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REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS

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REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS.

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ABSTRACT.

A certain partial order on the set of all partitions of a given natural number n describes many containment, specialization or degeneration relations in the, seemingly, rather disparate parts of mathematics dealing with permutation representations of S_n, the existence of (0,1)-matrices with prescribed row and column sums, symmetric mean inequalities, orbits of nilpotent matrices under similarity, Kronecker indices of control systems, doubly stochastic matrices and vectorbundles over the Riemann sphere. In this paper we discuss relations between all these subjects which show why the same ordering <u>must</u> appear all the time. Central to the discussion is the Schubert-cell decomposition of a Grassmann manifold and the associated (closure) ordering which is a quotient of the Bruhat ordering on the Weyl group S_n.

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Contents.

1. INTRODUCTION.

Let κ be a partition of n, $\kappa = (\kappa_1, \cdots, \kappa_m)$, $\kappa_1 \geq \cdots \geq \kappa_m \geq 0$, $\Sigma\kappa_i = n$. We identify partitions $(\kappa_1, \ldots, \kappa_m)$ and $(\kappa_1, \ldots, \kappa_m, 0, \ldots, 0)$. Quite a few classes of objects in mathematics are of course classified by partitions and often inclusion, specialization or degeneration relations between these objects are described by a certain partial order on the set of partitions. This partial order on the set of all partitions of n is defined as follows:

$$(1.1) \qquad \begin{aligned} &(\kappa_1, \ldots, \kappa_m) > (\kappa_1', \ldots, \kappa_m') \\ &\text{iff } \sum_{i=1}^{r} \kappa_i \leq \sum_{i=1}^{r} \kappa_i', \ r = 1, \ldots, m. \end{aligned}$$

Thus, for example $(2,2,1) > (3,2)$. If $\kappa > \kappa'$ we say that κ specializes to κ' or that κ is more general than κ'. The reverse order has been variously called the dominance order [2] , the Snapper order [35,42] or the natural order [36]. It occurs naturally in several seemingly rather unrelated parts of mathematics. Some of these occurrences are the

- (i) Snapper, Liebler-Vitale, Lam, Young theorem (on the permutation representations of the symmetric groups)
- (ii) Gale-Ryser theorem (on existence of (0,1)-matrices)
- (iii) Muirhead's inequality (a symmetric mean inequality)
- (iv) Gerstenhaber-Hesselink theorem (on orbit closure properties of SL_n acting on nilpotent matrices)
- (v) Kronecker indices (on the orbit closure, or degeneration, properties of linear control systems acted on by the socalled feedback group)
- (vi) Double stochastic matrices (when is a partition "an average" of another partition)
- (vii) Shatz's theorem (on degeneration of vectorbundles over the

Riemann sphere)

These will be described in more detail in section 2 below. In addition the same ordering, via the representation theory of the symmetric groups, plays a considerable role in theoretical chemistry (in the theory of chiral molecules), i.e., molecules that are optically active [11,16,18]. Finally the same order plays an important role in thermodynamical considerations. Consider an (isolated) system described by a probability vector $p = (p_1, p_2, \dots)$, where p_i is the probability that a particle is in state i, evolving according to some "master equation". Then in [37,38] it is shown that the system evolves in the direction of increasing $\bar{p} = (\bar{p}_1, \bar{p}_2, \dots)$ (with respect to the specialization order), where $\bar{p}$ is the unique rearrangement of p such that $\bar{p}_1 \geq \bar{p}_2 \geq \dots$. This statement is a good deal stronger, in fact infinitely stronger [39], than the statement that the

entropy $-\sum_{i=1}^{\infty} p_i \ln p_i$ must always increase.

Certain occurrences of the specialization order are known to be intimately related. Thus (i), (ii), (iii) and (vi) are very much related [2,5,13], cf. also section 2 below, and so are (v) & (vii) [15] and section 8 below. This paper will show that all these manifestations of this order are intimately related. Their common meeting ground seems to be the ordering defined by closure relations of the Schubert-cells (with respect to a standard basis) of a Grassmann manifold. I.e. a Schubert-cell SC(λ) is more general than SC(λ') ; in symbols: $SC(\lambda) > SC(\lambda')$, iff $\overline{SC(\lambda)} \supset SC(\lambda')$. This order in turn is much related to the Bruhat ordering (sometimes called Bernstein-Gelfand-Gelfand ordering) on the Weyl group S_n. It is, in fact, the quotient ordering induced by the canonical map of the manifold of all flags in $\mathbb{R}^{n+m}$ to the Grassmann manifold of n-planes in (n+m)-space.

It should be said that in all probability there is much more to be said. The diagram of interrelations between the manifestations of the specialization order (cf. section 5.1 below) has overlap with another (functorial relationship) diagram centering around the irreducible quotients of Verma modules for sl_n, the Jantzen conjecture (now proved by A. Joseph) and the Bruhat ordering, and involving, among others, work of Kazhdan-Lusztig, Gelfand-MacPherson (relations with Schubert cells), Borho-Kraft and the same relation between orbits of nilpotent matrices and permutation representations which plays a role in this paper. (We owe these remarks to W. Borho).

2. SEVERAL MANIFESTATIONS OF THE SPECIALIZATION ORDER.

A schematic overview of the various relations of the specialization order to be described below can be found in section 5 of this paper.

2.1. The Snapper, Liebler-Vitale, Lam, Young theorem (formerly the Snapper conjecture).

Let S_n be the group of permutations on n letters. Let $\kappa = (\kappa_1,\dots,\kappa_m)$ be a partition of n and let S_κ be the corresponding Young subgroup $S_\kappa = S_{\kappa_1} \times \dots \times S_{\kappa_m}$, where S_{κ_i} is seen as the subgroup of S_n acting on the letters $\kappa_1 + \dots + \kappa_{i-1} + 1, \dots, \kappa_1 + \dots + \kappa_i$. (If $\kappa_m = 0$ the factor S_{κ_m} is deleted). Take the trivial representation of S_k and induce this up to S_n. Let $\rho(\kappa)$ denote the resulting induced representation. It is of dimension $\binom{n}{k} = n!/\kappa_1! \dots \kappa_m!$ and it can be easily described as follows. Take m symbols $a_1, \dots, a_m$ and consider all associative (but non-commutative) words $\varepsilon_1 \dots \varepsilon_n$ of length n in the symbols $a_1, \dots, a_m$ such that a_i occurs precisely κ_i times. Let $W(\kappa_1,\dots,\kappa_m) = W(\kappa)$ denote this set, then S_n acts on $W(\kappa)$ by $\sigma^{-1}(\varepsilon_1\dots\varepsilon_n) = \varepsilon_{\sigma(1)}\varepsilon_{\sigma(2)} \dots \varepsilon_{\sigma(n)}$. Let $V(\kappa)$ be the vector space with the elements of $W(\kappa)$ as basis vectors. Extending the action of S_n linearly to $V(\kappa)$ gives a representation of S_n and this representation is $\rho(\kappa)$.

Now the irreducible representations of S_n are also labeled by partitions. Let $[\kappa]$ be the irreducible representation belonging to the partition κ. Snapper [21] proved that $[\kappa]$ occurs in $\rho(\kappa')$ only if $\kappa < \kappa'$ and conjectured the reverse implication. Liebler and Vitale [14] proved that $\kappa < \kappa'$ implies that $\rho(\kappa)$ is a direct summand of $\rho(\kappa')$ which, of course, implies that $\kappa < \kappa'$ which in turn implies that $[\kappa]$ occurs in $\rho(\kappa')$. Another proof of the implication (via a different generalization) is given in Lam [13]. Still another proof can be based on Young's rule, cf. section 6 below, and a completely elementary proof can be found in [7].It is probably correct to ascribe the result in the first place to Young.

2.2. Th<u>e Gale-Ryser Theorem ([19])</u>. Let μ and υ be two partitions of n. Then there is a matrix consisting of zeros and ones whose columns sum to μ and whose rows sum to υ iff $\upsilon > \mu^*$. Here μ^* is the dual partition of μ defined by $\mu_i^* = \#\{j \mid \mu_j \geq i\}$. For example, $(2,2,1)^* = (3,2)$.

2.3. <u>Doubly Stochastic Matrices</u>. A matrix $M = (m_{ij})$ is called doubly stochastic if $m_{ij} \geq 0$ for all i,j and if all the columns and all the rows add up to 1. Let μ and ν be two partitions of n. One says that μ is an average of ν if there is a doubly stochastic matrix M such that $\mu = M\nu$. Then there is the theorem that μ is an average of ν iff $\mu > \nu$ in the specialization order.

2.4. <u>Muirhead's Inequality</u>. One of the best-known inequalities is $(x_1,\ldots,x_n)^{1/n} \leq n^{-1}(x_1+\ldots+x_n)$. A far-reaching generalization due to Muirhead [22] goes as follows. Given a vector $p = (p_1,\ldots,p_n)$, $p_i \geq 0$, one defines a symmetrical mean (of the nonnegative variables $x_1,\ldots,x_n$) by the formula

$$(2.5) \qquad [p](x) = (n!)^{-1} \sum_{\sigma} x_1^{p_{\sigma(1)}} \cdots x_n^{p_{\sigma(n)}}$$

where the sum runs over all permutations $\sigma \in S_n$. Then one has Muirhead's inequality which states that $[p](x) \leq [q](x)$ for all nonnegative values of the variables $x_1, \ldots, x_n$ iff p is an average of q, so that in case p and q are partitions of n this happens iff $p > q$. The geometric mean-arithmetic mean inequality thus arises from the specialization relation $(1,\ldots,1) > (n,0,\ldots,0)$.

2.6. <u>Completely Reachable Systems.</u>

Let $L_{m,n}$ denote the space of all pairs of real matrices (A,B) of sizes n x n and n x m respectively. To each pair (A,B) one associates a control system given by the differential equations

$$(2.7) \qquad \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

where the u's are the inputs or controls. The pair (A,B), or equivalently, the system (2.7), is said to be <u>completely reachable</u> if the reachability matrix $R(A,B) = (B \; AB \; \ldots \; A^nB)$ consisting of the n+1 (nxm)-blocks A^iB, $i = 0, \ldots, n$, has maximal rank n. In system theoretic terms this is equivalent to the property that for any two points $x, x' \in \mathbb{R}^n$ one can steer $x(t)$ to x' in finite time starting from $x(0) = x$ by means of suitable control functions $u(t)$.

Let $L^{cr}_{m,n}$ denote the space of all completely reachable pairs of matrices (A,B). The Lie-group F of all block lower diagonal matrices $\begin{pmatrix} S & 0 \\ K & T \end{pmatrix}$, $S \in GL_n(\mathbb{R})$, $T \in GL_m(\mathbb{R})$, K an m x n matrix, acts on $L^{cr}_{m,n}$ according to the formula

$$(2.8)\qquad (A,B)^g = (SAS^{-1} + SBTS^{-1}K, SBT),\ g = \begin{pmatrix} S & 0 \\ K & T \end{pmatrix}$$

The "generating transformations" $(A,B) \to (SAS^{-1}, SB)$ (base change in state space), $(A,B) \to (A, BT^{-1})$ (base change in input space) and $(A,B) \to (A+BK, B)$ (state space feedback), occur naturally in design problems (of control loops) in electrical engineering. It is a theorem of Brunovsky [31] and Kalman [10] and Wonham and Morse [32] that the orbits of F acting on $L^{cr}_{m,n}$ correspond bijectively with partitions of n. The partition belonging to $(A,B) \in L^{cr}_{m,n}$ is found as follows. Let d_j be the dimension of the subspace of $\underline{\underline{R}}^n$ spanned by the vectors $A^i b_r$, $r = 1, \ldots, m$, $i \leq j$ where b_r is the r-th column of B. Let $e_j = d_j - d_{j-1}$, $d_{-1} = 0$. The partition corresponding to (A,B) is the dual partition of $(e_o, e_1, e_2, \ldots, e_n)$, i.e. $\kappa(A,B) = (e_o, e_1, \ldots, e_n)^*$. The numbers $\kappa_1 \geq \ldots \geq \kappa_m$ making up $\kappa(A,B)$ are called the Kronecker indices of (A,B). (Because the problem of classifying pairs (A,B) up to feedback equivalence, i.e. up to the action of F, is a subproblem of the problem of classifying pencils of matrices studied by Kronecker: to (A,B) one associates the pencil $(A-sI\ B)$. The partition $(e_o, \ldots, e_n)$ corresponds to the dimensions of the filtration of <u>controllability</u> subspaces.

Let Θ_κ be the orbit of F acting on $L^{cr}_{m,n}$ labeled by κ. Then a second theorem, noted by a fair number of people independently of each other (Byrnes, Hazewinkel, Kalman, Martin,...), but never yet published, states that $\bar{\Theta}_\kappa \supset \Theta_{\kappa'}$ iff $\kappa > \kappa'$. Some of the control theoretic implications of this are contained in Martin [33].

2.9. <u>Vectorbundles over the Riemann sphere</u>. Let E be a holomorphic vectorbundle over the Riemann sphere $S^2 = \underline{\underline{P}}^1(\underline{\underline{C}})$. Then according to Grothendieck [4] E splits as a direct sum of line bundles.

$$E \simeq L(\kappa_1) \oplus \ldots \oplus L(\kappa_m) \tag{2.10}$$

Where $L(i)$ is the unique (up to isomorphism) line bundle over $\mathbb{P}^1(\mathbb{C})$ of degree i, $L(i) = L(1)^{\otimes i}$, $i \in \mathbb{Z}$, where $L(1)$ is the canonical very ample bundle of $\mathbb{P}^1(\mathbb{C})$. Thus each holomorphic vectorbundle E over $\mathbb{P}^1(\mathbb{C})$ defines a m-tuple of integers $\kappa(E)$ (in decreasing order). The bundle is called positive if $\kappa_i(E) \geq 0$ for all $i = 1, \ldots, m$. Concerning these positive bundles there is now the following degeneration result of Shatz [20]. Let E_t be a holomorphic family of m-dimensional vectorbundles over $\mathbb{P}^1(\mathbb{C})$. Then for all small enough t, $\kappa(E_t) > \kappa(E_o)$. And inversely if $\kappa > \kappa'$ then there is a holomorphic family E_t such that $\kappa(E_\tau) = \kappa$ for t small $t \neq 0$ and $\kappa(E_o) = \kappa'$.

2.11. <u>Orbits of Nilpotent Matrices</u>. Let N_n be the space of all nxn complex nilpotent matrices. Consider $\mathbb{SL}_n(\mathbb{C})$ or $\mathbb{GL}_n(\mathbb{C})$ acting on N_n by similarity, i.e.

$$A^S = SAS^{-1}, (A \in N_n, \ S \in \mathbb{GL}_n(\mathbb{C})).$$

By the Jordan normal form theorem the orbits of this action are labelled by partitions of n. Let $O(\kappa)$ be the orbit consisting of all nilpotent matrices similar to the one consisting of the Jordan blocks $J(\kappa_i)$, $i = 1, \ldots, m$, where $J(\kappa_i)$ is the $\kappa_i \times \kappa_i$ matrix with 1's just above the diagonal and zeros everywhere else. Then the Gerstenhaber-Hesselink theorem says that $\overline{O(\kappa)} \supset O(\kappa')$ iff $\kappa < \kappa'$. (Note the reversion of the order with respect to the result on orbits described in 2.6 above).

3. GRASSMANN MANIFOLDS AND CLASSIFYING VECTORBUNDLES.

In order to describe how the various manifestations of the specialization order are connected to each other we need to define Grassmann manifolds, the classifying vectorbundles over them and their Schubert cell decompositions (in section 4 below).

3.1. Grassmann Manifolds. Fix two numbers $m, n \in \mathbb{N}$. Then the Grassmann manifold $G_n(\mathbb{C}^{n+m})$ consists of all n-dimensional subspaces of $\mathbb{C}^{n+m}$. Thus for example $G_1(\mathbb{C}^{m+1})$ is the m-dimensional complex projective space $P^m(\mathbb{C})$. Let $\mathbb{C}_{reg}^{n\times(n+m)}$ be the space of all complex n x (n+m) matrices of rank n. Let $GL_n(\mathbb{C})$ act on this space by multiplication on the left. Then the quotient space $\mathbb{C}_{reg}^{n\times(n+m)}/GL_n(\mathbb{C})$ is $G_n(\mathbb{C}^{n+m})$. The identification is done by associating to $M \in \mathbb{C}_{reg}^{n\times(n+m)}$ the subspace of $\mathbb{C}^{n+m}$ generated by the rows of M.

$G_n(\mathbb{C}^{n+m})$ inherits a natural holomorphic manifold structure from $\mathbb{C}^{n\times(n+m)}$. For a detailed description of $G_n(\mathbb{C}^{n+m})$ see e.g. [17] or [24].

3.2. The Classifying bundle. We define a holomorphic vectorbundle ξ_m over $G_n(\mathbb{C}^{n+m})$ as follows. For each x let the fibre over x, $\xi_m(x)$, be the quotient space $\mathbb{C}^{n+m}/x$. More precisely define the bundle η_n over $G_n(\mathbb{C}^{n+m})$ by

$$\eta_n = \{(x,v) \in G_n(\mathbb{C}^{n+m}) \times \mathbb{C}^{n+m} \mid v \in x\} \tag{3.3}$$

with the obvious projection $(x,v) \mapsto x$. Then ξ_m is the quotient bundle of the trivial vectorbundle $G_n(\mathbb{C}^{n+m}) \times \mathbb{C}^{n+m}$ over $G_n(\mathbb{C}^{n+m})$ by η_n. Both ξ_m and η_n can be used as universal or classifying bundles (cf. [17] for η_n as a universal bundle). Let E be an m-dimensional vectorbundle over a complex analytic manifold M. Let $\Gamma(E) = \Gamma(E,M)$ be the space of all holomorphic sections of E, i.e. the space of all holomorphic maps $s: M \to E$ such that $ps = id$, where $p: E \to M$ is the bundle projection. The universality, or classifying, property of ξ_m in the setting of complex analytic manifolds now takes the following form. Suppose $V \subset \Gamma(E)$ is an (n+m)-dimensional subspace such that for each $x \in M$ the vectors $s(x)$, $s \in V$ span $E(x)$, the fibre of E over x.

Now identify $V \simeq \underline{\underline{C}}^{n+m}$ and associate to $x \in M$ the point of $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ represented by Ker $(V \to E(x))$. This gives a holomorphic map $\Psi_E: M \to \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ such that the pullback of ξ_m by means of Ψ_E is isomorphic to E, $\Psi_E^! \xi_m \simeq E$. It is universality properties such as this one which account for the importance of the bundles ξ_m and η_n in differential and algebraic topology [17], algebraic geometry and also system and control theory (cf. [23,24] and the references therein for the last mentioned).

The bundle ξ_m has a number of obvious holomorphic sections, viz. the sections defined by $\varepsilon_i(x) = e_i$ mod x where e_i is the i-th standard basis vector of $\underline{\underline{C}}^{n+m}$, i = 1, ..., n+m. And, as a matter of fact, it is not difficult to show that $\Gamma(\xi_m, \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m}))$ is (n+m)-dimensional and that the $\varepsilon_1,\ldots,\varepsilon_{n+m}$ form a basis for this space of holomorphic sections.

4. SCHUBERT CELLS.

4.1. Schubert Cells. Consider again the Grassmann manifold $\underline{\underline{G}}_n(\underline{\underline{C}}^{m+n})$. Let $\underline{A} = (A_1,\ldots,A_n)$ be a sequence of n-subspaces of $\underline{\underline{C}}^{n+m}$ such that $0 \neq A_1 \subset A_2 \subset \cdots \subset A_n$ with each containment strict. To each such sequence $\underline{A}$ we associate the closed subset

$$(4.2) \qquad SC(\underline{A}) = \{x \in \underline{\underline{G}}_n(\underline{\underline{C}}^{m+n}) \mid \dim(x \cap A_i) \geq i\}$$

and call it the closed Schubert-cell of the sequence $\underline{A}$. In particular if $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n \leq n+m$ is a strictly increasing sequence of natural numbers less than or equal to n+m then we define (setting $\gamma = (\gamma_1,\ldots,\gamma_n)$)

$$(4.3) \qquad SC(\gamma) = SC(\underline{\underline{C}}^{\gamma_1},\ldots,\underline{\underline{C}}^{\gamma_n})$$

where $\underline{\underline{C}}^r$ is viewed as the subspace of all vectors in $\underline{\underline{C}}^{n+m}$ whose last n + m − r coordinates are zero.

4.4. Flag Manifolds and the Bruhat Decomposition. A flag in $\underline{\underline{C}}^{n+m}$ is a sequence of subspaces $\underline{F} = F_1 \subset \cdots \subset F_{n+m} \subset \underline{\underline{C}}^{n+m}$ such

that
$\dim F_i = i$. Let $F\ell(\underline{\underline{C}}^{n+m})$ denote the analytic manifold of all flags in $\underline{\underline{C}}^{n+m}$. There is a natural holomorphic mapping $F\ell(\underline{\underline{C}}^{n+m}) \to \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ given by associating to a flag $\underline{F}$ its n-th element F_n. The flag manifold can be seen as the space of all cosets Bg, $g \in \underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$ where B is the Borel subgroup of all lower triangular matrices in $\underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$. The mapping $\underline{\underline{GL}}_{n+m}(\underline{\underline{C}}) \to F\ell(\underline{\underline{C}}^{n+m})$ associates to a matrix g the flag $\underline{F}(g)$ whose i-th element is the subspace of $\underline{\underline{C}}^{n+m}$ spanned by the first i row vectors of g.

Now view S_{n+m}, the symmetric group on n + m letters as a subgroup of $\underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$ by letting it permute the basis vectors $(\sigma(e_i) = e_{\sigma(i)})$. Then in $\underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$ we have the so-called Bruhat decomposition.

(4.5) $$\underline{\underline{GL}}_{n+m}(\underline{\underline{C}}) = \bigcup_{\sigma} B\,\sigma\,B \quad \text{(disjoint union)}$$

Where σ runs through the Weyl group S_{n+m} of $\underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$. An analogous decomposition holds in a considerable more general setting (reductive groups, cf. [25], section 28).

4.6. The Bruhat order (also sometimes called Bernstein-Gelfand-Gelfand, or BGG order).

The closure of a double coset $B\,\sigma\,B$ is necessarily a union of other double cosets (by continuity). This defines an ordering on the Weyl group S_{n+m} defined by

(4.7) $$\sigma > \tau \leftrightarrow \overline{B\sigma B} \supset B\tau B$$

This ordering plays a considerable role in the study of cohomology of flag spaces [1] and also in the theory of highest weight representations [27,26].

Let H be the subgroup of $\underline{\underline{G}}_{n+m}(\underline{\underline{C}})$ consisting of all block lower triangular matrices of the form $\begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$,

$S_{11} \in \mathbf{GL}_n(\mathbf{C})$, $S_{22} \in \mathbf{GL}_m(\mathbf{C})$, S_{21} an arbitrary $m \times n$ matrix. Then, using the remarks made in subsection 4.4 above, one sees that $\mathbf{G}_n(\mathbf{C}^{n+m})$ is the coset space $\{Hg \mid g \in \mathbf{GL}_{n+m}(\mathbf{C})\}$. Now let $\sigma \in S_{n+m}$ and let $\gamma_1 < \ldots < \gamma_n$ be the n natural numbers in increasing order determined by $\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$, $i = 1, \ldots, n$. Then one easily sees that the image of $B\sigma B$ under $\mathbf{GL}_{n+m}(\mathbf{C}) \to \mathbf{G}_n(\mathbf{C}^{n+m})$, i.e. the set of all spaces spanned by matrices of the form $h\,\sigma\, b$, $h \in H$, $b \in B$, is the open Schubert cell of all elements in $\mathbf{G}_n(\mathbf{C}^{n+m})$ spanned by the rows of a matrix of the form

$$\begin{array}{ccccccccccccc}
* & \ldots & * & 0 & \ldots & 0 & 0 & & \ldots & & 0 \\
* & \ldots & * & * & \ldots & * & 0 & & \ldots & & 0 \\
 & & & & & & & & & & \\
* & \ldots & * & * & \ldots & * & * & \ldots & * & 0 \ldots & 0 \\
 & & \text{column } \gamma_1 & & & \text{column } \gamma_2 & & & & &
\end{array}$$

where the last * in each row is nonzero. The closure of this open Schubert-cell is the Schubert-cell $SC(\gamma)$ defined in (4.3) above.

One easily checks that

$$\text{(4.8)} \qquad SC(\mu) \subset SC(\gamma) \leftrightarrow \mu_i \leq \gamma_i, \; i = 1, \ldots, n$$

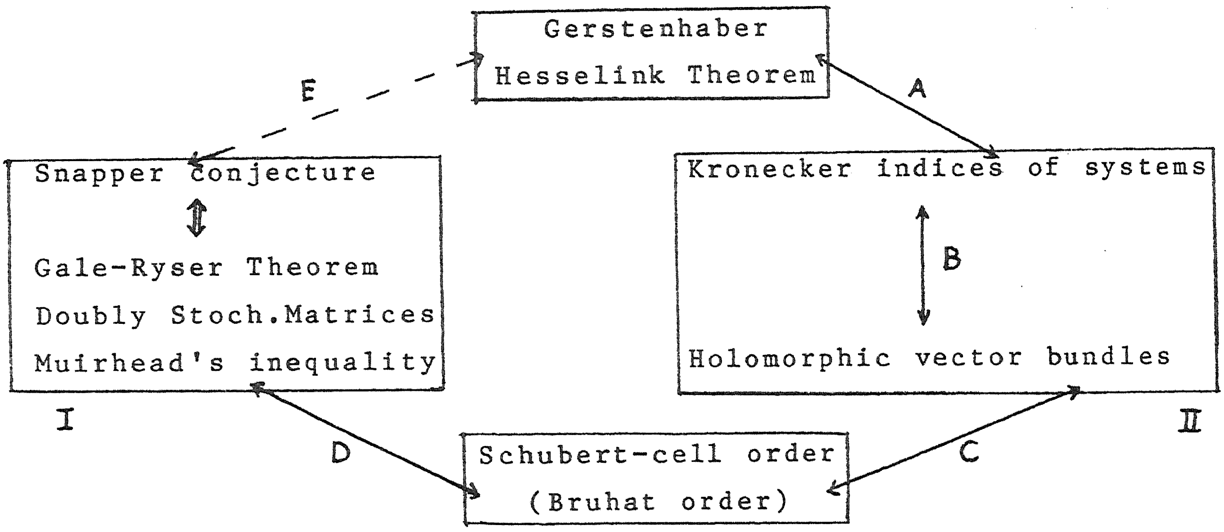
and this order on the Schubert cells $SC(\gamma)$, or the equivalent ordering on n-tuples of natural numbers, is therefore a quotient of the Bruhat order on the Weyl group S_{n+m}. It is the induced order on the set of cosets $(S_n \times S_m)\sigma$, $\sigma \in S_{n+m}$. (Obviously if $\tau \in S_n \times S_m$, then $\tau\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$ if $\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$).

(And inversely the Bruhat order is determined by the associated orders of Schubert cells in the sense that $\sigma > \tau$ in S_n iff for all $k = 1, \ldots, n-1$ we have for the associated Schubert cells in $\mathbf{G}_k(\mathbf{C}^n)$ that $SC(\sigma) \supset SC(\tau)$; this is a rather efficient way of calculating the Bruhat order on the Weyl group S_n).

5. INTERRELATIONS.

Now that we have defined the concepts we need we can start to describe some interrelations between the various manifestations of the specialization order we discussed in section 2 above.

5.1. Overview of the Various Relations. A schematic overview of the various interconnections is given by the following diagram. In this diagram we have put together in boxes the manifestations which are more or less known to be intimately related and have explicitly indicated the new relations to be discussed in detail below.



5.2. On the various Relations. The manifestations of the specialization order in box I are wellknown to be intimately related [2,5,11,13,19]. In particular, cf. [5] for the relations between doubly stochastic matrices, Muirheads inequality and the specialization order, which brings in also the marriage theorem and the Birkhoff-v. Neumann theorem that every doubly stochastic matrix is a convex linear combination of permutation matrices. For the relations of the Gale-Ryser theorem with the more or less combinatorial entities just mentioned cf [13,19] and also [2] which also contains lattice theoretic information on the partially ordered set of partitions with the specialization order.

Besides the Snapper conjecture (i.e. the Snapper, Liebler-Vitale, Lam, Young theorem) the Ruch-Schönhofer theorem [18], cf. also [21] also belongs in box I. This theorem states that $\langle \rho(\kappa), \bar{\rho}(\mu) \rangle = 1$ if and only if $\kappa > \mu^*$ where $\langle , \rangle$ denotes the usual inner product (which counts how many irreducible representations there are in common), and where $\bar{\rho}(\mu)$ is the representation of S_n obtained by inducing up the alternating representation of the Young subgroup S_μ. One way to link this theorem with the Gale-Ryser theorem is via Mackey's intertwining number theorem [29, 44] and Coleman's characterization [28] of double cosets of Young subgroups, cf. [11]. Another way goes via a beautiful formula of Snapper which we now explain (in a somehwat simplified case). Let $X = \{1,2,\dots,n\}$ with S_n acting on it in the natural way. Let Y be a finite set. A <u>weight</u> on Y is simply a function $w\colon Y \to \mathbb{N} \cup \{0\}$. Given a function $f\colon X \to Y$ its weight $w(f)$ is defined by $w(f)(y) = \# f^{-1}(y)$, where $\#$ denotes cardinality. For each weight w on Y let $I(w) = \{f\colon X \to Y \mid w(f) = w\}$. Now S_n acts on Y^X the space of functions from X to Y by $\sigma(f)(x) = f(\sigma^{-1}(x))$ and $I(w)$ is obviously invariant under this action. This associates a permutation representation $\rho(w)$ with each weight w on Y. Now consider two finite sets Y_1 and Y_2 with weights w_1 and w_2. Let $Y_1 \times Y_2$ be the product and π_1, π_2 the natural projections on Y_1 and Y_2. Define $M(w_1,w_2)$ as the set of all weights w on $Y_1 \times Y_2$ such that $w_i(y_i) = w(\pi_i^{-1}(y_i))$ for all $y_i \in Y_i$, $i = 1,2$. Finally let $M(w_1,w_2)$ be the sum of the characters belonging to the weights $w \in M(w_1,w_2)$. Then Snapper's formula says

$$\langle M(w_1,w_2), \chi \rangle = \langle \rho(w_1)\rho(w_2), \chi \rangle \tag{5.3}$$

for all characters χ. To connect this result with statements on integral matrices, it remains to note that $\langle M(w_1,w_2), 1 \rangle$ is the number of integral matrices with row sums w_1 and column sums w_2 and to prove that $\langle M(w_1,w_2),\delta\rangle$ is the number of (0,1)-matrices with row sums w_1 and column sums w_2. Here δ is the alternating character of S_n.

Relation A in the diagram is essentially established by giving two virtually identical (but dual) proofs of the theorems, and these results can then be used to give natural continuous isomorphisms between feedback orbits of systems and similarity orbits of nilpotent matrices. More details are in section 7 below. For connection B one associates to a system $\Sigma \in L^{cr}_{m,n}$ a vector bundle $E(\Sigma)$ of dimension m over $\underline{\underline{P}}^1(\underline{\underline{C}})$. The construction used is a modification of the one in [15], cf. section 8 below. It has the advantage that one sees immediately that $\kappa(\Sigma) = \kappa(E(\Sigma))$. For connection C one uses the classifying morphism Ψ_E: $\underline{\underline{P}}^1(\underline{\underline{C}}) \to \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ attached to a positive bundle E over $\underline{\underline{P}}^1(\underline{\underline{C}})$ (cf. section 3.2 above). It turns out that the invariants of E can be recovered from Ψ_E by considering the dimensions of the spaces $A_1, \ldots, A_n$ such that $\mathrm{Im}\Psi_E \subset SC(\underline{A})$, cf. section 9 below. To establish a link between representations of S_{n+m} and Schubert-cells we construct a family of representations of S_{n+m} parametrized by $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$, which can be used to give a deformation type proof of the Snapper conjecture (in the Liebler-Vitale form) (cf. section 12. below). This is not the shortest proof but it contains in it a purely elementary proof which uses no representations theory at all [7]. Combining the links A, C, D gives of course a link from the Gerstenhaber-Hesselink theorem to the Snapper conjecture, albeit a tenuous one. However, there is also a very direct link, due to Kraft [12], cf. section 6. below, and this gives yet another proof of the Snapper conjecture.

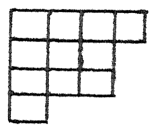
One possible approach to the Snapper conjecture is, of course, via Young's rule (discussed below in section 6), which states that the irreducible representation $[\kappa]$ occurs in $\rho(\lambda)$ with a multiplicity equal to the number of semistandard κ-tableaux of type λ. This can be made the basis of a proof and gives yet another link between the Snapper, Liebler-Vitale, Lam, Young theorem and the Gerstenhaber-Hesselink theorem. Both can be seen as consequences of the statement that there exists a semistandard λ-tableau of type μ iff $\lambda < \mu$, cf. section 7.6 below.

Finally let us remark that the proof of the increasing mixing character theorem for thermodynamic processes of Ruch and Mead follows readily from the theorem about doubly stochastic matrices described in 2.3 above.

6. YOUNG'S RULE, THE SPECIALIZATION ORDER AND NILPOTENT MATRICES.

6.1. Young Diagrams and Semistandard Tableaux. Let $\kappa = (\kappa_1, \ldots, \kappa_m)$ be a partition of n. As usual we picture κ as a Young diagram; that is an array of n boxes arranged in m rows with κ_i boxes in row i, as in the following example

(6.2) $\kappa = (4,3,3,2)$



Let $\lambda = (\lambda_1,\ldots,\lambda_s)$ be another partition of n. Then a semistandard κ-tableaux of type λ is the Young diagram of κ with the boxes labelled by the integers 1, ..., s such that i occurs λ_i times, i = 1, ..., s and such that the labels are nondecreasing in each row of the diagram and strictly increasing along each column. An example of a (5,3,2)-tableaux of type (4,2,2,2) is

$$(6.3) \qquad \begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & & \\ 3 & 4 & & & \end{array}$$

We shall use $K(\kappa,\lambda)$ to denote the number of different semistandard κ-tableaux of type λ; these numbers are sometimes called Kostka numbers.

6.4. Young's Rule. Let $[\rho]$ denote the irreducible representation associated to the partition ρ. Then Young's rule (cf. [30]) says that

6.5. Theorem. Let κ and λ be partitions of n. Then the number of times that the irreducible representation $[\lambda]$ occurs in the permutation representation $\rho(\kappa)$ is equal to the number $K(\lambda,\kappa)$ of semistandard λ-tableaux of type κ.

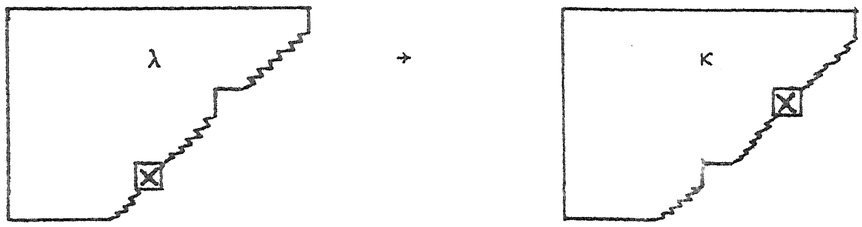
6.6. The Specialization order and Semistandard Tableaux.

The implication $\kappa > \lambda \rightarrow \rho(\lambda)$ is a direct summand of $\rho(\kappa)$ follows easily from this. First, however, we state a lemma which is standard and seemingly unavoidable when dealing with the specialization order. Its proof is easy.

6.7. Lemma. Let $\lambda = (\lambda_1,\ldots,\lambda_m)$ and $\kappa = (\kappa_1,\ldots,\kappa_m)$ be two partitions of n and suppose that $\lambda > \kappa$ and $(\lambda > \mu > \kappa) \Rightarrow (\lambda = \mu \text{ or } \mu = \kappa)$ for all partitions μ. Then there are

an i and a j, $i < j$ such
that $\kappa_i = \lambda_i + 1$, $\lambda_i < \lambda_{i-1}$, $\kappa_j = \lambda_j - 1$, $\lambda_j > \lambda_{j+1}$,
$\kappa_s = \lambda_s$, $s \neq i, j$.

Pictorially the situation looks as follows



That is a box in row j which can be removed without upsetting $\#(\text{row } j) \geq \#(\text{row } j+1)$ (which means that we must have had $\lambda_j > \lambda_{j+1}$) is moved to a higher row i which is such that it can receive it without upsetting $\#(\text{row } i) \leq \#(\text{row } i-1)$(which means that we must have had $\lambda_i < \lambda_{i-1}$). We will say that λ covers κ. Of course not all transformations of the type described above result in a pair λ, κ such that there is no μ strictly between λ and κ.

6.8. Lemma. Let λ and κ be two partitions of n and suppose that there exists a semistandard λ-tableaux of type κ. Then $\kappa > \lambda$.

Proof. In a semistandard λ-tableaux of type κ all labels i must occur in the first i rows (because the labels in the columns must be strictly increasing). The number of labels j with $j \leq i$ is $\kappa_1 + \dots + \kappa_i$ and the number of places available in the first i rows is $\lambda_1 + \dots + \lambda_i$. Hence $\lambda_1 + \dots + \lambda_i \geq \kappa_1 + \dots + \kappa_i$ for all i so that $\lambda < \kappa$.

6.9. The Implication $[\kappa]$ occurs in $\rho(\lambda) \Rightarrow \kappa < \lambda$.

Now suppose that $[\kappa]$ occurs in $\rho(\lambda)$. Then there is semistandard κ-tableaux of type λ by Youngs rule so that $\kappa < \lambda$ by lemma 6.8.

This implies, of course, that: $\rho(\kappa)$ is a subrepresentation of $\rho(\lambda) \rightarrow (\kappa < \lambda)$. Because there is obviously a semistandard κ-tableaux of type κ (in fact precisely one).

6.10. The Implication $\kappa < \lambda \Rightarrow \rho(\kappa)$ is a subrepresentation of

$\rho(\lambda)$. To obtain this implication it suffices by Young's rule to show that the Kostka numbers satisfy $K(\mu,\kappa) \leq K(\mu,\lambda)$ if $\kappa < \lambda$ for all μ. To see this it is convenient to define $K(\mu,\nu)$ as the number of semistandard μ-tableaux of type ν for any sequence of nonnegative integers $\nu = (\nu_1,\ldots,\nu_s)$ such that $|\nu| = n$. Let $\bar{\nu} = (\bar{\nu}_1,\ldots,\bar{\nu}_s)$ denote the rearrangement of the ν_i such that $\bar{\nu}_1 \geq \bar{\nu}_2 \geq \cdots \geq \bar{\nu}_s$. Then $K(\mu,\nu) = K(\mu,\bar{\nu})$ and from this (non trivial) fact combined with lemma 6.7 it is easy to see that $K(\mu,\kappa) \leq K(\mu,\lambda)$ if $\kappa < \lambda$. (Assume λ covers κ and rearrange both so that the two changing entries are the first two). We owe these remarks (indirectly) to A. Lascoux.

6.11. <u>Nilpotent Matrices and Representations [12]</u>. Let N_κ be the set of nilpotent matrices labelled by the partition κ, cf. 2.11 above. Let $\bar{N}_\kappa$ be its closure and let C be the set of diagonal matrices. Now take the scheme theoretic intersection of the closed subvarieties $\bar{N}_\kappa$ and C of the scheme of n x n matrices over $\mathbb{C}$. This is a finite $\mathbb{C}$-algebra with an obvious S_n-action. This turns out to be the permutation representation $\rho(\kappa)$ and using results from [40] a proof of the Snapper, Liebler-Vitale, Lam, Young theorem can be deduced. One very nice thing about this construction is that it also makes sense for the other classical simple Lie algebras and their Weyl groups. There are also relations with the socalled Springer representations of Weyl groups, [43,41].

7. NILPOTENT MATRICES AND SYSTEMS.

As was remarked in section 5 above the connection A in the diagram above essentially consists of an almost identical proof of the two theorems. We start with a proof of the Gerstenhaber-Hesselink theorem. The first ingredient which we shall also need for the feedback orbits theorem is the following elementary remark on ranks of matrices.

7.1. <u>Lemma</u>. Let A(t) be a family of matrices depending polynomially on a complex or real parameter t. Suppose that rank $A(t) \leq$ rank $A(t_o)$ for all t. Then rank $A(t)$ = rank $A(t_o)$ for all but finitely many t.

This follows immediately from the fact that a polynomial in t has only finitely many zeros.

Let A be a nilpotent matrix. Then of course the similarity type of A is determined by the sequence of numbers.

$$n_i = \dim \operatorname{Ker} A^i.$$

The numbers $e_i = n_{i+1} - n_i$ form a partition of n and are dual to the partition formed by the sizes of the Jordan blocks.

The key to a simple proof of the Gerstenhaber-Hesselink theorem is in exploiting this filtration instead of the Jordan form. The following elem entary lemma is the key observation.

7.2. <u>Lemma</u>. Let A be a nilpotent n x n matrix and let F be such that

$$(7.3) \qquad F(\operatorname{Ker} A^i) \subset \operatorname{Ker} A^{i-1}, \; i = 1,2,\ldots, n.$$

Then $tA + (1-t)F$ is similar to A for all but finitely many t.

<u>Proof</u>. We show first that

$$(7.4) \qquad \operatorname{Ker}(tA+(1-t)F)^i \supset \operatorname{Ker} A^i$$

for all t. Indeed from (7.3) with i = 1 we see that F(Ker A) = 0 and it follows that (tA+(1-t)F)(Ker A) = 0 which proves (7.4) for i = 1. Assume with induction that (7.4) holds for all i < s. Then

$$(tA+(1-t)F)^s \operatorname{Ker} A^s = (tA+(1-t)F)^{s-1}(tA+(1-t)F)\operatorname{Ker} A^s \subset$$
$$\subset (tA+(1-t)F)^{s-1}\operatorname{Ker} A^{s-1} = 0$$

because $A \operatorname{Ker} A^s \subset \operatorname{Ker} A^{s-1}$ and $F(\operatorname{Ker} A^s) \subset A^{s-1}$ by (7.3). This proves (7.4). Using 7.4 we know by (7.1) that for almost all t (take t_o = 1)

$$(7.5) \qquad \operatorname{rank}(tA+(1-t)F)^i = \operatorname{rank}(A^i)$$

and because $tA + (1-t)F$ and A are both nilpotent it then follows that the conclusion of the lemma is satisfied.

Now let A be a nilpotent matrix. We say that A is of type $\kappa = (\kappa_1, \dots, \kappa_m)$ if the Jordan normal form of A consists of m Jordan blocks of sizes $\kappa_i \times \kappa_i$, $i = 1, \dots, m$. E.g. A is of type $(4,2)$ iff its Jordan form is

$$\left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Consider $\operatorname{Ker} A$, $\operatorname{Ker} A^2, \dots, \operatorname{Ker} A^n$. Then A is of type κ iff $\dim(\operatorname{Ker} A^i) = \kappa_1^* + \dots + \kappa_i^*$, $i = 1, \dots, n$ where κ^* is the dual partition of κ. Thus in the example the kernel spaces $\operatorname{Ker} A^i$ are spanned by the basis vectors $\{e_1, e_5\}$, $\{e_1, e_2, e_5, e_6\}$, $\{e_1, e_2, e_3, e_5, e_6\}$, $\{e_1, e_2, e_3, e_4, e_5, e_6\}$.

7.6. <u>Semistandard Tableaux and Nilpotent Matrices</u>. Let A be a nilpotent matrix of type κ. Let μ be another partition of n and suppose that there is a μ^*-tableaux of type κ^*. Then there is nilpotent matrix F of type μ such that $F(\operatorname{ker} A^i) \subset \operatorname{Ker} A^{i-1}$ for all i. This matrix F is constructed as follows. First choose a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ such that the first $\kappa_1^* + \dots + \kappa_i^*$ elements of this basis form a basis for $\operatorname{Ker} A^i$, $i = 1, \dots, n$. Now consider a semistandard μ^*- tableaux T of type κ^*. Take the Young diagram of μ^* and lable the boxes of it by the basis vectors $e_1, \dots, e_n$ in such a way that the boxes marked with i in the semistandard tableaux T are filled with the basis vectors

$$e_{\kappa_1^* + \dots + \kappa_{i-1}^* + 1}, \dots, e_{\kappa_1^* + \dots + \kappa_i^*}.$$

This can be done because T is of type κ^* so that there are precisely κ_i^* boxes labelled i in T. Call this new μ^*- tableaux T'.

Now define F by $F(e_i) = e_j$, if e_j is just above e_i in the μ^*-tableaux T' and $F(e_j) = 0$ if e_j occurs in the first row of T'. Then obviously $\dim \operatorname{Ker} F^i = \mu_1^* + \ldots + \mu_i^*$ so that F is of type μ and $F(\operatorname{Ker} A^i) \subset \operatorname{Ker} A^{i-1}$ because the μ^*-tableaux T was semistandard which implies that the labels are strictly increasing along columns.

An example may illustrate things. Let $\kappa^* = (2,2,2)$, $\mu^* = (4,1,1)$. A μ^*-tableaux of type κ^* is then

$$\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & & & \\ 3 & & & \end{array}$$

Inserting $e_1, \ldots, e_6$ in such a way that e_1, e_2 are put into boxes marked with 1, e_3, e_4 in boxes marked with 2 and e_5, e_6 in boxes marked with 3 gives for example

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_5 \\ e_4 & & & \\ e_6 & & & \end{array}$$

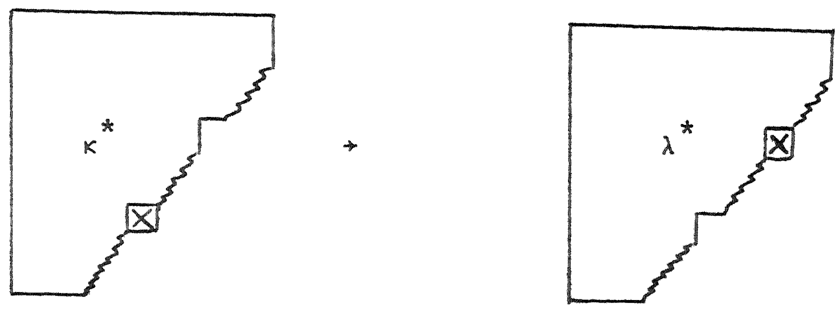
which yields an F defined by $F(e_6) = e_4$, $F(e_4) = e_1$,

$$F(e_1) = F(e_2) = F(e_3) = F(e_5) = 0.$$

7.7. Proof of the Gerstenhaber-Hesselink Theorem. (Cf. 2.11 above for a statement of the theorem).

The implication $\rightarrow$ is immediate. Indeed if $A_t \in O(\kappa)$ converges to $A_o \in O(\lambda)$ as $t \rightarrow 0$ then $\operatorname{rank}(A_t^i) \geq \operatorname{rank}(A_o^i)$ for small t and all $i = 1, \ldots, n$. Hence $\dim(\operatorname{Ker} A_t^i) \leq \dim(\operatorname{Ker} A_o^i)$ for small t so that $\kappa_1^* + \ldots + \kappa_i^* \leq \lambda_1^* + \ldots + \lambda_i^*$ for all i, hence $\kappa^* > \lambda^*$ and $\kappa < \lambda$. To prove the opposite implication it suffices to show this in case that κ is obtained from λ by a transformation of the type described in lemma 6.7. (Because if $\overline{O(\kappa)} \supset O(\lambda)$ and $\overline{O(\lambda)} \supset O(\mu)$, then $\overline{O(\kappa)} \supset \overline{O(\lambda)}$, and hence $\overline{O(\kappa)} \supset O(\mu)$). Then λ^* is obtained from κ^* by a similar transformation.

Recall the picture



Now take the unique semistandard κ^*-tableau of type κ^* and transform the box ⊠ together with its lable. The result is obviously a semistandard λ^*-tableau of type κ^*. Let A be a nilpotent matrix of type κ. Then by the construction of 7.6 above there is an F of type λ such that $F(\operatorname{Ker} A^i) \subset \operatorname{Ker} A^{i-1}$. Then $tA + (1-t)F$ is similar to A for almost all t by lemma 7.2 so that there is a sequence of A's in $O(\kappa)$ converging to $F \in O(\lambda)$, proving that $O(\lambda) \subset \overline{O(\kappa)}$, which finishes the proof of the theorem.

Incidentally it is quite easy to describe F directly without resorting to semistandard tableaux [8].

7.10. <u>Kronecker Indices of Systems</u>. Let $(A,B) \in L^{cr}_{m,n}$ be a completely reachable pair of matrices. Recall that this means the matrix $R(A,B) = (B\ AB\ \ldots\ A^nB)$ has rank n. Recall that the Kronecker indices $\kappa(A,B)$ of the pair (A,B) are defined as follows. Let for $i = 1, \ldots, n$

(7.11) $V_i(A,B)$ = space spanned by the column vectors of A^jB, $j = 0, \ldots, i-1$.

Let $d_i = \dim V_i(A,B)$, $e_i = d_i - d_{i-1}$, $d_o = 0$. Then $e_i \leq e_{i-1}$, $i = 1, \ldots, n-1$, and $\kappa(A,B)$ is defined as the dual partition of n

$$\kappa(A,B) = e(A,B)^* \tag{7.12}$$

where $e(A,B) = (e_1,\ldots,e_n)$.

The orbits of the feedback group (cf. 2.6 above) acting on $L^{cr}_{m,n}$ are precisely the subsets of $L^{cr}_{m,n}$ with constant $\kappa(A,B)$. Let $U(\kappa)$ be this orbit. The "degeneration of systems theorem" now says

7.13. Theorem. $\overline{U(\lambda)} \supset U(\kappa) \Leftrightarrow \lambda > \kappa$

Here follows a proof which is virtually identical with the proof of the Gerstenhaber-Hesselink theorem given above. First if $(A_t,B_t) \to (A_o,B_o)$ as $t \to 0$, $(A_t,B_t) \in U(\lambda)$, $(A_o,B_o) \in U(\kappa)$, then rank $(A_t^{i-1}B_t; \ \ldots \ ; A_tB_t;B_t) \geq \mathrm{rank}(A_o^{i-1}B_o; \ \ldots \ ;A_oB_o;B_o)$ for small t. Hence $\dim V_i(A_t,B_t) \geq \dim V_i(A_o,B_o)$ for small t. Hence $e(A_t,B_t) < e(A_o,B_o)$ for small t and $\kappa(A_t,B_t) > \kappa(A_o,B_o)$ for small t which proves the implication $\Rightarrow$.

To prove the inverse implciation it suffices to prove this in the case κ is obtained from λ by a transformation as described in lemma 6.7 (exactly as in the case of the Gerstenhaber-Hesselink theorem). Now let $(A,B) \in U(\lambda)$. Choose a basis $e_1,\ldots,e_n$ for $\mathbb{R}^n$ such that the first $\lambda_1^* + \ldots + \lambda_i^*$ elements of $e_1,\ldots,e_n$ form a basis for $V_i(A,B)$, $i = 1, \ldots, n$. Now write in the $e_1,\ldots,e_n$ in λ^* in the standard way and transform λ^* backwards to κ^*, moving $\blacksquare$ together with its label, cf the picture in section 7.7 above. E.g. if $\kappa^* = (4,3,2,2,1)$ and $\lambda^* = (4,4,2,1,1)$ then this would give

$$\begin{array}{llll} e_1 & e_2 & e_3 & e_4 \\ e_5 & e_6 & e_7 & \\ e_9 & e_{10} & & \\ e_{11} & e_8 & & \\ e_{12} & & & \end{array} \qquad\qquad \begin{array}{llll} e_1 & e_2 & e_3 & e_4 \\ e_5 & e_6 & e_7 & e_8 \\ e_9 & e_{10} & & \\ e_{11} & & & \\ e_{12} & & & \end{array}$$

The vectors in the first i rows of λ^* are a basis for $V_i(A,B)$. Now define a pair (F,G) in terms of κ^* as follows. G consists of the vectors in the first row of κ^* (plus a zero vector in case $\kappa_1^* < \lambda_1^*$), and F is defined by $F(e_i) = e_{i'}$ if $e_{i'}$ occurs just below e_i in κ^* and $F(e_i) = 0$ otherwise. Note the similarity with the construction in 7.6. One could put this in "Young tableaux" terms too. The relevant "Young tableaux are then the inverse semistandard ones with labels strictly decreasing from left to right along rows and decreasing from top to bottom along columns. Then (F,G) has the following properties (all immediate)

(i) $(F,G) \in U(\kappa) \subset L^{cr}_{m,n}$

(ii) $V_i(F,G) \subset V_i(A,B)$

(iii) $FV_i(A,B) \subset V_{i+1}(A,B)$

(of course (ii) follows from (iii) together with $V_1(F,G) \subset V_1(A,B)$). Now consider $A_t = tA + (1-t)F$, $B_t = tB + (1-t)G$. Then

(7.14) $\quad V_i(A_t,B_t) \subset V_i(A,B) \quad$ for all t

(7.15) $\quad V_i(A_t,B_t) = V_i(A,B) \quad$ for all but finitely many t

Indeed obviously $V_1(A_t,B_t) \subset V_1(A,B)$ because of (ii) above for i = 1. Now assume that (7.14) holds for all i < r. Then

$$\begin{aligned} V_r(A_t,B_t) &= (tA+(1-t)F)V_{r-1}(A_t,B_t) + V_{r-1}(A_t,B_t) \\ &\subset tAV_{r-1}(A,B) + (1-t)FV_{r-1}(A,B) + V_{r-1}(A,B) \\ &\subset V_r(A,B) + V_r(A,B) + V_{r-1}(A,B) = V_r(A,B) \end{aligned}$$

This proves (7.14) and (7.15) follows by means of lemma 7.1 (with $t_o = 1$) because

$$\dim V_i(A_t,B_t) = \operatorname{rank}\,(A_t^{i-1}B_t;\ \ldots\ ;B_t)$$

Now $(A_t, B_t) \to (F,G) \in U(\kappa)$ as $t \to 0$ and by (7.15) (and the theorem that the orbits under the feedback group are classified by the Kronecker indices) all but finitely many of the (A_t, B_t) are feedback equivalent to (A,B). Thus $(F,G) \in U(\kappa)$ and $(F,G) \in \overline{U(\lambda)}$ proving the theorem.

7.16. <u>Remarks</u>. The two proofs are very similar (up to duality in a certain sense). This can be given more precise form as follows. For a nilpotent matrix $N \in N_n$ let $\underline{s}(N) = \{(A,B) \in L^{cr}_{m,n} | N^i A^{i-1} B = 0,\ i = 1,\dots,n\}$ and for $(A,B) \in L^{cr}_{m,n}$ let $\underline{t}(A,B) = \{N \in N_n | N^i A^{i-1} B = 0,\ i = 1,\dots,n\}$. Then using the results above one shows that

$$\underline{t}\ \underline{s}(\overline{0(\kappa)}) = \overline{0(\kappa)},\quad \underline{s}\ \underline{t}(\overline{U(\kappa)}) = \overline{U(\kappa)}$$

so that $\underline{t}$ and $\underline{s}$ set up a bijective correspondence between the closures of orbits in the two cases and hence induce a bijective order preserving correspondence between the sets of orbits themselves.

8. <u>VECTORBUNDLES AND SYSTEMS</u>.

This section contains a modified version of the construction of Hermann-Martin [15] associating a vectorbundle $E(\Sigma)$ over the Riemann sphere $\underline{\underline{P}}^1(\underline{\underline{C}})$ to every $\Sigma = (A,B) \in L^{cr}_{m,n}$. This version makes it almost trivial to see that $E(\Sigma)$ splits as a direct sum of line bundles $L(\kappa_i)$, $i = 1, \dots, m$ where $\kappa = (\kappa_1, \dots, \kappa_m)$ is the set of Kronecker indices of Σ.

The first thing needed is some more information on the universal bundle ξ_m.

8.1. <u>On the Universal Bundle</u> $\xi_m \to \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$. Let V be a complex $n+m$ dimensional vector space and $V^* = \mathrm{Hom}_{\underline{\underline{C}}}(V, \underline{\underline{C}})$ its dual vector space. Given $x \in \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ define $x^* = \{y \in V^* | \langle y,v\rangle = 0$ $= 0$ for all $x \in V\}$ where $\langle , \rangle$ denotes the usual pairing $V^* \times V \to \underline{\underline{C}}$. Then x^* is m-dimensional and $x \to x^*$

defines a holomorphic isomorphism

(8.2) $$d\colon \underline{\underline{G}}_n(V) \to \underline{\underline{G}}_m(V^*).$$

Now $v \in V/x$ defines a unique homomorphism $v^T\colon x^* \to \underline{\underline{C}}$ as follows:

$v^T(a) = \langle a, \tilde{v}\rangle$ for all $a \in x^*$, where $\tilde{v} \in V$ is any representative of v. This is well defined because $\langle a,b\rangle = 0$ for all $b \in x$ if $a \in x^*$. This defines an isomorphism between the pullback $(d^{-1})^!\xi_m$ and the dual of the subbundle η_m on $G_m(V^*)$ defined by

$$\eta_m = \{(x^*,w) \in \underline{\underline{G}}_m(V^*) \times V^* \mid w \in x^*\}$$

It follows that ξ_m is a subbundle of an $n + m$ dimensional trivial bundle $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m}) \times \underline{\underline{C}}^{n+m}$. Because $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ is projective (compact) all holomorphic maps $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m}) \to \underline{\underline{C}}$ are constant so that the space of holomorphic sections $\Gamma(\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m}) \times \underline{\underline{C}}^{n+m}, \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m}))$ is of dimension $n + m$. As a subbundle of a trivial $(n+m)$-dimensional bundle ξ_m can therefore have at most $(n+m)$ linearly independent holomorphic sections. But we have already found $(n+m)$ linearly independent sections viz. the $\varepsilon_1, \ldots, \varepsilon_{n+m}$ defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i-th standard basis vector of $\underline{\underline{C}}^{n+m}$. Therefore

(8.3) $$\dim \Gamma(\xi_m, \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})) = n + m$$

Now let $A \in \underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$. Then A induces a holomorphic automorphism A^* of $\underline{\underline{G}}_m(\underline{\underline{C}}^{n+m})$ defined by $x \mapsto Ax$. Then, of course, there is an induced isomorphism $A^{-1}\colon \underline{\underline{C}}^{n+m}/Ax \to \underline{\underline{C}}^{n+m}/x$ which for varying x induces an isomorphism

(8.4) $$A_*^!\xi_m \simeq \xi_m, \quad A \in \underline{\underline{GL}}_{n+m}(\underline{\underline{C}})$$

8.5. <u>The Line Bundles</u> $L(i)$ <u>over</u> $\underline{\underline{P}}^1(\underline{\underline{C}})$. The Riemann sphere $\underline{\underline{P}}^1(\underline{\underline{C}}) = S^2$ can be obtained by gluing together two copies of $\underline{\underline{C}}$ along the open subsets $\underline{\underline{C}}\setminus\{0\}$ by the isomorphism

$$\mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}, \quad s \mapsto t = s^{-1}$$

A line bundle over $\mathbb{P}^1(\mathbb{C})$ is then obtained by giving a holomorphic isomorphism $\mathbb{C}\setminus\{0\}\times\mathbb{C} \to \mathbb{C}\setminus\{0\}\times\mathbb{C}$ linear in the second variable compatible with the above isomorphism. Obviously the only possibilities are $(s,v) \to (s^{-1}, s^i v)$ for $i \in \mathbb{Z}$. This gives us the following commutative diagram identifications

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{C} \arrow[d] & \supset & \mathbb{C}\setminus\{0\}\times\mathbb{C} \arrow[rr, "{(s,v)\to(s^{-1},s^i v)}"] \arrow[d] & & \mathbb{C}\setminus\{0\}\times\mathbb{C} \arrow[d] & \subset & \mathbb{C}\times\mathbb{C} \arrow[d] \\
\mathbb{C} \arrow[u, bend left, dashed, "s_1"] & \supset & \mathbb{C}\setminus\{0\} \arrow[rr, "s \to s^{-1} = t"'] & & \mathbb{C}\setminus\{0\} & \subset & \mathbb{C} \arrow[u, bend right, dashed, "s_2"']
\end{tikzcd}$$

The corresponding holomorphic line bundle is denoted $L(-i)$. A section of $L(-i)$ consists of two holomorphic mappings s_1, s_2 of the form $s \to (s, f(s))$, $t \to (t, g(t))$ such that $s^i f(s) = g(s^{-1})$. It readily follows that $f(s)$ must be a polynomial of degree $\leqslant -i$. Thus

(8.6) $\qquad \dim \Gamma(L(i)) = 0 \qquad$ if $i < 0$

(8.7) $\qquad \dim \Gamma(L(i)) = i + 1 \qquad$ if $i \geq 0$

8.8. <u>The (modified) Hermann-Martin vectorbundle of a system.</u> Let $\Sigma = (A,B)$ be a pair of real of complex matrices of sizes $n\mathrm{x}n$ and $n\mathrm{x}m$. Then (A,B) is completely reachable (cr) iff the $n \times (n+m)$ matrix $(sI-A;B)$ is of rank n for all complex values of s. So if $\Sigma = (A,B)$ is cr one can define a holomorphic map ψ_Σ by

(8.9) $\quad \psi_\Sigma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{G}_n(\mathbb{C}^{n+m}), \quad s \mapsto \mathrm{Row}(sI-A;B), \quad \infty \mapsto \mathrm{Row}(I;0)$

where $\mathrm{Row}(M)$ for an $n \times (m+n)$ matrix M denotes the subspace of $\mathbb{C}^{n+m}$ generated by the rows of M. The vectorbundle $E(\Sigma)$ over $\mathbb{P}^1(\mathbb{C})$ is now defined by

(8.10) $\qquad E(\Sigma) = \psi_\Sigma^! \xi_m$

8.11. <u>Proposition.</u> $E(\Sigma)$ depends only on the feedback orbit of Σ.

Indeed one easily checks that $\Sigma = (A,B)$, $\Sigma' = (A',B') \in L^{cr}_{m,n}$ are feedback equivalent (cf. 2.6 above) iff there are constant invertible matrices P,Q such that $P(sI-A;B)Q = (sI-A';B')$. Now $\mathrm{Row}(PM) = \mathrm{Row}(M)$ and postmultiplication with Q changes ψ_Σ to $Q_* \circ \psi_\Sigma$ and $E(\Sigma') = \psi^!_{\Sigma'}(\xi_m) = \psi^!_\Sigma(Q^!_* \xi_m) \simeq (\psi^!_\Sigma(\xi_m) = E(\Sigma)$ by 8.4 above, proving the proposition.

Thus to determine $E(\Sigma)$ we can assume that $\Sigma = (A,B)$ is in Brunowsky canonical form which means that A,B takes the form

$$\left(\begin{array}{c|c|c}
\begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix} & 0 & 0 \\ \hline
0 & \begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix} & 0 \\ \hline
0 & 0 & \begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix}
\end{array}\right)
\left(\begin{array}{ccc}
0 & 0 & 0 \\ & \vdots & \\ 0 & & \\ 1 & 0 & 0 \\ \hline
0 & 0 & 0 \\ & \vdots & \\ & 0 & \\ 0 & 1 & 0 \\ \hline
0 & 0 & 0 \\ & \vdots & \\ & & 0 \\ 0 & 0 & 1
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\}\kappa_1 \\
\left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\}\kappa_2 \\
\left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\}\kappa_3
\end{array}$$

in case $m = 3$, where $(\kappa_1, \kappa_2, \kappa_3) = \kappa(A,B)$ are the Kronecker indices of $\Sigma = (A,B)$. (The general case is evident from this example); every $(A,B) \in U(\kappa)$ is feedback equivalent to such a pair [31,10]. The matrix $(sI-A;B)$ is now easily written down, and one observes that for all $s \neq 0, \infty$, $e_1 \equiv e_2 \equiv \ldots \equiv e_{\kappa_1} \equiv e_{n+1}$ mod $\mathrm{Row}(sI-A;B)$, i.e. mod $\psi_\Sigma(s)$ and for $s = 0$,

$e_2 \equiv \ldots \equiv e_{\kappa_1} \equiv e_{n+1} \equiv 0$ but $e_1 \neq 0$ and for $s = \infty$,

$e_1 \equiv \ldots \equiv e_{\kappa_1} \equiv 0$ and $e_{n+1} \neq 0$. It follows that the vectors $\varepsilon_1(\psi_\Sigma(s)), \ldots, \varepsilon_{\kappa_1}(\psi_\Sigma(s)), \varepsilon_{n+1}(\psi_\Sigma(s))$ span a one-dimensional subspace of $\xi_m(\psi_\Sigma(s))$ for all s so that $E(\Sigma) \simeq \psi_\Sigma^! \xi_m$ contains a line bundle L_1 which admits at least $\kappa_1 + 1$ linearly independent holomorphic sections viz. the $\varepsilon_1, \ldots, \varepsilon_{\kappa_1}, \varepsilon_{n+1}$. Similar relations hold for $\varepsilon_{\kappa_1+\ldots+\kappa_{i-1}+1}, \ldots, \varepsilon_{\kappa_1+\ldots+\kappa_i}, \varepsilon_{n+i}$ for all $i = 1, \ldots, m$ giving us subbundles L_i, $i = 1, \ldots, m$ which admit at least $\kappa_i + 1$ linearly independent holomorphic sections. This exhausts the ε_i and because the $\varepsilon_1(x), \ldots, \varepsilon_{n+m}(x)$ span $\xi_m(x)$ for all $x \in G_n(C^{n+m})$ it follows that $E(\Sigma) = \oplus L_i$. As the pullback of the bundle ξ_m, $E(\Sigma)$ itself is a subbundle of an $(n+m)$-dimensional trivial bundle. Because $P^1(C)$ is projective it follows (as before) that $E(\Sigma)$ has at most $n + m$ linearly independent holomorphic sections. But L_i has at least $\kappa_i + 1$ linearly independent sections, hence $\oplus L_i$ has at least $\Sigma(\kappa_i+1) = n + m$ linearly independent sections which proves that L_i has precisely $\kappa_i + 1$ linearly independent sections and hence identifies L_i as the bundle $L(\kappa_i)$ described above in (8.5). We have reproved the theorem of Hermann and Martin [15].

8.12. Theorem. Keeping the notations introduced above in (8.10) and (8.5) we have $E(\Sigma) \simeq \bigoplus_{i=1}^{m} L(\kappa_i)$. Still another proof of this theorem, using the Riemann-Roch theorem is found in Byrnes [34].

8.13. The Correspondence B . (cf. the diagram in section 5 above). The mapping $\Sigma \mapsto E(\Sigma)$ is obviously continuous. Thus the result $\overline{U(\kappa)} \supset U(\lambda) \Leftrightarrow \kappa > \lambda$ can be deduced from Shatz's theorem (cf.2.9). Inversely Shatz's theorem for positive bundles over $P^1(C)$ can be deduced from the result on feedback orbits because every positive bundle arises as an $E(\Sigma)$. By tensoring with a suitable $L(r)$, r high enough, the result is then extended to arbitrary bundles over $P^1(C)$.

9. VECTOR BUNDLES, SYSTEMS AND SCHUBERT CELLS.

9.1. Partitions and Schubert-cells. Let κ be a partition of n. To κ we associate the following increasing sequence of n numbers $\tau(\kappa)$.

$$(9.2)\qquad \tau(\kappa) = (\underbrace{2,3,\ldots,\kappa_1+1}_{\kappa_1},\underbrace{\kappa_1+3,\ldots,\kappa_1+\kappa_2+2}_{\kappa_2},\ldots,$$

$$\underbrace{\kappa_1+\ldots+\kappa_{m-1}+m+1,\ldots,\kappa_1+\ldots+\kappa_m+m)}_{\kappa_m},$$

Let $\tau_j(\kappa)$, $j = 1, \ldots, n$, be the j-th element of this sequence. It is an easy exercise to check that

$$(9.3)\qquad \kappa > \lambda \Rightarrow \tau_i(\kappa) \geq \tau_i(\lambda) \qquad \text{for all } i = 1, \ldots, n$$

Thus the specialization order is a suborder of the inclusion ordering between closed Schubert cells, because $SC(\tau) \supset SC(\tau') \Rightarrow \tau_i \geq \tau_i'$, $i = 1, \ldots, n$. And in turn, as we saw above in section 4, the Schubert-cell order is a quotient of the Bruhat order on the Weyl group S_{n+m}.

9.4. Systems and Schubert Cells. Let $(A,B) \in L^{cr}_{m,n}$ be a system and as in section 8.8 consider the associated holomorphic morphism
$\psi_\Sigma\colon \underline{\underline{P}}^1(\underline{\underline{C}}) \to \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$. Suppose that (A,B) are in Brunovsky canonical form. Then simple inspection of the matrix $(sI-A;B)$ (cf. the example below proposition 8.11) shows that $\operatorname{Im} \psi_\Sigma \subset SC(\tau(\kappa))$, where $\kappa = \kappa(A,B)$. Now let (A,B) be any system in $L^{cr}_{m,n}$. Then it is feedback equivalent to one in Brunovsky canonical form so that $(sI-A;B) = P(sI-A_o;B_o)Q$ for certain constant invertible matrices P,Q where (A_o,B_o) is a canonical pair. Premultiplication with P does not change ψ_Σ and postmultiplication with Q induces an automorphism of $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ taking Schubert-cell $SC(\tau(\kappa))$ into another Schubert-cell of the same dimension type. Thus we have shown:

9.5. Theorem. Let $\Sigma \in L^{cr}_{m,n}$, $\kappa = \kappa(\Sigma)$ and let $\psi_\Sigma\colon \underline{\underline{P}}^1(\underline{\underline{C}}) \to \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ be the Hermann-Martin morphism of Σ. Then there is a Schubert-cell $SC(A)$, $\underline{A} = (A_1,\dots,A_n)$ such that $\operatorname{Im} \psi_\Sigma \subset SC(\underline{A})$ and $\dim A_i = \tau_i(\kappa)$, where $\tau_i(\kappa)$ is defined by (9.2).

We will now show that the Schubert-cell SC($\underline{A}$) obtained in 9.5 is the smallest possible in the sense of the associated sequence of dimension numbers. We first prove a technical lemma.

9.6. Lemma. Let $\underline{X}(s)$ be the matrix, defined by a partition $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m$, $\kappa_1 + \dots + \kappa_m = n$, consisting of blocks $\underline{X}_i(s)$ where

$$\underline{X}_i(s) = \begin{pmatrix} s & -1 & & & 0 \\ & s & -1 & & 0 \\ & & \ddots & -1 & \\ 0 & 0 & & s & 1 \end{pmatrix} \qquad \kappa_i \times (\kappa_i+1)$$

and

$$\underline{X}(s) = \begin{pmatrix} \underline{X}_1(s) & & 0 \\ & \ddots & \\ 0 & & \underline{X}_1(s) \end{pmatrix} \qquad n \times (n+m)$$

Let B be an $(m+n) \times \tau$ matrix of rank τ. Then X(s)B has rank greater than or equal to $\tau - t$ for almost all s where t is the largest number such that

$$\kappa_m + \kappa_{m-1} + \dots + \kappa_{m-t+1} + t \leq \tau.$$

Proof. We first consider the case that there is only one κ, i.e., $m = 1$. We can assume that B is in column echelon form by postmultiplying by a nonsingular matrix if necessary. So B has the following form:

$$\begin{array}{c} \begin{pmatrix} 0 & \dots & & 0 \\ I_{\lambda_1} & 0 & \dots & 0 \\ x & 0 & \dots & 0 \\ 0 & I_{\lambda_2} & & 0 \end{pmatrix} \end{array} \begin{array}{l} r_1 \\ \lambda_1 \\ r_2 \\ \lambda_2 \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} x & & x & 0 & \cdots & 0 \\ & \vdots & & & \vdots & \\ 0 & \cdots & 0 & & & I_{\lambda_u} \\ x & & & & & x \end{pmatrix} & \begin{array}{l} \\ \\ \lambda_u \\ r_{u+1} \end{array} \end{array}$$

The x's stand for possibly nonzero blocks.

Write $X(s) = s\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & & 0 \\ & & \ddots & \\ & & & -1 \\ 0 & & 0 & 1 \end{pmatrix} = sA_1 + A_2$

and write $B = \begin{pmatrix} b_1 \\ \vdots \\ b_{n+1} \end{pmatrix}$ where b_i is the i-th row.

Now $X(s)B = \begin{pmatrix} sb_1 - b_2 \\ \vdots \\ sb_{n-1} - b_n \\ sb_n + b_{n+1} \end{pmatrix}$. We need to prove that X(s)B has the required

rank. Assume that B has rank τ and $\tau \leq n$. Let x be a τ vector and assume that

$$X(s)Bx = 0$$

We will show that either x = 0 or the equation only holds for finitely many values of s. We first note that

$$\begin{aligned} b_2x &= sb_1x \\ &\vdots \\ b_nx &= s^{n-1}b_1x \\ b_{n+1}x &= -s^nb_1x \end{aligned}$$

Thus if $b_1x = 0$ then $b_ix = 0$ for all x. But since B has full rank this implies that x = 0. Thus we may assume that $b_1x = 1$ and thus

that $r_1 = 0$. So we have that $x_1 = 1$, $x_2 = s$, ..., $x_{\lambda_1} = s^{\lambda_1 - 1}$. If $r_2 = 0$, B is of the form $\left(\begin{smallmatrix} I \\ x \end{smallmatrix}\tau\right)$ and the result is obvious, so we can assume $r_2 \neq 0$. Then we have

$$sb_{\lambda_1}x = b_{\lambda_1+1}x$$

so that

$$s^{\lambda_1} = b_{\lambda_1+1,1} + b_{\lambda_1+1,2}s + \dots + b_{\lambda_1+1,\lambda_1}s^{\lambda_1-1}$$

and this equation is satisfied for only finitely many s. Therefore we have shown that if there is a nonzero solution of $X(s)Bx = 0$ then $b_1x \neq 0$ and the solution can exist only for finitely many values of s. Thus in this case the rank of $X(s)B$ is equal to τ for almost all s. If B is invertible (rank of B equal to n+1) then the rank of $X(s)B$ is equal to n = rank $X(s)$ = (rank B) - 1.

Now let m be greater than or equal to two. Again put B into column echelon form and partition B in such a way that the pieces $B_1, \dots, B_m$ are still in column echelon form.

$$\begin{array}{cccccc} B_1 & 0 & \cdots & 0 & & \kappa_1 + 1 \\ x & B_2 & \cdots & 0 & & \kappa_2 + 1 \\ & & \vdots & & & \\ x & x & \cdots & B_m & & \kappa_m + 1 \end{array}$$

The product $X(s)B$ has the form

$$\begin{array}{cccc} X_1(s)B_1 & 0 & \cdots & 0 \\ ? & X_2(s)B_2 & 0 \cdots & 0 \\ & & \ddots & \\ ? & & & X_m(s)B_m \end{array}$$

It follows that the rank of $X(s)B$ is equal to the sum of the ranks of the $X_i(s)B_i$. From before we have that rank $X_i(s)B_i$ = rank B_i for all but finitely many s unless B_i is invertible in which case $X_i(s)B_i$ = rank B_i - 1. This proves the proposition. We can now prove the theorem that relates the ordering on the Schubert cells to the ordering on the orbits of the feedback group.

9.7. Theorem. Let (F,G) be a controllable pair and let ψ be the associated morphism from $\mathbb{P}^1(\mathbb{C})$ into $\mathbb{G}_n(\mathbb{C}^{n+m})$. Let $A_1 \ \ldots \ A_n$ be a sequence of subspaces of $\mathbb{C}^{n+m}$ such that $\psi(\mathbb{P}^1(\mathbb{C}))$ is contained in the Schubert cell $SC(A_1,\ldots,A_n)$. Let $\kappa_1, \ldots, \kappa_m$ be the Kronecker indices of (F,G) and for each i let $p(i) = j$ iff $\kappa_1 + \ldots + \kappa_j < i \leq \kappa_1 + \ldots + \kappa_{j+1}$. Then $\dim A_i \geq i + p(i) = \tau_i(\kappa)$.

Proof. It is a simple matter to check that $\tau_i(\kappa)$ (cf. (9.2) above) is equal to $i + p(i)$. We can assume that (F,G) is in Brunovsky canonical form. Suppose that $\dim A_i = t < i+p(i)$. Then $A_i = \{x \in \mathbb{C}^{n+m}: \langle b_j,x\rangle = 0,\ j = 1, \ldots, n+m-t\}$ for certain linearly independent b_j. Let B be matrix whose columns are the b_i's. Let $P(s)$ be space spanned by the rows of $X(s)$. Since $\psi(\mathbb{P}^1(\mathbb{C}))$ is contained in $SC(A_1,\ldots,A_n)$ we must have that $\dim(A_i \cap P(s)) \geq i$. Thus the dimension of $P(s)B$ is less than or equal to $n-i$ which is the same as

$$\operatorname{rank} X(s)B \leq n-i.$$

Now by the previous proposition $\operatorname{rank} X(s)B \geq n+m-t-\ell$ where ℓ is the largest number such that

$$\kappa_m + \kappa_{m-1} + \ldots + \kappa_{m-\ell+1} + \ell \leq n + m - t.$$

So we have the following

(1) $t < i + p(i)$ (by hypothesis)
(2) $n-i \geq n + m - t - \ell$ or equivalently $i \leq t + \ell - m$
(3) $\kappa_m + \ldots + \kappa_{m-\ell+1} + \ell \leq n + m - t$
(4) $\kappa_1 + \ldots + \kappa_{p(i)} < i \leq \kappa_1 + \ldots + \kappa_{p(i)+1}$.

Using (2) and (3) we have
that $\kappa_m + \ldots + \kappa_{m-\ell+1} \leq n-i = \kappa_1 + \ldots + \kappa_m - i$ so we have $i \leq \kappa_1 + \ldots + \kappa_{m-\ell}$ which implies $m - \ell \geq p(i) + 1$ thus $p(i) + i \leq m - \ell - 1 + i \leq (m-\ell-1) + (t+\ell-m) = t-1$ which contradicts (1).

This proves the theorem.

9.7. Vectorbundles and Schubert cells. Because every positive vectorbundle over $\underline{\underline{P}}^1(\underline{\underline{C}})$ arises as the bundle $E(\Sigma)$ of some system
Σ one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over $\underline{\underline{P}}^1(\underline{\underline{C}})$. Here the morphism ψ_Σ must, of course, be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle E, and n + m and m are determined respectively as dim $\Gamma(E,\underline{\underline{P}}^1(\underline{\underline{C}}))$ and dim E.

10. DEFORMATIONS OF REPRESENTATIONS HOMOMORPHISMS AND SUBREPRESENTATIONS.

10.1. On proving Inclusion Results for Representations. Suppose we have given a continuous family of homomorphisms of finite dimensional representations over $\underline{\underline{C}}$ of a finite group G

(10.2) $$\pi_t: M \to V$$

Suppose that Im $\pi_t \simeq \rho$ for $t \neq 0$ (and small) and that Im $\pi_o \simeq \rho_o$. Then the representation ρ_o is a direct summand of the representation ρ. This is seen as follows. Because the category of finite dimensional complex representations of G is semisimple there is a homomorphism of representations ϕ_o: Im $\pi_o \to M$ such that $\pi_o \circ \phi_o = \mathrm{id}$. Then $\pi_t \circ \phi_o$: Im $\phi_o \to$ Im π_t is still injective for small t (by the continuity of π_t) which gives us ρ_o as a subrepresentation and hence a direct summand of ρ.

It is almost equally easy to construct a surjective homomorphism Im $\pi_t \to$ Im π_o.

10.3. The Inverse Result. Inversely if ρ_o is a subrepresentation of ρ then there is a family of representations (10.3) such that
Im $\pi_t \simeq \rho$ for $t \neq 0$ and Im $\pi_o \simeq \rho_o$, and if ρ is generated (as a $\underline{\underline{C}}[G]$-module) by one element one can take for M in (10.2) the

regular representation. Indeed if ρ_o is a subrepresentation of ρ then $\rho = \rho_o \oplus \rho_1$. Let π: $M \to \rho = \rho_o \oplus \rho_1$ be a surjective map of representations. Let π_o, π_1 be the two components of π. Let $s = (s_o, s_1)$ be a section of π. Then $\pi_o s_o = \mathrm{id}$, $\pi_1 s_1 = \mathrm{id}$, $\pi_o s_1 = 0$, $\pi_1 s_o = 0$ and it follows that $\pi(t)$ consisting of the components π_o and $t\pi_1$ is still surjective. Hence $\mathrm{Im}\ \pi(t) = \rho$ and $\mathrm{Im}\ \pi(0) = \rho_o$.

11. A FAMILY OF REPRESENTATIONS OF S_{n+m} PARAMETRIZED by $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$.

11.1. Construction of the Family. Let M be the regular representation of S_{n+m}. That is M has a basis $e_\sigma, \sigma \in S_{n+m}$ and S_{n+m} acts on M by the formula $\tau(e_\sigma) = e_{\tau\sigma}$, for all $\tau \in S_{n+m}$. Now consider the universal bundle ξ_m over $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ and the n + m holomorphic sections $\varepsilon_1, \ldots, \varepsilon_{n+m}$ defined by

$\varepsilon_1(x) = e_i \bmod x \in \underline{\underline{C}}^{n+m}/x$, where e_i is the i-th standard basis vector. Take the (m+n)-fold tensor product of ξ_m and define a family of homomorphisms parametrized by $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ by

$$(11.2)\quad \pi_x\colon M \to \xi_m(x)^{\otimes(n+m)},\ e_{\sigma^{-1}} \mapsto \varepsilon_{\sigma(1)}(x)\otimes\ldots\otimes\varepsilon_{\sigma(n)}(x)$$

More precisely (11.2) defines a homomorphism of vectorbundles

$$(11.3)\qquad \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m}) \times M \to \xi_m^{\otimes(n+m)}$$

The group S_{n+m} acts on $\xi_m(x)^{\otimes(n+m)}$ by permuting the factors and it is a routine exercise to see that π_x is equivariant with respect to this action, i.e. that $\pi_x(\tau v) = \tau\pi_x(v)$ for all $v \in M$, $\tau \in S_{n+m}$. (Here the product $\tau\sigma \in S_{n+m}$ is interpreted as first the automorphism σ of 1,...,n+m and then the automorphism τ).

Thus $\mathrm{Im}\ \pi_x = \pi(x)$ is a representation of S_{n+m} for all x giving us a family of representations parametrized by $\underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$.

Fixing a point $x_o \in \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ and choosing m independent sections of ξ_m in a neighbourhood U of x_o, this gives us families of homomorphisms of representations

$$(11.4) \qquad M \overset{\pi'_x}{\rightarrow} (\underline{\underline{C}}^m)^{\otimes(n+m)}, \quad x \in U \subset \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$$

such that Im $\pi'_x \simeq \pi(x)$ for $x \in U$.

11.5. Permutation Representations and Schubert-cells. (On connection D). Let $x \in \underline{\underline{G}}_n(\underline{\underline{C}}^{n+m})$ be a subspace of C^{n+m} spanned by the rows of a matrix of the form (n=3, n=5)

$$\begin{matrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 \end{matrix}$$

where all the *'s are nonzero. Then obviously the representation $\pi(x)$ of S* is isomorphic to $\rho(\tilde{\kappa})$ with $\tilde{\kappa} = (4,3,1)$. Note that x is in the standard Schubert-cell $SC(\tau(\kappa))$, with $\kappa = (3,2,0)$. This holds in general and it is not difficult to extend this to

11.6. Proposition. Let κ be an m-part partition of n, $\tilde{\kappa} = (\kappa_1+1,\ldots,\kappa_m+1)$. Then for almost all $x \in SC(\tau(\kappa))$, the representation $\pi(x)$ of S_{n+m} contains the representation $\rho(\tilde{\kappa})$ and for some $x \in SC(\tau(\kappa))$, $\pi(x) \simeq \rho(\tilde{\kappa})$.

Conjecturally the reverse holds also. That is if for almost all x in a standard Schubert-cell $SC(\lambda)$ we have that $\pi(x)$ contains $\rho(\tilde{\kappa})$ then $\lambda_i \geq \tau_i(\kappa)$, i = 1, ..., n. And I am even inclined to believe that if $x \in SC(\lambda)$ and $\pi(x)$ contains (or is equal to) $\rho(\tilde{\kappa})$ then $\lambda_i \geq \tau_i(\kappa)$.

Note also that the matrices (11.5) are precisely the type of matrices (sI-A;B) for a system $\Sigma = (A,B)$ in feedback canonical form ($s \neq 0, \infty$) suggesting that there is a natural representation of S_{n+m} attached to Σ awaiting interpretation.

11.7. <u>On a proof of the Snapper, Liebler-Vitale, Lam, Young Theorem vice the Universal Family</u> (11.2).

The structure of the family of representations (11.2) rather quickly suggests a way of proving the Snapper etc. theorem by deformation arguments as in 10.1. The argument is, however, more complicated than one would like perhaps. It is perhaps best illustrated by means of an example.

Consider an $x \in \underline{\underline{G}}_3(\underline{\underline{C}}^5)$ spanned by the rows of a matrix of the form

$$\begin{matrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ z & 0 & 0 & -1 & t \end{matrix}$$

Let $f_1, \ldots, f_5$ be the images of the standard basis vectors $e_1, \ldots, e_5$ in $\underline{\underline{C}}^5/x$. Then $f_1 = f_2 = f_3 \neq f_4 = zf_1 + tf_5$ so that

f_1 and f_5 are a basis for $\underline{\underline{C}}^5/x$ for all values of z and t. Let $(1) \in S_5$ be the identity permutation. The image of $e_{(1)} \in M$ in

$(\underline{\underline{C}}^5/x)^{\otimes 5}$ is by the definition (11.2) equal to

$$f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 = zf_{11115} + tf_{11155} \tag{11.8}$$

Where f_{11115} is short for $f_1 \otimes f_1 \otimes f_1 \otimes f_1 \otimes f_5$ and similarly for other 5-tuples of indices. Symmetrizing the element (11.8) with respect to the permutation (45) gives us

$$z(f_{11115} + f_{11151}) + 2tf_{11155} \tag{11.9}$$

Let V_1 be the subrepresentation of $\mathrm{Im}\,\pi_x$ generated by the element (11.9). (The representation $\mathrm{Im}\,\pi_x$ is the subrepresentation of $(\underline{\underline{C}}^5/x)^{\otimes 5}$ generated by (11.8)). Now (11.9) is invariant under the Young subgroup $S_3 \times S_2$. Hence dim $V_1 \leq 5!/3!2!$. On the other hand, if $t \neq 0$ then setting $z = 0$ in (11.9) (which corresponds to the surjective map mentioned just above 10.2 associated to a family of representations) obviously maps V_1 onto the vector space with as basis all symbols $f_{\cdot\cdot\cdot\cdot\cdot}$ with three of the indices equal to 1 and 2 equal to 5. This is $\rho(3,2)$ of dimension $5!/3!2!$ so that $V_1 \simeq \rho(3,2)$ if $t \neq 0$. Now for $z \neq 0$ set $t = 0$ in (11.8) to obtain a homomorphism of representations

$$\mathrm{Im}\ \pi_x \to \pi(4,1)$$

It is now not hard to prove that (cf. [7] for a detailed proof).

11.10. Proposition. The composed homomorphism of representations
$\rho(3,2) \simeq V_1 \subset \mathrm{Im}\ \pi_x \to \rho(4,1)$ is surjective.

This then proves that $\rho(4,1)$ is a direct summand of $\rho(3,2)$. The argument generalizes without difficulty for partitions $\kappa > \lambda$ such that λ is obtained from κ by a transformation of the type described in 6.7 above.

This is by no means the easiest way to prove this theorem. It is perfectly easy to describe the morphism $\rho(\kappa) \to \rho(\lambda)$ directly and then the general analogue of proposition 11.10 yields the Snapper, Liebler-Vitale, Lam, Young result. This proof uses no representation theory at all (except the definition of the permutation representations $\rho(\kappa)$; cf. [7] for details).

11.11. Remarks. It is conceivable that the family (11.2) contains all the families of representations one needs to prove the Snapper etc. result by means of deformation argument. Quite generally we would like to pose the question which representations occur in this family and investigate universal families (for continuous families) of homomorphisms of representations from some fixed representation space into another.

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