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ON THE SNAPPER / LIEBLER–VITALE / LAM THEOREM ON PERMUTATION REPRESENTATIONS OF THE SYMMETRIC GROUP

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1. Introduction

Let $\kappa=(\kappa_1,\ldots,\kappa_m)$, $\kappa_i\in\mathbb{N}\cup\{0\}$, $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_m\geq 0$ be a descending partition of n. We identify partitions which differ only by the addition of some additional zero's. An ordering, which we call the specialization order, is defined on the set of all partitions by

$$\kappa>\lambda \Leftrightarrow \sum_{i=1}^{r}\kappa_i\leq\sum_{i=1}^{r}\lambda_i, \quad r=1,2,\ldots. \tag{1.1}$$

The reverse order has been called the dominance order. It occurs in many, seemingly unrelated parts of mathematics [1, 2, 3], and one of the central occurrences is in the representation theory of the symmetric groups in characteristic zero.

Let $S_\kappa=S_{\kappa_1}\times\cdots\times S_{\kappa_m}$ be the Young subgroup of S_n (S_{κ_i} is viewed as the permutation subgroup of S_n permuting the letters $\kappa_1+\cdots+\kappa_{i-1}+1,\ldots,\kappa_1+\cdots+\kappa_i$) corresponding to the partition κ and let $\varrho(\kappa)$ be the representation of S_n obtained by inducing the trivial representation of S_κ up to S_n. Also let $[\kappa]$ be the irreducible representation of S_n (in characteristic zero) associated to the partition κ. Snapper [5] proved that $[\kappa]$ occurs in $\varrho(\lambda)$ implies $\kappa<\lambda$ (this also follows readily from Young's rule) and conjectured the reverse, which he proved for $m=2$. Proofs of the conjecture were given by Liebler–Vitale [4] and Lam [3]. Liebler and Vitale proved more precisely that $\kappa<\lambda$ implies that $\varrho(\kappa)$ is a subrepresentation of $\varrho(\lambda)$ (which obviously implies the conjecture because $[\kappa]$ occurs in $\varrho(\kappa)$).

In this note we give a completely elementary direct proof of the Liebler–Vitale result which requires no representation theory at all (beyond the definition of the permutation representations $\varrho(\kappa)$) by constructing explicit homomorphisms of representations.

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2. The Snapper/Liebler–Vitale/Lam Theorem

2.1. *Description of the permutation representation* $\varrho(\kappa)$. Let $W(\kappa)$ be the set of all words of length n in the symbols $a_1, \ldots, a_m$ such that each a_i occurs exactly κ_i times. The group S_n acts in the obvious way on $W(\kappa)$ ($\sigma^{-1}(b_1 \cdots b_n) = b_{\sigma(1)} \cdots b_{\sigma(n)}$, $\sigma \in S_n$) and the vector-space $V(\kappa)$ with the elements of $W(\kappa)$ as basis and the action extended linearly is the representation $\varrho(\kappa)$. We shall denote the elements of $W(\kappa)$ and the corresponding basis elements of $V(\kappa)$ with the same symbols.

2.2. *Reduction to the case* $m=2$. It obviously suffices to prove the statement "$\kappa < \lambda \Rightarrow \varrho(\kappa)$ is a subrepresentation of $\varrho(\lambda)$" in the case that $\kappa < \lambda$ and $\kappa < \mu < \lambda \Rightarrow \kappa = \mu$ or $\lambda = \mu$. In this case there exist i and j, $i > j$, such that $\lambda_i = \kappa_i + 1$, $\lambda_j = \kappa_j - 1$ and $\lambda_r = \kappa_r$ for $r \neq i, j$.

This statement is standard and its proof is easy, but we give it for completeness sake. We use induction on m. For $m=2$ the statement is trivial. If $\lambda_m = \kappa_m$ then $(\lambda_1, \ldots, \lambda_{m-1}) > (\kappa_1, \ldots, \kappa_{m-1})$ as partitions of $n - \lambda_m$ and we have reduced to $m-1$. If $\lambda_m > \kappa_m$ consider $\mu = (\mu_1, \ldots, \mu_m) = (\lambda_1, \ldots, \lambda_{s-1}, \lambda_s + 1, \lambda_{s+1}, \ldots, \lambda_m - 1)$ where s is such that $\lambda_{s-1} \neq \lambda_s = \lambda_{m-1}$. Clearly $\lambda > \mu$, $\lambda \neq \mu$. It remains to prove that $\mu > \kappa$ and hence $\mu = \kappa$. For $r \leq s$ we have

$$\sum_{i=r}^{m} \mu_i = \sum_{i=r}^{m} \lambda_i \geq \sum_{i=r}^{m} \kappa_i.$$

For $r > s$ we have

$$1 + \sum_{i=r}^{m} \mu_i = \sum_{i=r}^{m} \lambda_i \geq \sum_{i=r}^{m} \kappa_i.$$

But if $\sum_{i=r}^{m} \kappa_i = \sum_{i=r}^{m} \lambda_i = (m-r)\lambda_{m+1} + \lambda_m$, then $\lambda_{r-1} \geq \kappa_{r-1} \geq \kappa_r > \lambda_r$ because $\sum_{i=r-1}^{m} \kappa_i \leq \sum_{i=r-1}^{m} \lambda_i$ and $\kappa_m < \lambda_m$ hence $r \leq s$. So we must have $\sum_{i=r}^{m} \kappa_i < \sum_{i=r}^{m} \lambda_i$ which implies that $\sum_{i=r}^{m} \kappa_i \leq \sum_{i=r}^{m} \mu_i$. This proves the statement. The idea of the proof is that of [4] but there the details are not entirely correctly written down.

In the case described above we define a linear map

$$\beta_{\lambda,\kappa} : V(\lambda) \to V(\kappa) \tag{2.3}$$

by the formula

$$\beta_{\lambda,\kappa}(b_1 \cdots b_n) = \sum b'_1 \cdots b'_n \tag{2.4}$$

where the sum extends over all words $b'_1 \cdots b'_n$ such that $b'_t = b_t$ for all but one t. And for that one t we have $b_t = a_i$ and $b'_t = a_j$. That is, the words in the sum on the right are obtained by replacing precisely one occurrence of a_i by a_j. This is obviously an S_n-equivariant map.

We shall prove that $\beta_{\lambda,\kappa}$ is surjective if (and only if) $\lambda > \kappa$. This proves the theorem because the category of S_n-modules (in characteristic zero) is semisimple. Alternatively observe that if $\alpha_{\kappa,\lambda} : V(\kappa) \to V(\lambda)$ is defined as $\beta_{\lambda,\kappa}$ with the letters a_j

and a_i interchanged then $\alpha_{\kappa,\lambda}$ and $\beta_{\lambda,\kappa}$ are adjoint to each other in the sense that

$$\langle \alpha_{\kappa,\lambda} v, \omega \rangle = \langle v, \beta_{\lambda,\kappa} \omega \rangle, \quad v \in V(\kappa), \omega \in V(\lambda) \tag{2.5}$$

where the inner products on $V(\lambda)$ and $V(\kappa)$ are the ones for which $W(\lambda)$ and $W(\kappa)$ form orthonormal bases. This $\alpha_{\kappa,\lambda}$ is an S_n-equivariant injection iff $\beta_{\lambda,\kappa}$ is surjective and it remains to prove that $\beta_{\lambda,\kappa}$ is surjective if $\kappa < \lambda$.

To do this observe that as a vectorspace $V(\lambda)$ is the direct sum of

$$\binom{n}{\lambda}\binom{\lambda_i+\lambda_j}{\lambda_i}^{-1}$$

copies of $V(\lambda_j, \lambda_i)$ indexed by all words in the symbols $a_1, \ldots, \hat{a}_j, \ldots, \hat{a}_i, \ldots, a_m, c$ ($\hat{}$ denotes deletion) such that a_t occurs λ_t times and c occurs $\lambda_i + \lambda_j$ times. Similarly $V(\kappa)$ is the direct sum of

$$\binom{n}{\kappa}\binom{\kappa_i+\kappa_j}{\kappa_i}^{-1} = \binom{n}{\lambda}\binom{\lambda_i+\lambda_j}{\lambda_i}^{-1}$$

copies of $V(\kappa_j, \kappa_i)$ and the homomorphism (2.4) maps the copies of $V(\lambda_j, \lambda_i)$ and $V(\kappa_j, \kappa_i)$, labelled by the same word in $a_1, \ldots, \hat{a}_j, \ldots, \hat{a}_i, \ldots, a_m, c,$ into each other and is in fact the direct sum of these induced maps. Hence it is sufficient to prove the surjectivity of $\beta_{\lambda,\kappa}$ in the case $m=2$.

2.6. *Proof of the surjectivity of $\beta_{\lambda,\kappa}$ in the case $m=2$.* Let $\lambda = (r-1, s+1)$, $\kappa = (r,s)$, $r+s=n$ and write x for a_i and y for a_j. Then $W(r-1, s+1)$ consists of words of length n in $(r-1)$ x's and $(s+1)$ y's and $\beta = \beta_{\lambda,\kappa}$ changes such a word into the sum of all words which can be obtained from this word by chancing precisely one y into an x. For example,

$$\beta(xxxyyy) = xxxxyy + xxxyxy + xxxyyx. \tag{2.7}$$

We shall now show that β is surjective if $r \geq s+1$. (We only need the case $r \geq s+2$). Let $W = W(r-1, s+1) \cup W(r,s)$. For each pair $\omega_1 = b_1 \cdots b_n$, $\omega_2 = b'_1 \cdots b'_n$ in W we define the distance $d(\omega_1, \omega_2)$ by

$$d(\omega_1, \omega_2) = \#\{t \mid b_t \neq b'_t\}. \tag{2.8}$$

(This distance is called Hamming distance in coding theory). Now for $\omega_0 = x \cdots xy \cdots y \in W(r,s)$ let

$$E_t = \{\omega \in W \mid d(\omega_0, \omega) = t\} \tag{2.9}$$

Then $E_t \subset W(r,s)$ if t is even and $E_t \subset W(r-1, s+1)$ if t is odd. Note that $\omega \in E_{2t}$ iff there are precisely t y's among the first r letters and t x's among the second s letters and similarly $\omega \in E_{2t+1}$ iff there are precisely $t+1$ y's among the first r letters of ω and t x's among the last s letters.

Now let

$$f=r^{-1}\left(c_0\sum_{\omega\in E_1}\omega+c_1\sum_{\omega\in E_3}\omega+\cdots+c_s\sum_{\omega\in E_{2s+1}}\omega\right) \tag{2.10}$$

where

$$c_t=(-1)^t\binom{r-1}{t}^{-1}. \tag{2.11}$$

We claim that $\beta(f)=\omega$. To see this observe that since $\omega\in W(r,s)$ and $r\geq s+1$ the maximum distance of an $\omega\in W$ to ω_0 is $2s+1$. Observe that if $\omega'\in E_{2t+1}$ then $\beta(\omega')$ is a sum of elements in E_{2t} and E_{2t+2} (except when $t=s$, then only elements of E_{2s} can occur by the maximum distance observation).

Now let $\omega''=b_1\cdots b_n\in E_{2t}(t\geq 1)$ then the coefficient of $\omega''\in\beta(f)$ is equal to

$$\begin{aligned}&r^{-1}c_t(\#\{i\in\{1,\dots,r\}\mid b_i=x\})\\&\quad+r^{-1}c_{t-1}(\#\{i\in\{r+1,\dots,r+s\}\mid b_i=x\})\\&\quad=r^{-1}c_t(r-t)+r^{-1}c_{t-1}t.\end{aligned} \tag{2.12}$$

(The first contribution comes from the elements in E_{2t+1} whose ith element was y and is transformed to x to decrease the distance to ω_0; the second contribution comes from elements of E_{2t-1} whose ith element was y and is transformed to x to increase the distance). By definition of c_t the right-hand side of (2.12) is zero.

The coëfficient of $\omega_0\in\beta(f)$ is equal to

$$r^{-1}c_0(\#\{i\in\{1,\dots,r\}\mid b_i=x\}=r^{-1}\cdot 1\cdot r=1. \tag{2.13}$$

This proves that $\omega_0=\beta(f)\in\operatorname{Im}\beta$ and hence $\omega\in\operatorname{Im}\beta$ for all $\omega\in W(r,s)$ because β is S_n-equivariant and S_n acts transitively on $W(r,s)$. This concludes the proof.

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