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On families of systems: pointwise–local–global isomorphism problems†

MICHIEL HAZEWINKEL‡ and ANNA-MARIA PERDON§

Let Σ and Σ' be two families of linear dynamical systems, or, almost equivalently, let Σ and Σ' be two systems over a ring. This paper addresses itself to the question, what, if anything, can be said about the relations between Σ and Σ' if it is known that Σ and Σ' are pointwise isomorphic for all or almost all of the parameter values.

1. Introduction

(and motivational remarks for studying families rather than single systems)

A linear dynamical system is a system of differential or difference equations

$$\left.\begin{array}{ll} \dot{x}=Fx+Gu, & x(t+1)=Fx(t)+Gu(t) \\ y=Hx, & y(t)=Hx(t) \end{array}\right\} \tag{1}$$

$x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $y\in\mathbb{R}^p$, i.e. we have state space dimension n, m inputs and p outputs. The theory of linear dynamical systems deals with various properties of and constructions pertaining to such sets of equations, with the coefficients, i.e. the entries of the matrices F, G, H, assumed known. Yet in many circumstances these coefficients are imperfectly known at best and it becomes important to examine what happens to various notions and constructions as the coefficients vary (slightly).

To make things more precise let Q be a topological space. Roughly a family of linear dynamical systems over Q consist of a collection of such equations (1), one for each $q\in Q$, such that the matrices F, G, H depend continuously on the parameter q. More generally (and also more properly), a family over Q consists of a vector bundle E over Q (of dimension n), a vector bundle endomorphism $F: E\to E$ and two vector bundle homomorphisms $G: Q\times\mathbb{R}^m\to E$, $H: E\to Q\times\mathbb{R}^p$. The two definitions agree locally (i.e. over small enough open subsets of Q) and for the purposes of this paper the first definition mostly suffices.

In the discrete time case (i.e. the difference equation case) one can consider systems of equations

$$x(t+1)=Fx(t)+Gu(t), \quad y(t)=Hx(t) \tag{2}$$

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‡ Department of Mathematics, Erasmus University, Rotterdam 3000, Postbus 1738, The Netherlands.

§ Institute di Mat. Appl., Universita di Padova, Italy.

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where now the matrices F, G, H can have their coefficients in any ring R (and $t=0, 1, 2, \ldots$, say). For each prime ideal $\mathfrak{p}$ of R let $R(\mathfrak{p})$ be the quotient field of the integral domain $R/\mathfrak{p}$. This gives us a family of systems

$$x(t+1)=F(\mathfrak{p})x(t)+G(\mathfrak{p})u(t), \quad y(t)=H(\mathfrak{p})x(t) \tag{3}$$

which is the local algebraic-geometric analogue of the topological concept of a family introduced above. The main goal of the theory of families of systems is now to develop techniques and prove theorems which do for families all the nice things one can do for a single linear dynamical system, for example :

(i) Realization theory for a family of input/output maps (cf. also Byrnes 1977 a, 1978, Hazewinkel 1979 b).
(ii) Pole placement and stabilization by feedback (cf. also Byrnes 1978, Hazewinkel 1979 b, Tannenbaum 1978, Wyman 1978).
(iii) Decomposition (e.g. construction of the ' canonical ' completely reachable subsystem (cf. Hazewinkel 1979 b, 1980 a).
(iv) Controllability subspaces and their applications.
(v) Disturbance decoupling.

The general philosophy of, and motivation for, the study of families of (linear) dynamical systems rather than single ones is discussed more extensively in (Hazewinkel 1980 a, 1979 b, Kamen 1978). Results pertaining to different aspects than those of the present paper are in Hazewinkel (1980 b, c).

In view of the reinterpretation (sketched above) of a system (2) over a ring R as an algebraic-geometric family of systems over Spec(R), the general project encompasses trying to do all the things listed above for systems over rings, and this constitutes an important bit of motivation for studying families of systems.

A related, and important, bit of motivation comes from linear delay differential dynamical systems, for example :

$$\left.\begin{aligned} \dot{x}_1(t) &= x_1(t)+x_2(t-1)+u(t-1) \\ \dot{x}_2(t) &= x_1(t-1)+u(t) \\ y(t) &= x_1(t)+x_2(t-2) \end{aligned}\right\} \tag{4}$$

Introducing the delay operator σ, $\sigma x(t)=x(t-1)$, we can write (4) formally as a linear system over the ring $R[\sigma]$, viz :

$$\left.\begin{aligned} \dot{x}(t) &= F(\sigma)x(t)+G(\sigma)u(t) \\ y(t) &= H(\sigma)x(t) \end{aligned}\right\} \tag{5}$$

where $F(\sigma)$, $G(\sigma)$, $H(\sigma)$ are the following matrices with coefficients in the ring of polynomials $\mathbb{R}[\sigma]$

$$F(\sigma)=\begin{bmatrix} 1 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad G(\sigma)=\begin{bmatrix} \sigma \\ 1 \end{bmatrix}, \quad H(\sigma)=[1, \ \sigma^2]$$

As it turns out this rather formal-looking procedure is most useful (Kamen 1975). For instance in a very nice paper Kamen (1978) has worked out some of the relationships between the spectral properties of (4) and the commutative algebra which goes into the study of (5). And using this, and the re-interpretation of (5) as a family of systems, Byrnes (1977 b) has been able to do things about the feedback stabilization theory of (4).

Other bits of motivation for studying families come, for example, from identification theory (Hazewinkel 1979 a) and the study of high-gain feedback systems (Kar-Keung *et al.* 1977). In both these cases it is important to know in what ways a family of systems can suddenly degenerate, which is the subject matter of Hazewinkel (1980 c) and also of the present paper (Theorems 3 and 4).

Ideally one would like to write down explicit local (uni)versal deformations for each system as Arnol'd (1971) did for matrices. On general principles one expects that this is possible and for pairs of matrices (F, G), i.e. ' input systems ' or ' control systems ' this has recently been done by Tannenbaum (1980).

To extend these constructions *à la* Arnol'd of versal deformations to the case of triples of matrices may involve non-trivial difficulties. A reason for thinking this is that the stabilizer subgroup (see § 3 for a definition) of a system which is completely observable (CO) *or* completely reachable (CR) is trivial. Yet there is no (fine) moduli space for families of CO or CR systems as examples (8) and (9) show. That is, the stabilizer subgroup, which is at the heart of Arnold's constructions may be an insufficient guide in the setting of triples of matrices. For completely reachable or completely observable systems universal deformations result from the fine moduli space of Hazewinkel (1977 a, b). And in fact the original starting point for this paper was the far too optimistic idea that these moduli spaces might quite well be extendable to some extent. Thus the main problem considered in this paper became : given two families of linear dynamical systems Σ, Σ' over a manifold Q. Suppose that pointwise the systems Σ_q, Σ'_q are isomorphic for all or almost all $q \in Q$. What can be said about the relation between Σ and Σ' as families and what can be said about the relations between Σ_q and Σ'_q at the remaining points of Q.

The first question is of course entirely analogous to the one studied by Wasow (1962) and later in an algebraic setting by Ohm and Schneider (1964), with respect to similarity of families of matrices which depend (holomorphically) on a parameter.

2. Almost everywhere isomorphic families of systems

We use the abbreviations CR for completely reachable and CO for completely observable. Recall that the system (1) is CR if and only if the matrix

$$R(F, G) = [G \;\; FG \ldots F^n G] \tag{6}$$

is of full rank n, and that (1) is CO if and only if the matrix $Q(F, H)$ is of full rank n. Here $Q(F, H)$ is defined as

$$Q(F, H)^{\mathrm{T}} = [H^{\mathrm{T}} \;\; F^{\mathrm{T}} H^{\mathrm{T}} \ldots F^{n\mathrm{T}} H^{\mathrm{T}}] \tag{7}$$

where the symbol T means ' transpose '.

Let $L_{m,n,p}$ be the space of all linear dynamical systems (1) of state space dimension n and with m inputs and p outputs. That is, $L_{m,n,p}$ is the space of all triples of matrices (F, G, H) over $\mathbb{R}$ of dimensions $n \times n$, $n \times m$, $p \times n$ respectively. We give $L_{m,n,p}$ the corresponding topology, i.e. the topology of $\mathbb{R}^{n(n+m+p)}$. For the purposes of this paper a family of systems over a topological space Q is simply a continuous map $Q \to L_{m,n,p}$. A more general (and better) definition of family of systems is given in Hazewinkel (1980 a, 1979 b) and there the reader will also find a discussion of the reasons why the present definition is inadequate in some contexts. The theorems of the present paper extend with no trouble to this more general setting. This is automatic for the local Theorems 5 and 6, because locally (i.e. over a small enough open neighbourhood) the naive definition and the proper one agree. For the global versions of Theorems 1–4 it suffices to appeal to the same rigidity phenomenon (= uniqueness of (iso)morphisms if they exists at all) which is the basis of the corresponding local results.

If $\Sigma = (F, G, H)$ is a family of linear dynamical systems over a topological space Q we denote by $\Sigma(q)$ the system $(F(q), G(q), H(q))$. Completely analogously if $\Sigma = (F, G, H)$ is a (discrete time) system over a ring R then

$$\Sigma(\mathfrak{p}) = (F(\mathfrak{p}),\ G(\mathfrak{p}),\ H(\mathfrak{p}))$$

is the induced system over $R(\mathfrak{p})$, the quotient field of $R/\mathfrak{p}$.

Theorem 1

Let Σ and Σ' be two families over a topological space Q. Let

$$U_1 = \{q \in Q : \ \Sigma(q) \quad \text{and} \quad \Sigma'(q) \quad \text{are both CR}\}$$

and

$$U_2 = \{q \in Q : \ \Sigma(q) \quad \text{and} \quad \Sigma'(q) \quad \text{are both CO}\}$$

Suppose that $U_1 \cup U_2 = Q$ and suppose that $\Sigma(q)$ and $\Sigma'(q)$ are pointwise isomorphic for a dense set Z of points q in Q. Then Σ and Σ' are isomorphic as families over Q, (which, by definition, means that there is a *continuous* map $Q \to GL_n(\mathbb{R})$, $q \mapsto S(q)$, such that $F'(q) = S(q)F(q)S(q)^{-1}$, $G'(q) = S(q)G(q)$, $H'(q) = H(q)S(q)^{-1}$ for all $q \in Q$).

It follows in particular that $\Sigma(q)$ and $\Sigma'(q)$ are also isomorphic in all the remaining points, i.e. the points of $Q \backslash Z$. The (local) algebraic geometric version of this theorem is

Theorem 2

Let Σ and Σ' be two systems over a ring R. Let $U_1 = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are both CR$\}$, $U_2 = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are both CO$\}$. Suppose that $U_1 \cup U_2 = \mathrm{Spec}(R)$ and that there is a dense subset $Z \subset \mathrm{Spec}(R)$ such that $\Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are isomorphic for all $\mathfrak{p} \in Z$. Then Σ and Σ' are isomorphic as systems over R.

This means in particular that if R is an integral domain and $\Sigma = (F, G, H)$, $\Sigma' = (F', G', H')$ are two n-dimensional systems over R which are isomorphic over K, the quotient field of R, and if moreover for all maximal ideals $\mathfrak{m} \subset R$ we have that the rank of both $R(F, G)$, $R(F', G')$ or of both $Q(F, H)$, $Q(F', H')$ stays n mod $\mathfrak{m}$ then Σ and Σ' are also isomorphic as systems over R.

Both Theorems 1 and 2 are almost trivial consequences of the existence of fine moduli spaces for CR families and for CO families. These exist both in the topological case (cf. Hazewinkel 1977 a) and the algebraic-geometric case. This last fact is proved in Byrnes and Hurt (1979), Byrnes (1977 b), Hazewinkel (1977 b) for families of systems over an (algebraically closed) field k. For the proof of Theorem 2 one needs the stronger statement that the moduli space exists and has the fine moduli property as a scheme over $\mathbb{Z}$, which is proved in Hazewinkel 1980 a.

The proofs of Theorems 1 and 2 now go as follows. (We write out the details in the topological case only.) Recall that the fine moduli space M^{CR} is the quotient space $L^{\mathrm{CR}}{}_{m,n,p}/GL_n(\mathbb{R})$. Now let $\Sigma: S \to L^{\mathrm{CR}}{}_{m,n,p}$ be a family of CR systems. Assign to Σ the composed map $S \to L^{\mathrm{CR}}{}_{m,n,p} \to M^{\mathrm{CR}}$, which assign to $s \in S$ the point of M^{CR} corresponding to $\Sigma(s)$ (= the orbit of $\Sigma(s)$). Then part of the fine moduli property of M^{CR} says that two systems over S are isomorphic (as systems) if and only if they give rise to the same map $S \to M^{\mathrm{CR}}$. (This part of the fine moduli theorem is in fact almost trivial.) Thus under the hypothesis of Theorem 1 the families Σ and Σ' give rise to the same continuous map

$$U_1 \cap Z \to M^{\mathrm{CR}}$$

and because $U_1 \cap Z$ is dense in U_1 these two maps agree on all of U_1 which (by the fine moduli property) means that Σ and Σ' are isomorphic over U_1, i.e. there exists a continuous map

$$\phi_1: \ U_1 \to GL_n(\mathbb{R})$$

such that for all $q \in U_1$

$$\Sigma'(q) = \Sigma(q)^{\phi_1(q)}$$

where Σ^S is short for (SFS^{-1}, SG, HS^{-1}) if $\Sigma = (F, G, H)$, $S \in GL_n(\mathbb{R})$, the group of invertible $n \times n$ matrices.

Similarly there exists a fine moduli space for families of CO systems M^{CO} which similarly permits us to conclude that Σ and Σ' are isomorphic over U_2, so that there is a continuous map

$$\phi_2: \ U_2 \to GL_n(\mathbb{R})$$

such that

$$\Sigma'(q) = \Sigma(q)^{\phi_2(q)}, \quad q \in U_2$$

Now systems which are CR or CO enjoy the following rigidity property: if they are isomorphic the isomorphism is unique. Indeed if (F, G, H), $(F', G', H') \in L_{m,n,p}$ are isomorphic via $S \in GL_n(\mathbb{R})$ then S satisfies

$$SR(F, G) = R(F', G') \quad \text{and} \quad Q(F, H)S^{-1} = Q(F', H')$$

and if (F, G, H) and (F', G', H') are both CR or if both are CO then these relations determine S uniquely.

It follows that in the setting above $\phi_1(q) = \phi_2(q)$ for all $q \in U_1 \cap U_2$. That is, ϕ_1 and ϕ_2 agree on $U_1 \cap U_2$ proving that Σ and Σ' are isomorphic over all of Q.

The proof of Theorem 2, the algebraic-geometric version is completely analogous: it suffices essentially to replace the words 'continuous map' with 'morphism of algebraic varieties' everywhere in the above.

The trouble with Theorems 1 and 2 is that, unless one demands something like pointwise isomorphism everywhere, or CR everywhere, or CO everywhere, the condition $U_1 \cup U_2 = Q$ cannot be stated in terms of the separate families Σ and Σ'. So one is led to ask whether or not a condition like everywhere CO or CR would be sufficient. It is not ; as is more or less predictable from the well known fact that as a rule it is perfectly possible for two non-isomorphic systems Σ and Σ' over an integral domain R to become isomorphic over the quotient field (Sontag 1976). The simplest such example is undoubtedly the following one-dimensional one over $\mathbb{R}[\sigma]$.

$$\left.\begin{aligned} \Sigma: \quad F=1, \quad G=\sigma, \quad H=1 \\ \Sigma': \quad F'=1, \quad G'=1, \quad H'=\sigma \end{aligned}\right\} \tag{8}$$

Considered as families over $Q=\mathbb{R}$, parametrized by σ, we have that Σ is CO everywhere and CR everywhere except in 0, while Σ' is CR everywhere and CO everywhere except in 0. Thus $U_1 = U_2 = \mathbb{R}\backslash\{0\}$. Also $\Sigma(q)$ and $\Sigma'(q)$ are isomorphic for all $q \neq 0$. But of course Σ and Σ' are not isomorphic as families nor as systems over the ring $\mathbb{R}[\sigma]$.

Another example, which is slightly more illustrative of what goes on is given by the families

$$\left.\begin{aligned} \Sigma = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ \sigma & b \end{bmatrix}, [1, 0] \right) \\ \Sigma' = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sigma \\ 1 & b \end{bmatrix}, [1, 0] \right) \end{aligned}\right\} \tag{9}$$

which have essentially the same properties as the families (8). And here we note that though $\Sigma(0)$ and $\Sigma'(0)$ are of course not isomorphic, they are also not totally unrelated. In fact they agree on the completely reachable subsystem of $\Sigma(0)$. (For a more precise description of what this means, see below.) Note also that these examples largely destroy all hope about extending the fine moduli spaces ${M^{CR}}_{m,n,p}$ and ${M^{CO}}_{m,n,p}$ a bit.

Morphisms

Let Σ and Σ' be two families over Q. A morphism $\Sigma \to \Sigma'$ over Q then consists of a continuous map $\psi: Q \to M^{n \times n}$ the space of $n \times n$ matrices such that for all $q \in Q$, $\psi(q)G(q) = G'(q)$, $F'(q)\psi(q) = \psi(q)F(q)$, $H'(q)\psi(q) = H(q)$.

Completely analogously a morphism $\Sigma \to \Sigma'$ between two systems over a ring R is an $n \times n$ matrix T such that $TG = G'$, $F'T = TF$, $H'T = H$.

Using this notion one can now state the two following (dual) ' mildness of degeneracy ' results.

Theorem 3

Let Σ and Σ' be two families over Q. Suppose that $\Sigma(q)$ is CR for all $q \in Q$. Suppose moreover that $\Sigma'(q)$ and $\Sigma(q)$ are isomorphic for all q in a dense subset Z of Q. Then there is a morphism $T: \Sigma \to \Sigma'$ over Q such that $T(q): \Sigma(q) \to \Sigma'(q)$ is an isomorphism for all $q \in Z$ and such that $T(q): \Sigma(q) \to \Sigma'(q)$ maps the state space of $\Sigma(q)$ onto the completely reachable subspace of the state space of $\Sigma'(q)$ for all $q \in Q$.

Theorem 4

Let Σ and Σ' be two families over Q. Suppose that $\Sigma(q)$ is CO for all $q \in Q$. Suppose moreover that $\Sigma'(q)$ and $\Sigma(q)$ are isomorphic for all q in a dense subset Z of Q. Then there is a morphism $T: \Sigma' \to \Sigma$ over Q such that $T(q): \Sigma(q) \to \Sigma'(q)$ is an isomorphism for all $q \in Z$ and such that for all $q \in Q \backslash Z$ two states x, x' in state space of $\Sigma'(q)$ are indistinguishable (by means of observations) if and only if their difference $x - x'$ is in ker $(T(q))$.

There are of course the obvious analogous results for systems over rings. In this case Theorem 3 says, among other things, that the system over a ring R which is CR everywhere is maximal in the lattice of all realizations of minimal rank over R which realize the same input/output behaviour; similarly Theorem 4 says that the everywhere CO realization is the minimal element of this lattice. See Sontag (1977) for a discussion of the lattice of realizations of a linear response map over a ring.

Proof of Theorem 3

Let $q \in Q$. Because Σ is CR in q there exists a nice selection (Kalman 1971, Hazewinkel and Kalman 1976, Hazewinkel 1977 a, 1979 b) and an open subset $U \subset Q$ containing q such that $R(F(q'), G(q'))_\alpha$ is invertible for all $q' \in U$. Now let $z_1, z_2, \ldots,$ be a sequence of points of $Z \cap U$ converging to q.

Define the matrix $T(q)$ as the limit

$$T(q) = \lim_{i \to \infty} R(F'(z_i), G'(z_i))_\alpha (R(F(z_i), G(z_i))_\alpha^{-1}$$

It is not difficult to check that $T(q)$ does not depend on the choice of α or on the choice of the sequence $z_1, z_2, \ldots$.

Now for all i we have $z_i \in Z$ so that $\Sigma'(z_i)$ and $\Sigma(z_i)$ are isomorphic, say by $S_i \in GL_n(\mathbb{R})$. Then S_i satisfies

$$S_i R(F(z_i), G(z_i)) = R(F'(z_i), G'(z_i))$$

so that

$$S_i = R(F'(z_i), G'(z_i))_\alpha (R(F(z_i), G(z_i))_\alpha^{-1}$$

Writing out that S_i is an isomorphism we find

$$S_i F(z_i) = F'(z_i) S_i, \quad S_i G(z_i) = G'(z_i), \quad H'(z_i) S_i = H(z_i)$$

and taking the limit for $i \to \infty$ we find the relations

$$T(q) F(q) = F'(q) T(q), \quad T(q) G(q) = G'(q), \quad H'(q) T(q) = H(q)$$

so that $T(q)$ is a morphism $\Sigma(q) \to \Sigma'(q)$. It is easy to check that $T(q)$ depends continuously on q so that the $T(q)$ combine to define a morphism $T: \Sigma \to \Sigma'$. If $q \in Z$ then $T(q)$ is of course the unique isomorphism $\Sigma(q) \to \Sigma'(q)$. The relations written out above which are satisfied by $T(q)$ imply

$$T(q) R(F(q), G(q)) = R(F'(q), G'(q))$$

and, using that $(F(q), G(q))$ is completely reachable, it follows that the completely reachable subspace of $(F'(q), G'(q))$ is equal to the image of $T(q)$ (because the completely reachable subspace of a system (F, G, H) is the image of the map $R(F, G): \mathbb{R}^{(n+1)m} \to \mathbb{R}^n$).

The Proof of Theorem 4

This is similar to the proof of Theorem 3 (or we may appeal to duality).

Example

Let Σ and Σ' be two families over Q, which are pointwise isomorphic over a dense subset Z of Q. Then, without any further assumptions, we know of course that for all $q \in Q$, $\Sigma(q)$ and $\Sigma'(q)$ are related in the sense that their CR and CO subquotients are isomorphic. This follows from the continuity of the Laplace transform. Beyond this there seems little one can say (without making some sort of stableness hypothesis as in Theorems 3 and 4), as the following example shows.

$$\left.\begin{aligned} \Sigma &= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & a \\ \sigma & 2 \end{bmatrix}, [\sigma, 1] \right) \\ \Sigma' &= \left(\begin{bmatrix} \sigma \\ 1 \end{bmatrix}, \begin{bmatrix} 1-\sigma a & \sigma^2 a \\ -a & \sigma a + 2 \end{bmatrix}, [0, \sigma] \right) \end{aligned}\right\} \tag{10}$$

These families are pointwise isomorphic for all $\sigma \neq 0$. But for $\sigma = 0$ there is not even a morphism $\Sigma(0) \to \Sigma'(0)$, in fact there is not a morphism between the input parts of the completely reachable subsystems of $\Sigma(0)$ and $\Sigma'(0)$.

3. Everywhere pointwise isomorphic families of systems

Now let Σ and Σ' be families of systems over Q (resp. Spec(R)) which are pointwise isomorphic everywhere. Then it does not necessarily follow that Σ and Σ' are isomorphic as families over Q (resp. are isomorphic as systems over R), as the following example shows.

Example

Consider the two families over $\mathbb{R}$ (or the two systems over $\mathbb{R}[\sigma]$) defined by

$$\Sigma = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \sigma^2 & 1 \end{bmatrix}, [1, 2] \right)$$

$$\Sigma' = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix}, [1, 2\sigma] \right)$$

These two families are pointwise isomorphic for all σ (resp. the systems $\Sigma(\not p)$, $\Sigma'(\not p)$ are isomorphic for all prime ideals $\not p \subset \mathbb{R}[\sigma]$) but they are not isomorphic as families over $\mathbb{R}$ (resp. as systems over $\mathbb{R}[\sigma]$); indeed Σ and Σ' are not isomorphic in any neighbourhood of 0 (resp. not isomorphic over any localization $\mathbb{R}[\sigma]_f$ of $\mathbb{R}[\sigma]$ for which $f(0) \neq 0$).

So we shall need some sort of extra condition to ensure that pointwise isomorphism implies isomorphism as families.

Stabilizer subgroups

Let Σ be a family over Q. Then for each $q \in Q$ we define

$$N(q) = \{S \in GL_n(\mathbb{R}) : SF(q) = F(q)S, SG(q) = G(q), H(q)S = H(q)\}$$

This is the stabilizer subgroup in $GL_n(\mathbb{R})$ of the system $\Sigma(q)$. The Lie algebra of $N(q)$ is

$$L(q) = \{T \in M^{n \times n} : TF(q) = F(q)T, TG(q) = 0, H(q)T = 0\}$$

We use $r(q)$ to denote the dimension of $N(q)$ which is of course equal to the dimension of $L(q)$. Completely analogously one defines in the case of a system $\Sigma = (F, G, H)$ over a ring R the subgroup $N(\mathfrak{p})$ of $GL_n(R(\mathfrak{p}))$ consisting of all invertible matrices S over the field $R(\mathfrak{p})$ (= quotient field of $R/\mathfrak{p}$), such that $SF(\mathfrak{p}) = F(\mathfrak{p})S$, $SG(\mathfrak{p}) = G(\mathfrak{p})$, $H(\mathfrak{p})S = H(\mathfrak{p})$ and $L(\mathfrak{p})$ as the Lie algebra of all $n \times n$ matrices T with coefficients in $R(\mathfrak{p})$ such that $TF(\mathfrak{p}) = F(\mathfrak{p})T$, $TG(\mathfrak{p}) = 0$, $H(\mathfrak{p})T = 0$.

Differentiable families of systems

Topologically the space of all n dimensional systems with m inputs and p outputs is homeomorphic with $\mathbb{R}^{n(n+m+p)}$, cf. § 2. We now give $L_{m,n,p}$ also the differentiable structure of $\mathbb{R}^{n(n+m+p)}$. Now let Q be a differentiable manifold. Then a family of systems $\Sigma : Q \to L_{m,n,p}$ is a *differentiable family of systems* if the map Σ is differentiable. Two differentiable families of systems Σ and Σ' are isomorphic as differentiable families if there is a differentiable map $\phi : Q \to GL_n(\mathbb{R})$ such that $\Sigma(q)^{\phi(q)} = \Sigma'(q)$ for all $q \in Q$. Here, of course, $GL_n(\mathbb{R})$ is given the differentiable structure of an open subset of $\mathbb{R}^{n^2}$. The space of orbits M^{CR} of completely reachable systems has a natural differentiable structure and with this structure it is a fine moduli space for the appropriate notion (based on vector bundles) of differentiable families of CR systems (in the differentiable category), (Hazewinkel 1977 a, 1980 a).

Theorem 5

Let Σ and Σ' be two differentiable families over the differentiable manifold Q. Suppose that Σ and Σ' are pointwise isomorphic everywhere. Suppose moreover that $r(q) = \dim N(q)$ ($= \dim L(q)$) is constant in some neighbourhood U of $q_0 \in Q$. Then there is a (possibly smaller) neighbourhood V of q_0 such that Σ and Σ' are isomorphic as differentiable families over V.

Proof

The proof is not difficult (and more or less standard). Consider the map $\phi : GL_n(\mathbb{R}) \times Q \to L_{m,n,p} \times Q$ given by $(S, q) \to (\Sigma(q)^S, q)$. It follows from the assumption of constancy of the dimension of $N(q)$ that $d\phi$ is of constant rank, so that ϕ is a submersion onto its image. In particular ϕ locally admits sections; i.e. if $(\Sigma_0, q_0) \in \mathrm{Im}\, \phi$ then there is an open neighbourhood U of (Σ_0, q_0) and a differentiable map $s : U \to GL_n(\mathbb{R}) \times Q$ such that $\phi \circ s = id$. Now consider $\psi : Q \to L_{m,n,p} \times Q$ given by $\psi(q) = (\Sigma'(q), q)$; this is simply the graph of Σ'. By assumption for each q we know that $\psi(q) \in \mathrm{Im}\ \phi(GL_n(\mathbb{R}) \times \{q\})$ and the fibre of ϕ over $\psi(q)$ is precisely $\Phi(q) \times \{q\}$ where $\Phi(q)$ is set of all possible isomorphisms $\Sigma(q) \to \Sigma'(q)$. (Of course $\Phi(q)$ is a left coset of $N(q)$.) Now let s be a local section of ϕ defined in some neighbourhood of $(\Sigma'(q_0), q_0)$. Restricting s to the graph of Σ' (i.e. the image of ϕ) gives us a map $U_0 \to GL_n(\mathbb{R}) \times U_0$ of the form $q' \mapsto (S(q'), q')$ (because s is a section). The map $q' \mapsto S(q')$ is then the desired isomorphism $\Sigma \to \Sigma'$ (over U_0). For this proof at least, some sort of differentiability restriction is necessary. There are analogous theorems for holomorphic families and real analytic families. The corresponding theorem for systems over rings is

Theorem 6

Let Σ and Σ' be two systems over a ring R. Suppose that $\Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are isomorphic for all prime ideals $\mathfrak{p}$ contained in some open subset U of $\mathrm{Spec}(R)$ Suppose moreover that $r(\mathfrak{p}) = \dim N(\mathfrak{p})$ is constant for some neighbourhood U' of $\mathfrak{p}_0 \in U$. Then there exists an open neighbourhood $V = \mathrm{Spec}(R_f)$, $f \in R$, of $\mathfrak{p}_0$ such that Σ and Σ' are isomorphic as systems over R_f (or, equivalently, as families over V).

For both Theorems 5 and 6 it is in general not true that Σ and Σ' are necessarily isomorphic over all of Q (resp. isomorphic as systems over R) as the following example shows.

Example

Consider the following two systems, either as families over $\mathbb{R}$ or as systems over the ring $\mathbb{R}[\sigma]$

$$\Sigma = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sigma \\ 0 & \sigma^2 \end{bmatrix}, [\sigma^2 - 1, -\sigma] \right)$$

$$\Sigma' = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sigma + 2 \\ 0 & \sigma^2 \end{bmatrix}, [\sigma^2 - 1, -\sigma - 2] \right)$$

These two families are pointwise isomorphic everywhere ; the dimension of the stabilizer subgroups is 1 everywhere ; in addition one has rank $R(F(\sigma), G(\sigma))$ and rank $Q(F(\sigma), H(\sigma))$ are also equal to 1 everywhere. As families the two systems are isomorphic over $\mathbb{R} \backslash \{-1\}$ and also over $\mathbb{R} \backslash \{1\}$. As systems over rings they are isomorphic over $\mathbb{R}[\sigma]_{\sigma-1}$ and $\mathbb{R}[\sigma]_{\sigma+1}$, but not, as is easily checked, as systems over $\mathbb{R}[\sigma]$ itself. The systems Σ and Σ' are not even isomorphic as differentiable (or topological) families. Indeed such an isomorphism must necessarily be of the form

$$\sigma \mapsto \begin{bmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{bmatrix}$$

because the isomorphism matrices must take $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = G(\sigma)$ into $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = G'(\sigma)$.

Here $c_{12}(\sigma)$, $c_{22}(\sigma)$ are continuous functions of σ such that $c_{22}(\sigma)$ is nowhere zero on $\mathbb{R}$. From

$$\begin{bmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{bmatrix} \begin{bmatrix} 1 & \sigma \\ 0 & \sigma^2 \end{bmatrix} = \begin{bmatrix} 1 & \sigma+2 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{bmatrix}$$

$$[\sigma^2-1, \ -\sigma-2] \begin{bmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{bmatrix} = [\sigma^2-1, \ -\sigma]$$

one then sees that the sole remaining condition on the $c_{12}(\sigma)$, $c_{22}(\sigma)$ is that

$$(\dagger) \qquad (\sigma^2-1)c_{12}(\sigma) = c_{22}(\sigma)(\sigma+2) - \sigma$$

This means that $3c_{22}(1) = 1$ and $c_{22}(-1) = -1$. But there is no real continuous function assuming these values in 1 and -1 and which is non-zero everywhere.

And of course the matrix

$$\begin{bmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{bmatrix}$$

defines an isomorphism over the ring $\mathbb{R}[\sigma]$ if and only if $c_{22}(\sigma)$ is a non-zero constant which is also incompatible with (†).

The main ingredient of the proof of Theorem 6 is the following generalization of the central lemma of Ohm and Schneider (1964).

Lemma

Let R be a ring without nilpotents, let A be an $m \times n$ matrix with coefficients in R and let $a \in R^m$. Consider the equation $Ax = a$. Suppose that the equation $A(\mathfrak{p})y = a(\mathfrak{p})$ over the field $R(\mathfrak{p})$ can be solved for all prime ideals $\mathfrak{p}$. Suppose moreover that $r(\mathfrak{p}) = \text{rank } A(\mathfrak{p})$ is constant (as a function of $\mathfrak{p}$). Then $Ax = a$ is solvable over R. Moreover if $\mathfrak{m}$ is a maximal ideal of R and $y(\mathfrak{m})$ is any pre-given solution of $A(\mathfrak{m})y = a(\mathfrak{m})$, then there is a solution x of $Ax = a$ over R such that $x \equiv y(\mathfrak{m}) \bmod \mathfrak{m}$. Finally if $\mathfrak{p}$ is a prime ideal and $y(\mathfrak{p})$ is any given solution of $A(\mathfrak{p})y = a(\mathfrak{p})$ then there is an $f \in R \backslash \mathfrak{p}$ and a solution of $Ax = a$ over R_f such that $x \equiv y(\mathfrak{p}) \bmod \mathfrak{p}R_f$.

Proof

Let $P = \text{Im}(A)$, and let $Q = R^m/\text{Im}(A)$. Let $\mathfrak{p}$ be a prime ideal of R and consider the localized morphism of modules

$$A_{\mathfrak{p}}: \ R_{\mathfrak{p}}^n \rightarrow R_{\mathfrak{p}}^m$$

$A_{\mathfrak{p}}$ takes $\mathfrak{p}R_{\mathfrak{p}}^n$ into $\mathfrak{p}R_{\mathfrak{p}}^m$. Let $A(\mathfrak{p})$ be the induced quotient map

$$A(\mathfrak{p}): \ R(\mathfrak{p})^n \rightarrow R(\mathfrak{p})^m$$

where $R(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is the quotient field of $R/\mathfrak{p}$. By premultiplying and postmultiplying $A(\mathfrak{p})$ with invertible matrices S, T we can see to it that

$A(\mathfrak{p})$ is of the form

$$(*) \qquad \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{bmatrix}$$

(where there are $r = r(\mathfrak{p})$ 1's for the first r diagonal entries and zero's everywhere else). Let $\tilde{S}$, $\tilde{T}$ be any invertible matrices over R which reduce to S and $T \bmod \mathfrak{p}R_{\mathfrak{p}}$. Then $\tilde{S}A_{\mathfrak{p}}\tilde{T}$ looks like $(*)$ with the 1's replaced by $1 + a_{ii}$, $a_{ii} \in \mathfrak{p}R_{\mathfrak{p}}$ and the 0's replaced by a_{ij}, $a_{ij} \in \mathfrak{p}R_{\mathfrak{p}}$. Because the $1 + a_{ii}$ are invertible in R further pre- and post-multiplication with invertible matrices gives $A_{\mathfrak{p}}$ the form

$$(**) \qquad \left[\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline & & & * & \cdots & * \\ & 0 & & \vdots & & \vdots \\ & & & * & \cdots & * \end{array}\right]$$

where all the $(*)$-elements are in $\mathfrak{p}A_{\mathfrak{p}}$. But the rank hypothesis says that the rank of the matrix $(**)$ considered as a matrix over the quotient field of R is also r. Because R has no nilpotents it follows that all the $(*)$-elements in $(**)$ are zero. And from this it is of course immediate that $Q_{\mathfrak{p}} =$ co-ker $(A_{\mathfrak{p}} : R_{\mathfrak{p}}^{n} \to R_{\mathfrak{p}}^{m})$ is free of rank $m - r$.

It follows that Q is a projective R-module (because Q is locally free (Bourbaki 1961, Ch. II, § 5) and hence a direct summand of some free R-module R^p

$$i : Q \to R^p$$

Now consider the image $\bar{a}$ of a in $Q = R^m/\mathrm{Im}(A)$. The solvability of $A(\mathfrak{p})y = a(\mathfrak{p})$ means that $\bar{a}$ maps to zero under $Q \to Q(\mathfrak{p}) = Q \otimes R/\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. So for all coordinates $i_1(\bar{a}), \ldots, i_p(\bar{a})$ of $i(\bar{a})$ we have that $i_s(\bar{a}) \equiv 0 \bmod \mathfrak{p}$ for all $\mathfrak{p}$, i.e. $i_s(\bar{a}) \in \mathfrak{p}$ all $\mathfrak{p}$. Because R has no nilpotents this means that $i_s(\bar{a}) = 0$, $s = 1, \ldots, p$ and hence $\bar{a} = 0$ proving that $Ax = a$ is solvable over R.

Now let $y(\mathfrak{m})$ be any pre-given solution of $A(\mathfrak{m})y = a(\mathfrak{m})$ where $\mathfrak{m}$ is a maximal ideal of R. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & R^n & \xrightarrow{A} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 0 & \longrightarrow & C(\mathfrak{m}) & \longrightarrow & R(\mathfrak{m})^n & \longrightarrow & P(\mathfrak{m}) & \longrightarrow & 0 \end{array}$$

where C is the kernel of $A : R^n \to R^m$. The module P is also projective as the kernel of $R \to Q$. It follows that the lower sequence is also exact. Some diagram chasing, using the fact that j' is surjective now readily proves the second assertion of the lemma.

Indeed let x_1 be any solution of $Ax=a$. Then $x_1(\mathfrak{m})$ is also a solution of $A(\mathfrak{m})y=a(\mathfrak{m})$. It follows that $A(\mathfrak{m})(x_1(\mathfrak{m})-y(\mathfrak{m}))=0$ so that by the exactness of the lower sequence of the diagram above $x_1(\mathfrak{m})-y(\mathfrak{m})\in C(\mathfrak{m})$. Now let $x_2\in C$ be such that $j'(x_2)=x_1(\mathfrak{m})-y(\mathfrak{m})$. Because $x_2\in C=\ker(A)$, $x=x_1-x_2$ is also a solution of $Ax=a$. Moreover $x(\mathfrak{m})=j(x)=j(x_1)-j(x_2)=x_1(\mathfrak{m})-(x_1(\mathfrak{m})-y(\mathfrak{m}))=y(\mathfrak{m})$, so that this solution does indeed specialize to the given one mod $\mathfrak{m}$.

If $\mathfrak{p}\subset R$ is prime, one argues exactly the same. The only extra difficulty is that j' : $C\to C(\mathfrak{p})$ is not necessarily surjective. However, if $z\in C(\mathfrak{p})$ is any element, then there always is an $f\in R\backslash\mathfrak{p}$ such that z is in the image of $C_f\to C(\mathfrak{p})$.

Proof of Theorem 6

Given the lemma, the proof of Theorem 6 is entirely straightforward. Indeed one considers the linear map A : $R^k\to R$ given by $X\mapsto(XF-F'X,\ XG,\ H'X)$ where $k=n^2$ and X is a k-vector written as an $n\times n$ matrix. Here $l=n^2+nm+np$. Now let $a\in R^l$ be the vector $(0,\ G',\ H)$. The constancy of dim $N(\mathfrak{p})=\dim L(\mathfrak{p})$ means that rank $A(\mathfrak{p})=$ constant. Now let $\mathfrak{p}_0$ be any prime ideal and $S(\mathfrak{p}_0)$ an invertible matrix over $R(\mathfrak{p})$ taking $\Sigma(\mathfrak{p}_0)$ into $\Sigma'(\mathfrak{p}_0)$. Then $S(\mathfrak{p}_0)$ solves $A(\mathfrak{p}_0)y=a(\mathfrak{p}_0)$. So by the lemma there is a solution S over R_f for some $f\in R\backslash\mathfrak{p}_0$ of $Ax=a$ which moreover agrees with $S(\mathfrak{p}_0)$ mod $\mathfrak{p}_0$. Because $S(\mathfrak{p}_0)$ is invertible S is invertible over $R_{ff'}$ for some suitable $f'\in R\backslash\mathfrak{p}_0$.

Examples

It does not appear that the condition that the dimension of the stabilizer subgroups $N(q)$ remains constant as q varies has much to do with conditions which seem system-theoretically more natural like rank $R(F(q),\ G(q))$ being constant. Consider for example the family

$$\Sigma=\left(\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}1&1\\0&\sigma\end{bmatrix},\ [0,\ 2]\right)$$

For this family over $\mathbb{R}$ one has rank $R(F(q),\ G(q))=1=$ rank $Q(F(\sigma),\ H(\sigma))$ for all $\sigma\in\mathbb{R}$, but dim $N(\sigma)=1$ if $\sigma=1$ and dim $N(\sigma)=0$ otherwise. On the other hand the family

$$\Sigma=\left(\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}0&1\\\sigma&1\end{bmatrix},\ [1,\ 0]\right)$$

has dim $N(\sigma)=0$ everywhere but rank $R(F(\sigma),\ G(\sigma))=2$ if $\sigma\neq 0$ and $=1$ if $\sigma=0$ (and rank $Q(F(\sigma),\ H(\sigma))=2$ everywhere).

4. Conclusions

The main questions studied in this paper were :

1. Given two families of systems Σ and Σ' which are pointwise isomorphic. Are they then also isomorphic as families ?
2. Given two families of systems Σ and Σ' over Q which are pointwise isomorphic over Q or some dense subset Z of Q. What can be said about the relation between $\Sigma(q)$ and $\Sigma'(q)$ at the points of $Q\backslash Z$.

Question 1 received a positive answer which specializes to a theorem of Wasow's (1962) for holomorphic families of matrices under similarity. It seems also likely that the theorem is the best possible in the sense that if Σ is a family such that dim $N(q)$ is not constant then there is a family Σ' which is pointwise isomorphic to Σ everywhere but not isomorphic as families in any neighbourhood of a point q where dim $N(q)$ suddenly increases. As to question 2, there are definite relations between $\Sigma(q)$ and $\Sigma'(q)$ if either Σ or Σ' is CR or CO in a neighbourhood of q. If not then a number of examples show that the ways in which a family of systems can degenerate do not depend only on the isomorphism classes of the systems involved but also on the systems themselves (apart from the subquotients which are recoverable from the transfer functions (cf. also Hazewinkel 1979 a, 1980 c). Thus one has here the usual scaling and singular perturbation phenomena. It remains to construct local versal deformation of non-CR and non-CO systems.

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