb{R})$ act on N_n by conjugation, i.e. $N^S = SNS^{-1}$. Every $N \in N$ is similar to a Jordan normal form matrix with zeros on the diagonal and thus the orbits of $SL_n(\mathbb{R})$ acting on N_n are labelled by partitions $K = (K_1,\ldots,K_n)$ of n, where the K_i represent the sizes of the Jordan blocks. Let N_K be the orbit corresponding to K. Then the Gerstenhaber-Hesselink theorem [11], [17] says

1.9. Theorem

$\bar{N}_K \supset N_{K'}$, iff $K < K'$.

Note the reversal of the order in this statement with respect to the statement of Theorem 1.7.

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1.10. Degeneration of Vector Bundles

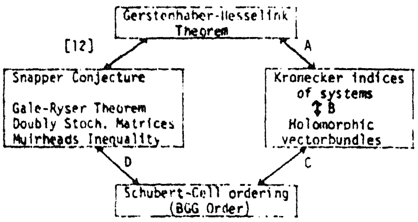
Let E be a holomorphic vectorbundle over the Riemann sphere S^2. Then according to [16] E splits as a direct sum of line bundles (i.e. vectorbundles of dimension 1) $E = L(K_1) \oplus \ldots \oplus L(K_m)$ and in turn line bundles are classified by their degree (or first Chern-class). Thus holomorphic vectorbundles over S^2 of dimension m are classified by an m-tuple of integers $K(E) = (K_1(E),\ldots,K_m(E))$, $K_i(E) \in \mathbb{Z}$, $K_1(E) \geq \ldots \geq K_m(E)$. The bundle E is called positive (or ample) if $K_i(E) \geq 0$ for all i. We have

1.11. Theorem

Let E_t be a holomorphic family of positive vectorbundles over S^2. Then $K(E_0) < K(E_t)$ for all small enough t. Inversely if $K < K'$ are two partitions of n, then there is a holomorphic family of bundles E_t such that $K(E_0) = K$ and $K(E_t) = K'$ for all $t \neq 0$.

1.12. Interrelations

It is well-known that Snapper conjecture implies the Gale-Ryser theorem, the result on doubly stochastic matrices as well as another combinatorial result known as Muirhead's inequality, cf. [1], [2]. On the other hand, the Hermann-Martin vectorbundle associated to a system provides the connection between theorems 1.11 and 1.7, cf. [13], [4], and explains why the same partial order occurs in the two theorems. In this paper we present a direct link between theorems 1.8 and 1.7 and show how the Snapper-conjecture and theorem 1.11 relate to the ordering of the Weyl group S_{n+m} of the semisimple Lie group $SL_{n+m}(\mathbb{C})$, the so-called BGG order [9], or, more precisely how these results relate to the natural "closure ordering" on the Schubert cells of the Grassmann manifold $G_n(\mathbb{C}^{n+m})$. These notions will be defined below. This explains why the same ordering occurred again and again above. It also gives us a new deformation type proof of the Snapper conjecture. In addition to these new connections there is also a direct connection between the Snapper conjecture and the Gerstenhaber-Hesselink theorem [12] which completes the picture in a very nice way, as illustrated by the following diagram



2. Grassmann Manifolds, the Canonical Bundle and Schubert Cells

The Grassmann manifold $G_n(\mathbb{C}^{n+m})$ is, as a set, the collection of all n-dimensional subspaces of $\mathbb{C}^{n+m}$. This set has a natural structure of an analytic manifold. We define a holomorphic vectorbundle ξ_m over $G_n(\mathbb{C}^{n+m})$ by taking as the fibre over x the m-dimensional quotient space $\mathbb{C}^{n+m}/x$. Let $p: \xi_m \to G_n(\mathbb{C}^{n+m})$ be the projection, and let $\Gamma(\xi_m)$ be the vector space of holomorphic sections of p, i.e. the space of all holomorphic $s: G_n(\mathbb{C}^{n+m}) \to \xi_m$ such that $p \cdot s = id$. There are $(n+m)$ obvious elements in $\Gamma(\xi_m)$ defined by $\varepsilon_i(x) = e_i \bmod x \in \xi_m(x) = \mathbb{C}^{n+m}/x$ where e_i is the i-th canonical basis vector of $\mathbb{C}^{n+m}$. These elements are linearly independent (obviously) and, though we shall not need this, they form a basis for $\Gamma(\xi_m)$.

For each sequence of n subspaces $A = (0 \subsetneq A_1 \subsetneq A_2 \cdots \subsetneq A_n)$ of $\mathbb{C}^{n+m}$ we define the closed Schubert cell

$$SC(A) = \{x \in G_n(\mathbb{C}^{n+m}) | \dim(x \cap A_i) \geq i\} . \quad (2.1)$$

In particular if $\lambda = (\lambda_1,\ldots,\lambda_n)$ is a strictly increasing sequence of natural numbers we define

$$SC(\lambda) = SC(\mathbb{C}^{\lambda_1} \subset \ldots \subset \mathbb{C}^{\lambda_n}) .$$

One easily checks that $SC(\lambda) \subset SC(\lambda')$ if and only if $\lambda_i \leq \lambda'_i$ for all i. Now assign to an m-part partition $K = (K_1,\ldots,K_m)$ the sequence of natural numbers

$$\lambda(K) = (\underbrace{2,3,\ldots,K_1+1}_{K_1}, \underbrace{K_1+3,\ldots,K_1+K_2+2}_{K_2},\ldots, \underbrace{K_1+\ldots+K_{m-1}+m+1,\ldots,K_1+\ldots+K_m+m}_{K_m}) \quad (2.2)$$

Then, clearly $K > K'$ if and only if $\lambda_i(K) \geq \lambda_i(K')$, $i = 1,\ldots,n$ so that the mapping $K \mapsto \lambda(K)$ exhibits the specialization order as a suborder of the ordering defined by the inclusion relations between the Schubert cells $SC(\lambda)$. This ordering in turn is a quotient ordering of the Bernstein-Gelfand-Gelfand ordering on the Weyl group S_{n+m}, cf. [9].

3. Vectorbundles and Systems (Connection B)

Consider a system $\Sigma = (A,B) \in L^{cr}_{m,n}$. Assign to it the holomorphic map $\phi_\Sigma: S^2 = \mathbb{C} \cup \{\infty\} \to G_n(\mathbb{C}^{n+m})$

$$s \mapsto [sI_n - A,B], \qquad \infty \mapsto [I_n \quad 0] \quad (3.1)$$

where I_n is the $n \times n$ unit matrix and $[M]$ for an $n \times (n+m)$ matrix M denotes the subspace of $\mathbb{C}^{n+m}$ spanned by the rows of M. This is modified version of the map defined in [13]. And correspondingly one has

3.2. Theorem

Let $E(\Sigma)$ be the pullback vectorbundle $\phi_\Sigma^! \xi_m$. Then $K(E(\Sigma)) = K(\Sigma)$.

With the present definitions the proof turns out to be almost a triviality, cf. [14].

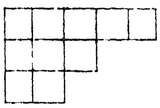
4. Systems and Nilpotent Matrices (Connection A)

This connection takes the form of a common proof of both theorems. The idea of the proof is two exhibit a small closed set that intersects each orbit in the closure of some fixed orbit. This closed set is constructed in terms of certain filtrations that uniquely define the orbit. We first consider the case of nilpotent matrices.

Let λ be the partition $\lambda_1,\dots,\lambda_n$ and associate with λ the Young tableaux numbered from left to right i.e.

1	2	3	4	5
6	7	8		
9				
10				

Let γ be a partition such that $\gamma > \lambda$ and $\gamma > \tau \geq \lambda$ implies $\tau = \lambda$. Then as in the introduction we know that the Young diagram for γ is obtained from the Young diagram for λ by shifting an end block to the first possible row above. For example



Associate with the diagram the Young tableaux numbered from left to right as above

1	2	3	4	5
6	7	8		
9	10			

Now define a function on the first n integers in terms of Young Tableaux for γ by $f(i)$ is the number assigned to the box immediately above the i-th box, if such a box exists, if not let $f(i) = 0$. Note that $f(i) = 0$ iff i is a number in the first row. Also that if k is in the i-th row of λ then $f(k)$ is in a row of λ with number less than or equal to i-1. We will occassionally refer to f as the upward shift operator.

Let A be a nilpotent matrix with associated filtration $\operatorname{Ker} A \subseteq \operatorname{Ker} A^2 \subseteq \dots \subseteq \operatorname{Ker} A^n$ of type λ. Choose a basis for $\operatorname{Ker} A^n$ such that $e_1,\dots,e_{\lambda_1}$ generate $\operatorname{Ker} A$ and in general $e_{\lambda_1+\dots+\lambda_{i-1}+1},\dots,e_{\lambda_1+\dots+\lambda_i}$ generate $\operatorname{Ker} A^i$. Now define a linear function F by defining $F(e_i) = e_{f(i)}$, where we take $e_0 = 0$, and extending F linearly. Now from the definition of f we have the following two facts.

1) $\operatorname{Ker} F^i \supseteq \operatorname{Ker} A^i$

2) $F \operatorname{Ker} A^{i+1} \subseteq \operatorname{Ker} A^i$.

We first prove a lemma about ranks of sums matrices.

4.1. Lemma

Let A and B be arbitrary matrices. The rank of $(tA + sB)^i$ is constant except on a finite number of lines in $\mathbb{C}^2\backslash\{(0,0)\}$ and $\operatorname{rk}(tA + SB)^i \geq \max \{\operatorname{rk} A^i, \operatorname{rk} B^i\}$.

Proof. Suppose the $\max\limits_{t,s} \operatorname{rank} (tA + sB)^i = k$. Then there is a $k \times k$ minor that evaluated at t_0, s_0 does not vanish. Since the minor is polynomial in t,s then there is a Zariski open set on which it doesn't vanish. The polynomial is homogeneous so we can conclude that it is defined on $\mathbb{P}^1(\mathbb{C})$ and doesn't vanish on a Zariski open set of $\mathbb{P}^1(\mathbb{C})$ and hence it vanishes at a finite number of points on $\mathbb{P}^1(\mathbb{C})$ hence on a finite number of lines. Thus the rank can only go down at these isolated points. The Lemma follows by choosing $t = 0$ $s = 1$ and $t = 1$ $s = 0$.

The next lemma will be the key for the proof of the theorem.

4.2. Lemma

Let A and F be as above, then $tA + F$ is conjugate to A for all but finitely many values of t.

Proof. We will prove by induction that $\operatorname{Ker}(tA+F)^i \supseteq \operatorname{Ker} A^i$. For $i = 1$ let $x \in \operatorname{Ker} A$. Then $x \in \ker F$ and hence $(tA + F)x = 0$ for all t. Now assume $i = k$ that $\operatorname{Ker} (tA + F)^k \supseteq \operatorname{Ker} A^k$. Let $x \in \operatorname{Ker} A^{k+1}$ and note that $x \in \operatorname{Ker} F^{k+1}$. We calculate $(tA + F)^{k+1} x = (tA + F)^k (tAx + Fx)$ but $Ax \in \operatorname{Ker} A^k$ and $Fx \in \operatorname{Ker} A^k$ and by the induction hypothesis $\operatorname{Ker} (tA + F)^k \supseteq \operatorname{Ker} A^k$. Thus $x \in \operatorname{Ker} (tA + F)^{k+1}$ for all t. Thus we have proven that $\operatorname{rk}(tA + F)^i \leq \operatorname{rk} A^i$ for all i. By lemma 4.1 we know that $\operatorname{rk} A^i \leq \operatorname{rk}(tA + F)$ for all but finitely many t. Thus for all but finitely many t we have equality of rank and this proves conjugacy.

Define a set $M = \{F : F^n = 0 \text{ and for all } i \operatorname{Ker} F^i \supseteq \operatorname{Ker} A^i\}$. M is clearly an algebraic subvariety of the nilpotent matrices defined in terms of n homogeneous equations. [Let a be a basis element in $\operatorname{Ker} A^i$ then $F^i a = 0$ is one such equation.] Let τ be any partition greater than λ. Then there is an element of type τ in M and further more there is a sequence of line segments in M from A to an element of type τ. Thus M is contained in the closure of the orbit of A.

4.3. Lemma

The closure of the oribt of A is contained in the set

$$M = \bigcap_{i=1}^{r} \{F : \operatorname{rk} F^i \leq \operatorname{rk} A^i\}.$$

Proof. If F is conjugate to A then $\operatorname{rk} F^i = \operatorname{rk} A^i$ for all i and hence the orbit of A is contained in M. Each of the sets in the intersection is closed (even algebraic) and hence M is closed and the lemma follows.

The main theorem now follows trivially.

4.3. Theorem (Gerstenhaber-Hesselink)

A matrix B is contained in the closure of the orbit of A iff the filtration type of B is larger than the filtration type of A.

Proof. If $B \in M$ then there is an F in M of the same type and F is in the closure of the orbit of A.

We now consider the case of pairs of matrices and the feedback group. Again we must define a shift function but this time we need a down shift instead of an up shift. Let λ be a partition with Young tableaux T. Let γ be less than λ and again have the property that $\gamma < \tau \leq \lambda$ implies $\tau = \lambda$. Let T' be the tableaux for γ obtained by moving the appropriate box of the diagram for λ. Define a function on the first n integers by $f(i)$ is the number of the box in the tableaux T' immediately below the box of i if such a box exists and zero otherwise.

Let (A,B) be a controllable pair and let the filtration of controllable subspaces have type λ. Recall that this filtration is defined by B_1 is the space spanned by the columns of B and $B_{k+1} = AB_k + B_1$. One of the standard theorems is that (A,B) is controllable iff $B_n = E^n$. See [4] for a survey. Choose a basis for E^n such that the first λ_i in B_i for all i. Let the tableaux for γ be defined as above. We will define a pair (F,G) in terms of the tableaux of γ. Let G be the matrix whose columns are the basis elements numbered by the first row of γ. Define F by defining F on the basis by $F(e_i) = e_{f(i)}$ with $e_0 = 0$ and extend F to a linear function. Now note that F and G have the following properties. Let $G_1 \subseteq \dots \subseteq G_n$ be the filtration of (F,G).

1) (F,G) is controllable
2) $G_i \subseteq B_i$
3) $FB_i \subseteq B_{i+1}$

The following lemma is the counterpart of lemma 4.2.

4.4. Lemma

Let (A,B) and (F,G) be as above. Then the system $(tA+F,\ tB+G)$ is equivalent to (A,B) for all but finitely many t.

Proof. We use the fact that two systems are feedback equivalent iff the filtrations are of the same type [18]. Let $V_1 \subseteq V_2 \subseteq \dots \subseteq V_n$ be the filtration of $(tA+F,\ tB+G)$. First since $G_1 \subseteq B_1$ we have that $V_1 \subseteq B_1$ for all t. Assume $V_k \subseteq B_k$ and we are given that $G_k \subseteq B_k$. Let $x \in V_{k+1}$ then by construction there is a y_1 and $y_2 \in V_k$ such that

$$(F + tA)y_1 + y_2 = x$$

but $y_2 \in V_k \subseteq B_k \subseteq B_{k+1}$ and $y_1 \in V_k \subseteq B_k$ and hence $Fy_1 \in B_{k+1}$. By definition $Ay_1 \subseteq B_{k+1}$ so we have that $x \in B_{k+1}$. Thus we have that $V_k \subseteq B_k$ for all k. This proves that the $rk[(tA+F)^i(tB+G),\dots,(tB+G)] \leq rk[A^iB,\dots,B]$ for all i and all t. A slight modification of lemma 4.1 yields that for all but finitely many t the reverse inequality holds and thus the lemma is proven.

Now define a set of pairs $S = \{(F,G)$: the filtration of (F,G) is contained subspace by subspace in the filtration of (A,B) and (F,G) is controllable$\}$. Again S is an algebraic subvariety of the controllable pairs, but seen by choosing, with respect to some innerproduct, a complementary set of subspaces. Let τ be any partition less than γ then there is a pair $(F,G) \in S$ of type τ and furthermore the pair can be reached from (A,B) by a sequence of line segments as constructed in the previous lemma. Thus S is contained in the closure of the orbit of (A,B).

For a pair (F,G) denote the filtration by $V_1(F,G) \subseteq \dots \subseteq V_n(F,G)$.

4.5. Lemma

The closure of the orbit of (A,B) is contained in the set

$$S = \bigcap_{i=1}^{n} \{(F,G) : \dim V_i(F,G) = \dim V_i(A,B) \text{ and } (F,G) \text{ controllable}\}$$

Proof. Clearly the orbit of (A,B) is contained in S and since each set in the intersection is closed so is S.

The main theorem now follows trivially.

4.2. Theorem

A pair (F,G) is in the closure of the orbit of (A,B) if the filtration type of (F,G) is less than or equal to the filtration type of (A,B).

Proof. If $\tau \leq \gamma$ then there is a system of type τ in S and hence if (F,G) is of type τ then its equivalent to a system in S.

The two theorems have almost identical proofs. In both cases the key is that there is a map from each orbit onto a flag manifold that is really the crucial element. The set M and the set S are closely related to this map for let x be either a nilpotent matrix or a controllable system and let $\pi(x)$ be the corresponding element of the flag manifold. Let H be the stabilizer of the flag and consider the set in the original variety of $H \cdot x$. It is not hard to show that $H \cdot x \subseteq M$ or S as the case may be. The closure of Hx seems to be in general smaller than M or S, but if we do the same trick for each y in the closure of Hx then the union is M or S. Closing the stabilizer picks up those elements with adjacent types and perhaps a little more.

The key to the simplicity of these proofs was the fact that in both cases we were working with the corresponding filtration instead of the canonical forms.

5. Classifying Maps (Connection C)

Let $E = L(K_1) \oplus \dots \oplus L(K_m)$ be a positive vectorbundle of dimension m over S^2. Now $\Gamma(L(i))$ has dimension $i+1$ and it follows that $\Gamma(E)$ is of dimension $n+m$. For each $s \in S^2$ let $x(s)$ be the Kernel of the evaluation map $\gamma \to \gamma(x)$, $\gamma \in \Gamma(E)$, $\gamma(s) \in E(s)$ the m-dimensional fibre of E over s. The vectorspace homomorphism $\Gamma(E) \to E(s)$ is surjective (positivity of E) and $x(s)$ therefore has dimension n.

We can therefore define a morphism $\psi_E : S^2 \to G_n(\Gamma(E))$ by $s \to x(s)$. This map is classifying (meaning that $\psi_E^! \xi_m \simeq E$ (Easy) and moreover

5.1 Theorem

Let S^2, E, ψ_E be as above and let $K = (K_1, \ldots, K_m)$. Then

(i) There is a Schubert-cell $SC(A)$ such that $Im(\psi_E) \subset SC(A)$ and such that $\dim A_i = \lambda_i(K)$ $i = 1, \ldots, n$ (cf. (2.2) for the definition of $\lambda_i(K)$).

(ii) If a Schubert-cell $SC(B)$ is such that $Im \psi_E \subset SC(B)$ then $\dim B_i \geq \lambda_i(K)$, $i = 1, \ldots, n$.

6. Systems and Schubert Cells (the combined connection C - B)

Let $\Sigma = (A,B) \in L^{cr}_{m,n}$. There as in section 3 above we associate to Σ to holomorphic map $\phi_\Sigma : S^2 \to G_n(\mathbb{C}^{n+m})$ defined by

$$s \mapsto [sI - A, B], \quad \infty \mapsto [I, 0] \qquad (6.1)$$

This is the classifying map of the vectorbundle $E(\Sigma)$ of Σ (by definition of the latter). It follows that in terms of systems theorem 5.1 translates as

6.2. Theorem

Let Σ, ϕ_Σ be as above and let $K = (K_1(\Sigma), \ldots, K_m(\Sigma))$, $\lambda = \lambda(K)$.

(i) There is a Schubert-cell $SC(A)$ such that $\dim(A_i) = \lambda_i(K)$ such that $Im\ \phi_\Sigma \subset SC(A)$.

(ii) If $Im\ \phi_\Sigma \subset SC(B)$ then $\dim(B_i) \geq \lambda_i(K)$.

Assume $\Sigma = (A,B)$ to be in Brunovsky canonical form. Then after renumbering the usual basis of $\mathbb{C}^{n+m}$, which amounts to rearranging the columns of $(sI - A, B)$, the map ϕ_Σ looks particularly simple. For example if $K = (3,2,1)$ we find

$$s \to \left(\begin{array}{ccc:c:cc:c:c:c} s & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & s & 1 \end{array}\right) \qquad (6.3)$$

and we observe that indeed $Im\ \phi_\Sigma \subset SC(2,3,4,6,7,9)$.

7. A Family of Representations of S_{n+m} Parameterized by $G_n(\mathbb{C}^{n+m})$

Let M be the regular representation of S_{n+m}; i.e. M is a vector space with basis e_σ, $\sigma \in S_{n+m}$ and S_{n+m} acts on e_σ by $(e_\sigma^\tau) = e_{\tau\sigma}$. Let ξ_m be the classifying vectorbundle over $G_n(\mathbb{C}^{n+m})$ defined in section 2 above, whose fibre over x is equal to $\xi_m(x) = \mathbb{C}^{n+m}/x$.

Now for each $x \in G_n(\mathbb{C}^{n+m})$ we define an homomorphic of vector spaces

$$\chi_x : M \to \xi_m(x)^{\otimes(n+m)}, \quad e_\sigma \to \varepsilon_{\sigma(1)}(x) \otimes \ldots \otimes \varepsilon_{\sigma(n+m)}(x) \qquad (7.1)$$

where the $\varepsilon_1, \ldots, \varepsilon_{n+m}$ are the $n+m$ holomorphic sections of ξ_m defined in section 2, i.e. $\varepsilon_i(x) = e_i \bmod x$, e_i the i-th basis vector of $\mathbb{C}^{n+m}$. S_{n+m} acts on $\xi_m(x)^{\otimes(n+m)}$ be permuting the factors and with respect to this action (7.1) is S_{n+m}-equivariant and thus defines a continuous family of homomorphisms. More precisely we have a homomorphism of vector bundles

$$X : M \times G_n(\mathbb{C}^{n+m}) \to \xi_m^{\otimes(n+m)} \qquad (7.2)$$

which on each fibre is equivariant with respect to the S_{n+m} action on $M \times \{x\}$ and $\xi_m(x)^{\otimes(n+m)}$.

For each $x \in G_n(\mathbb{C}^{n+m})$ let $\pi(x)$ be the S_{n+m}-module $X_x(M)$. This gives us a family of representations of S_{n+m} which is "continuous" in the sense that it arises as the family of images of a continuous family of homomorphisms of representations.

Very many representations of S_{n+m} arise in this way. We have not yet determined completely which representations of S_{n+m} occur among the $\pi(x)$. But, for example, if K is a partition of n and $\tilde{K} = (K_1 + 1, \ldots, K_m + 1)$ then the induced representation $\rho(\tilde{K}) = \mathrm{Ind}_{S_{\tilde{K}}}^{S_{n+m}} 1$ occurs among the $\pi(x)$. For example if $K = (3,1,0)$ then $\rho(\tilde{K}) = \pi(x)$ if x is the row vector space of a matrix of the form

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 1 & 0 \end{pmatrix}$$

where the * elements are all nonzero. Indeed in this case the vectors e_1, e_2, e_3, e_4 are scalar multiples of each other mod x and so are e_5 and e_6, while $e_1 \bmod x$, $e_5 \bmod x$ and $e_7 \bmod x$ are linearly independent in $\xi_m(x)$.

By letting S_n be the group of permutations of various sets of n letters among the symbols on which S_{n+m} acts, many representations of S_n arise. Conjecturally all representations of S_n arise in this way.

It is perhaps also worth observing that for all $s \neq 0$ the representation $\pi(\phi_\Sigma(s))$, where Σ is a system in Brunowsky canonical form is the induced representation $\rho(\tilde{K})$. It would be nice to be able to interpret this in control theoretic terms.

8. Families of Representations and Snapper Type Results

Now let us see how "continuous" families of representations give us the type of result occurring in the Snapper conjecture. The relevant theorem is

8.1. Theorem

Let V and W be two S_n-modules. Suppose we have a continuous family of homomorphisms $\phi_t : V \to W$. Let $\sigma(0) = \mathrm{Im}\,\phi_0$, $\sigma(t) = \mathrm{Im}\,\phi_t$. Then the representation $\sigma(0)$ is a direct summand of the representation $\sigma(t)$ for small t.

The proof is easy. Because the category of S_n-modules is semi-simple, there exists a homomorphism of S_n-modules $\psi_0 : \mathrm{Im}(\phi_0) \to V$ such that $\phi_0 \cdot \psi_0 = \mathrm{id}$. Then because ϕ_t is continuous in t it follows that $\phi_t \cdot \psi_0$ is injective for small t. This gives us an embedding of S_n-modules $\sigma(0) \hookrightarrow \sigma(t)$ and hence, using semisimplicity again, $\sigma(0)$ is a direct summand of $\sigma(t)$.

9. On the Proof of the Snapper Conjecture

Thus to prove the Snapper conjecture it suffices to find families of maps of representations $\phi_t : V \to W$ such that for a given $K > K'$ we have $\mathrm{Im}\,\phi_t \simeq \rho(K)$ if $t \neq 0$ (and small) and $\mathrm{Im}\,\phi_0 \simeq \rho(K')$. Quite possibly such families can be found within the grand family constructed above in section 7. Certainly the grand-family contains all the representations $\rho(K)$ (as pointed out in section 7. To prove the Snapper conjecture we rely on a slightly more complicated construction which is perhaps best illustrated by means of the following example.

Consider the representation in the family of section 7 defined over an x_t, $t \in \mathbb{R}$ in $G_n(\mathbb{C}^{n+m})$ given by a matrix of the form

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & y & t & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 1 \end{pmatrix} \tag{9.1}$$

where y, z and all the $*$'s are nonzero elements. Consider the element

$$\begin{aligned} \alpha = {} & e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_7 \\ & - e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_1 \otimes e_5 \otimes e_7 \end{aligned} \tag{9.2}$$

in $(\mathbb{C}^{n+m})^{\otimes n+m}$, where e_i is the standard i-th basis vector. There $m = 2$, $n = 5$. Now consider the S_{n+m} submodule K_t of $\xi_m(x_t)^{\otimes(n+m)}$ generated by the image of the element α.

Now note that $te_6 + ye_5 + ze_1 = 0$. Using this and the extra relation that the image of (9.2) is zero mod K_t it follows readily that for $t \neq 0$ the images of the two elements

$$\begin{aligned} & e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_6 \otimes e_7 \\ & e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_6 \otimes e_5 \otimes e_7 \end{aligned} \tag{9.3}$$

are equal in mod K_t. From this it easily follows that the image of

$$M \xrightarrow{\phi_t} \xi_m(x_t)^{\otimes(n+m)} \xrightarrow{\pi_t} \xi_m(x_t)^{\otimes(n+m)}/K_t \tag{9.4}$$

is $\rho(K)$ for $t \neq 0$ where $K = (9.3)$.

But for $t = 0$, $ye_5 + ze_1 = 0$ so that $K_0 = \{0\}$. Also $\mathrm{Im}\,\phi_0 \simeq \rho(K')$, where $K' = (5,2)$ as we saw in section 7 above. Now choose $\psi_0 : \mathrm{Im}\,\phi_0 \to M$ such that $\phi_0\psi_0 = \mathrm{id}$.

Let us take $y = -1$, $z = 1$ for convenience. Then

$$e_5 = e_1 + te_6 \mod \xi_m(x_t) \tag{9.5}$$

Consider $\phi_t\psi_0 : \mathrm{Im}\phi_0 \to \mathrm{Im}\phi_t$. A basis for $\xi_m(x_t)$ for all t is given by the images $\bar{e}_1$ and $\bar{e}_6$ of e_1 and e_6 respectively. Now because of (9.5) (and the other relations given by $\xi_m(x_t)$)

$$\phi_t(e_\sigma) = \phi_0(e_\sigma) + t\beta(e_\sigma) \tag{9.6}$$

where $\beta(e_\sigma)$ is a tensor product e of $\bar{e}_1$ and $\bar{e}_6$ involving 3 factors $\bar{e}_6$ and 4 factors $\bar{e}_1$ and $\phi_0(e_\sigma)$ involves 2 factors $\bar{e}_6$ and 5 factors $\bar{e}_1$.

Now observe that the image of α in $\xi_m(x_t)^{\otimes n+m}$ is a sum of terms involving 5 factors $\bar{e}_1$ and 2 factors $\bar{e}_6$. So that $\phi_t\psi_0(v) = v + t\beta(\psi_0(v))$ can be in K_t iff

$$v \in K_t, \quad \beta(\psi_0(v)) = 0, \quad v \in \mathrm{Im}\,\phi_0 \tag{9.7}$$

Using the usual lift ψ_0 (defined by $\bar{e}_1 \otimes \dots \otimes \bar{e}_1 \otimes \bar{e}_6 \otimes \bar{e}_6 \to (5!2!)^{-1} \sum_{\tau \in S_5 \times S_2} e_\tau$) it is a straightforward matter to check that $\beta\psi_0$ is injective on K_t. This proves that $\pi_t\phi_t\psi_0$ is injective so that $\mathrm{Im}\,\phi_0 \simeq \rho(K')$ is a direct summand of $\mathrm{Im}\,\pi_t\phi_t \simeq \rho(K)$.

In this vein one proves the Snapper conjecture for $K > K'$ with $K_1, K_1' \geq 1$. The remaining cases are handled by embedding $S_n \hookrightarrow S_{n+m}$ in in the obvious way and by letting K correspond to $\tilde{K} = (K_1 + 1, \dots, K_m + 1)$.

Observe that the representations we are using from the grand-family are precisely (up to taking a quotient of one of them) among those living over the Schubert-cells $SC(K)$ and $SC(K')$.

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