THE INCOMPLETE LU-DECOMPOSITION AS A RELAXATION METHOD IN MULTI-GRID ALGORITHMS

P.W. Hemker
Mathematical Centre, Amsterdam, The Netherlands

ABSTRACT

We consider relaxation methods for use with Multi-Grid (MG-) algorithms to solve a sparse linear system

$$A_\ell \, x_\ell = f_\ell, \qquad \ell > 0,$$

which is supposed to be the discretization of a continuous boundary value problem Ax = f. With the same equation coarser discretizations are related:

$$A_k \, x_k = f_k, \qquad k = 0,1,\ldots,\ell-1.$$

We first give a brief exposition of the framework of MG-methods. Next, we describe the incomplete LU-decomposition as a relaxation method and in the 3rd section we compare different variants of it. The conclusion is in favour of the ILU variant.

THE MULTI-GRID FRAMEWORK

Considering MG methods in which each iteration step consists of γ coarse grid correction steps, preceeded by p and followed by q relaxation sweeps, we see that in each MG iteration step the residual is multiplied by the operator (cf. [2])

$$\bar{M}_k^{MLA} = \bar{M}_k^{TLA} + (\bar{M}_k^{REL})^q \, A_k \, P_{k-1}^k \, A_{k-1}^{-1} \, (\bar{M}_{k-1}^{TLA})^\gamma \, R_k^{k-1} \, (\bar{M}_k^{REL})^p,$$

where

$$\bar{M}_0^{TLA} = 0$$

$$\bar{M}_k^{TLA} = (\bar{M}_k^{REL})^q \, (I - A_k \, P_{k-1}^k \, A_{k-1}^{-1} \, R_k^{k-1}) \, (\bar{M}_k^{REL})^p, \quad k = 1,2,\ldots,\ell.$$

P_{k-1}^k denotes the prolongation from level k-1 to level k,

R_k^{k-1} denotes the restriction from level k to level k-1,

A_{k-1} denotes the k-1 level discretisation of the operator A, e.g.

$$A_{k-1} = R_k^{k-1} \, A_k \, P_{k-1}^k.$$

$\bar{M}_k^{REL}$ is the operator on level k by which the residual is multiplied in one relaxation sweep (see next section).

Communication between finer and coarser grids takes place via the prolongations and restrictions. By the spectral decomposition of a gridfunction

$$u_h(jh) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\omega \in [-\pi/h,\pi/h]^n} e^{+ijh\omega} \, \hat{u}_h(\omega) \, d\omega$$

where

$$\hat{u}_h(\omega) = \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_{j \in \mathbb{Z}^n} e^{-ijh\omega} \, u_h(jh)$$

denotes the spectrum of the gridfunction u_h, defined on $[-\pi/h,\pi/h]^n$, we see that only low frequency components of a gridfunction can be represented on coarse grids. The relation between the spectra of a gridfunction, its prolongation and its restriction are given by ($H = qh$)

$$\widehat{Ru_h}(\omega) = \left(\frac{\sqrt{2\pi}}{h}\right)^n \sum_{0\le p<q} \hat{a}_h(\omega+2\pi p/H)\ \hat{u}_h(\omega+2\pi p/H),$$

$$\widehat{Pu_h}(\omega) = \left(\frac{\sqrt{2\pi}}{H}\right)^n \hat{b}_h(\omega)\ \hat{u}_H(\omega);$$

where a_h and b_h are grid functions that characterize the particular prolongations and restrictions (cf.[3]).

With the choice $A_{k-1} = RA_kP$, we find the relation

$$(I-\tilde{P}R)(I-A_k\ P\ A_{k-1}^{-1}\ R) = (I-A_k\ P\ A_{k-1}^{-1}\ R)$$

for any prolongation $\tilde{P}$. Hence, taking e.g. Shannon's interpolation for $\tilde{P}$, we see that - in the residual - coarse grid corrections anihilate all frequencies that can be represented on the coarser grid. Similar results hold for other reasonable choices of A_{k-1}.

From this it is clear that an efficient relaxation method in an MG-algorithm should damp those high frequencies that cannot be represented on the coarser grid (in the 2-D case (n=2) and with mesh doubling (H=qh=2h) this is the shadowed portion of fig. 1). This behaviour is analyzed by considering $\hat{M}(\omega)$, the spectrum of the operator $\tilde{M}_k^{REL}$. Following BRANDT [1], we define the smoothing rate

$$\mu = \sup_{\substack{\omega\in[-\pi/h,\pi/h]^n \\ \omega\notin[-\pi/2h,\pi/2h]^n}} |\hat{M}(\omega)|.$$

For any linear difference operator A_k with constant coefficients and each relaxation method this μ can easily be determined numerically. E.g. it is well known that for the usual 5-point discretization of Poisson's equation (n=2,q=2) and GS-relaxation we have $\mu = 0.5$.

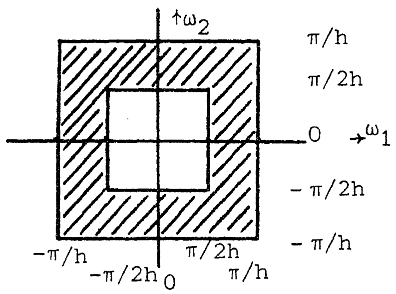


Fig. 1. Frequency region

We consider similar relaxation methods in order to find methods that take less work per iteration sweep and smaller values of μ.

INCOMPLETE LU-DECOMPOSITION AS A RELAXATION METHOD

By B_k we denote the approximate inverse in a stationary relaxation method for the solution of $A_k x_k = f_k$. Then $\bar{M}_k^{REL} = I - A_k B_k$ is the operator by which the residual is multiplied in each relaxation sweep. E.g. with Jacobi-iteration $B_k = (\text{diag}(A_k))^{-1}$ and with forward - or backward Gauss-Seidel iteration

$$B_k = (\text{lower triag } (A_k))^{-1} \quad \text{resp. } B_k = (\text{upper triag } (A_k))^{-1}.$$

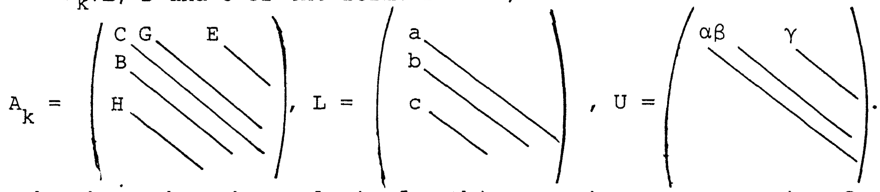
We are interested in incomplete LU-decomposition, where for some D, L and U (which are diagonal, lower and upper triangular respectively) we take

$$B_k = (LD^{-1}U)^{-1}$$

and we may write

$$A_k = LD^{-1}U + R.$$

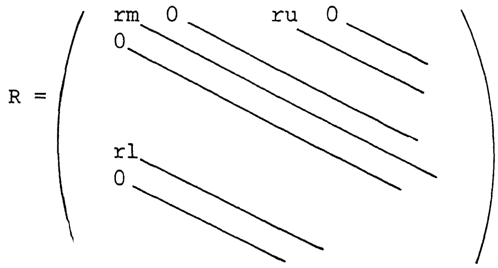
Although the treatment of incomplete LU-decomposition can be given for the general case, here we confine ourselves to infinite Toeplitz-matrices A_k, L, D and U of the form: D = wI,

(*) 

It can be shown that the analysis for this case is representative for the local behaviour in the interior of the domain of the PDE. Multiplication of $LD^{-1}U$ and identification of the sub-and superdiagonals yields

$$c = H, \ b = B, \ \beta = G, \ \gamma = E, \ a = \alpha = w^{-1},$$

and R has the form



$$ru = BE/a,$$
$$rl = HG/a,$$
$$rm = a - C + (GD+HE)/a.$$

Here one free parameter a is left. If no corrections are admitted to the main diagonal: rm = 0 and a is prescribed by

(ILU) $\quad a = \frac{1}{2}(C \pm \sqrt{C^2 - 4(GD+HE)}\,).$

To get minimal ru and rl the sign which yields maximal absolute value is used. Other choices for a are possible:

(SGS) $\quad a=C,$

or the 1-st order approximation

(MILU) $\quad a=C - (GD+HE)/C.$

Remark.
Notice that the choice a = C yields a method which is equivalent with symmetric Gauss-Seidel relaxation. In this case namely $A_k = L + U - D$ and we see

$$\bar{M}^{REL} = I - A_k(LD^{-1}U)^{-1} = (I - A_kU^{-1})(I - A_kL^{-1}),$$

i.e. one GS-forward step followed by one GS-backward step is identical with this particular incomplete LU-iteration step. *Symmetric GS-iteration (SGS) is one (simple) form of incomplete LU-iteration.*

THE EFFICIENCY OF INCOMPLETE LU-ITERATION

To study the efficiency of the incomplete LU-relaxation, first we compute the smoothing rate $\mu(a)$ for the usual 5-point discretization of Poisson's equation, depending on the parameter a. The maximum of

$$\hat{M}(\omega) = \frac{(a^2 - 4a + 2) + 2\cos(\omega_1 - \omega_2)}{(a^2+2) - 2a(\cos\omega_1 - \cos\omega_2) + 2\cos(\omega_1-\omega_2)}$$

on the shaded area of fig. 1 can be computed analytically and is (a>2)

$$\mu(a) = \frac{8 \pm A\sqrt{A^2 + 12}}{8 - 2Aa \pm (A+4a)\sqrt{A^2+12}} \quad , \text{ where } A = a^2 - 4a + 2,$$

and is attained at $\omega = (\arccos(\frac{8 + A\sqrt{A^2 + 12}}{A^2 + 16}), \frac{\pi}{2})$.

A graph of this function is given in fig. 2. For our three particular choices of a we find:

SGS : $a = 4$, $\mu(a) = 0.2500$, $\omega = (\arccos(4/5), \pi/2)$;
MILU: $a = 3.5$, $\mu(a) = 0.1649$, $\omega = (0.555\pi , \pi/2)$;
ILU : $a = 2 + \sqrt{2}$, $\mu(a) = 0.2035$, $\omega = (\pi/3 , \pi/2)$.

The minimal value of $\mu(a)$ is 0.1607 and is attained for a = 3.510 which is remarkably close to our choice MILU.

The amount of work in each iteration step is 5N for the solution and 3N for the computation of the residual (2N in the case ILU, where rm = 0). Summarizing, the efficiency of the different methods is

SGS : $\mu = 0.25$, work 8N, efficiency $\sqrt[8]{0.25} = 0.84$;
MILU: μ 0.16, work 8N, efficiency $\sqrt[8]{0.16} = 0.80$;
ILU : μ 0.20, work 7N, efficiency $\sqrt[7]{0.20} = 0.79$.

For the Poisson equation we conclude that, although MILU has optimal $\mu(a)$, ILU has slightly better efficiency because of the cheaper computation of the residual.

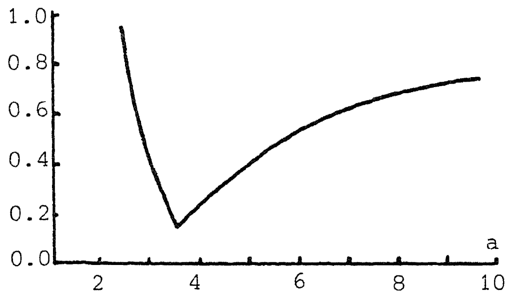


Fig. 2 The smoothing rate $\mu(a)$.

Beside Poisson's equation we also considered other model equations viz. the *convection-diffusion equation*

$$\varepsilon \, \Delta\phi + \cos(\alpha)\, \phi_x + \sin(\alpha)\, \phi_y = f,$$

discretized with Il'in's method and the *anisotrope Poisson equation*

$$\varepsilon \, \phi_{xx} + \phi_{yy} = f.$$

For the convection diffusion we see from (*) that, asymptotically for $\varepsilon \to 0$, either H or E and either B or G vanish. Hence, asymptotically the methods SGS, MILU and ILU coincide. Moreover, for $0 < \alpha < \pi/2$, mod(π), asymptotically $\mu \to 0$, i.e. the system is solved exactly by only one iteration sweep. However, for a convection direction α with $-\pi/2 \le \alpha \le 0$, mod(π), μ takes a positive value. For different values of ε and α, these μ are given in table 1. Taking into account the number of operations, we see again that, although $\mu_{MILU} \le \mu_{ILU} \le \mu_{SGS}$, ILU is in most cases more efficient than MILU.

α: $\varepsilon = 1.0$	$-\pi/4$	$-\pi/8$ $-3\pi/8$	$\alpha = 0$ $\pi/2$	$\pi/8$ $3\pi/8$	$\alpha = \pi/4$
SGS	0.261	0.258	0.243	0.225	0.215
MILU	0.200	0.195	0.174	0.153	0.144
ILU	0.229	0.226	0.205	0.182	0.171
$\varepsilon = 0.1$					
SGS	0.446	0.655	0.405	0.128	0.679
MILU } ILU	0.447	0.660	0.427	0.131	0.488
$\varepsilon = 0.001$					
SGS } MILU } ILU	0.447	0.679	0.499	0.000	0.000

Table 1. The smoothing rate μ for the convection diffuction equation with Il'in's discretization.

	ε = 1	ε = 0.5	ε = 0.1	ε = 0.01	ε = 0.001
SGS	0.250	0.321	0.697	0.961	0.996
MILU	0.165	0.190	0.387	0.889	0.988
ILU	0.204	0.250	0.477	0.768	0.916

Table 2. μ for the anisotrope Poisson equation.

For the anisotrope Poisson equation (table 2) we see that in the three cases $\mu \to 1$ as $\varepsilon \to 0$. However, MILU and ILU have smaller smoothing rates and for small ε we even have $\mu_{ILU} < \mu_{MILU}$.
We conclude that also in this case the ILU-decomposition is the best choice in our class of incomplete LU-decomposition relaxation methods and it is significantly more efficient than symmetric Gauss-Seidel relaxation.

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