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(FINE) MODULI (SPACES) FOR LINEAR SYSTEMS: WHAT ARE THEY AND WHAT ARE THEY GOOD FOR ?

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ABSTRACT

This tutorial and expository paper considers linear dynamical systems $\dot{x} = Fx + Gu$, $y = Hx$, or, $x(t+1) = Fx(t) + Gu(t)$, $y(t) = Hx(t)$; more precisely it is really concerned with families of such, i.e., roughly speaking, with systems like the above where now the matrices F,G,H depend on some extra parameters σ. After discussing some motivation for studying families (delay systems, systems over rings, n-d systems, perturbed systems, identification, parameter uncertainty) we discuss the classifying of families (fine moduli spaces). This is followed by two straightforward applications: realization with parameters and the nonexistence of global continuous canonical forms. More applications, especially to feedback will be discussed in Chris Byrnes' talks at this conference and similar problems as in these talks for networks will be discussed by Tyrone Duncan. The classifying fine moduli space cannot readily be extended and the concluding sections are devoted to this observation and a few more related results.

1. INTRODUCTION

The basic object of study in these lectures (as in many others at this conference) is a constant linear dynamical system, that is a system of equations

$$\begin{array}{ccc} \dot{x} = Fx + Gu & & x(t+1) = Fx(t) + Gu(t) \\ y = Hx & (\Sigma) & y(t) = Hx(t) \\ \text{(a): continuous time} & & \text{(b): discrete time} \end{array} \qquad (1.1)$$

C. I. Byrnes and C. F. Martin, Geometrical Methods for the Theory of Linear Systems, 125-193.

:h $x \in k^n$ = state space, $u \in k^m$ = input or control space, : k^p = output space, and F,G,H matrices with coefficients in of the appropriate sizes; that is, there are m inputs and p :puts and the dimension of the state space, also called the iension of the system Σ and denoted $\dim(\Sigma)$, is n. Here k an appropriate field (or possibly ring). In the continuous ;e of course k should be such that differentiation makes sense · (enough) functions $\mathbb{R} \to k$, e.g. $k = \mathbb{R}$ or $\mathbb{C}$. Often one ls a direct feedthrough term Ju, giving $y = Hx + Ju$ in case and $y(t) = Hx(t) + Ju(t)$ in case (b) instead of $y = Hx$ l $y(t) = Hx(t)$ respectively; for the mathematical problems be discussed below the presence or absence of J is essen- :lly irrelevant.

More precisely what we are really interested in are families objects (1.1), that is sets of equations (1.1) where now the :rices F,G,H depend on some extra parameters σ. As people 'e found out by now in virtually all parts of mathematics and : applications, even if one is basically interested only in igle objects, it pays and is important to study families of :h objects depending on a small parameter ε (deformation and ·turbation considerations). This could be already enough moti- :ion to study families, but, as it turns out, in the case of near) systems theory there are many more circumstances where iilies turn up naturally. Some of these can be briefly summed as delay-differential systems, systems over rings, continuous ionical forms, 2-d and n-d systems, parameter uncertainty, ngularly) perturbed systems. We discuss these in some detail ow in section 2.

To return to single systems for the moment. The equations .1) define input/output maps $f_\Sigma : u(t) \mapsto y(t)$ given ;pectively by

$$y(t) = \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau \ , \quad t \geq 0 \tag{1.2a}$$

$$y(t) = \sum_{i=1}^{t} A_i u(t-1-i), \ A_i = HF^{i-1}G,$$

$$i = 1,2,\ldots, \quad t = 1,2,3,\ldots \tag{1.2b}$$

:re we have assumed that the system starts in $x(0) = 0$ at ie 0. In both cases the input/output operator is uniquely :ermined by the sequence of matrices $A_1, A_2, \ldots$. Inversely, ilization theory studies when a given sequence $A_1, A_2, \ldots$ is :h that there exist F,G,H such that $A_i = HF^{i-1}G$ for all i. ilization with parameters is now the question: given a sequence matrices $A_1(\sigma)$, $A_2(\sigma)$, $A_3(\sigma)$, ... depending polynomially :sp. continuously, resp. analytically, resp. ...) on parameters

σ when do there exist matrices F,G,H depending polynomially (resp. continuously, resp. analytically, resp. ...) on the parameters σ such that $A_i(\sigma) = H(\sigma)F^{i-1}(\sigma)G(\sigma)$ for all i. And to what extent are such realizations unique? Which brings us to the next group of questions one likes to answer for families.

A single system Σ given by the triple of matrices F,G,H is completely reachable if the matrix $R(F,G)$ consisting of the blocks $G,FG,\ldots,F^nG$

$$R(\Sigma) = R(F,G) = (G \,\vdots\, FG \,\vdots\, \ldots \,\vdots\, F^nG) \tag{1.3}$$

has full rank n. (This means that any state x can be steered to any other state x' by means of a suitable input). Dually the system Σ is said to be completely observable if the matrix $Q(F,G)$ consisting of the blocks $H,HF,\ldots,HF^n$

$$Q(\Sigma) = Q(F,H) = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^n \end{pmatrix} \tag{1.4}$$

has full rank n. (This means that two different states $x(t)$ and $x'(t)$ of the system can be distinguished on the basis of the output $y(\tau)$ for all $\tau \geq t$). As is very well known if $A_1,A_2,\ldots$ can be realized then it can be realized by a co and cr system and any two such realizations are the same up to base change in state space. That is, if $\Sigma = (F,G,H)$ and $\Sigma' = (F',G',H')$ both realize $A_1,A_2,\ldots$ and both are cr and co then $\dim(\Sigma) = \dim(\Sigma') = n$ and there is an invertible $n\times n$ matrix S such that $F' = SFS^{-1}$, $G' = SG$, $H' = HS^{-1}$. (It is obvious that if Σ and Σ' are related in this way then they give the same input/output map). This transformation

$$\Sigma = (F,G,H) \mapsto \Sigma^S = (F,G,H)^S = (SFS^{-1},SG,HS^{-1})$$

corresponds of course to the base change in state space $x' = Sx$. This argues that at least one good notion of isomorphism of systems is: two systems Σ, Σ' over k are isomorphic iff $\dim(\Sigma) = \dim(\Sigma')$ and there is an $S \in GL_n(k)$, the group of invertible matrices with coefficients in k, such that $\Sigma' = \Sigma^S$. A corresponding notion of homomorphism is: a homomorphism from $\Sigma = (F,G,H)$, $\dim\Sigma = n$, to $\Sigma' = (F',G',H')$, $\dim\Sigma = n'$, is an $n'\times n$ matrix B (with coefficients in k) such that $BG = G'$, $BF = F'B$, $H'B = H$. Or, in other words, it is a linear map from the state space of Σ to the state space of Σ' such that the diagram below commutes.

$$\begin{tikzcd} & k^n \arrow[r, "F"] \arrow[d, "B"] & k^n \arrow[d, "B"] \arrow[dr, "H"] & \\ k^m \arrow[ur, "G"] \arrow[r, "G'"'] & k^{n'} \arrow[r, "F'"'] & k^{n} \arrow[r, "H'"'] & k^p \end{tikzcd} \tag{1.6}$$

The obvious corresponding notion of isomorphism for families $\Sigma(\sigma)$, $\Sigma'(\sigma)$ is a family of invertible matrices $S(\sigma)$ such that $\Sigma(\sigma)^{S(\sigma)} = \Sigma'(\sigma)$, where, of course, $S(\sigma)$ should depend polynomially, resp. continuously, resp. analytically, resp. ... on σ if Σ and Σ' are polynomial, resp. continuous, respt. analytical, resp. ... families. One way to look at the results of section 3 below is as a classification result for families, or, even, as the construction of canonical forms for families under the notion of isomorphism just described. As it happens the classification goes in terms of a universal family, that is, a family from which, roughly speaking, all other families (up to isomorphism) can be uniquely obtained via a transformation in the parameters.

Let $L_{m,n,p}(k)$ be the space of all triples of matrices (F,G,H) of dimensions $n\times n$, $n\times m$, $p\times n$, and let $L^{co,cr}_{m,n,p}$ be the subspace of cr and co triples. Then the parameter space for the universal family is the quotient space $L^{co,cr}_{m,n,p}(k)/GL_n(k)$, which turns out to be a very nice space.

The next question we shall take up is the existence or non-existence of continuous canonical forms. A continuous canonical form on $L^{co,cr}_{m,n,p}$ is a continuous map $(F,G,H)\mapsto c(F,G,H)$ such that $c(F,G,H)$ is isomorphic to (F,G,H) for all $(F,G,H)\in L^{co,cr}_{m,n,p}$ and such that (F,G,H) and (F',G',H') are isomorphic if and only if $c(F,G,H) = c(F',G',H')$ for all (F,G,H), $(F',G',H')\in L^{co,cr}_{m,n,p}$. Obviously if one wants to use canonical forms to get rid of superfluous parameters in an identification problem the canonical form had better be continuous. This does not mean that (discontinuous) canonical forms are not useful. On the contrary, witness e.g. the Jordan canonical form for square matrices under similarity. On the other hand, being discontinuous, it also has very serious drawbacks; cf. e.g. [GWi] for a discussion of some of these. In our case it turns out that there exists a continuous canonical form on all of $L^{co,cr}_{m,n,p}$ if and only if $m = 1$ or $p = 1$.

Now let, again, Σ be a single system. Then there is a canonical subsystem $\Sigma^{(r)}$ which is completely reachable and a canonical quotient system Σ^{co} which is completely observable.

Combining these two constructions one finds a canonical subquotient (or quotient sub) which is both cr and co. The question arises naturally whether (under some obvious necessary conditions) these constructions can be carried out for families as well and also for single time varying systems. This is very much related to the question of whether these constructions are continuous. In the last sections we discuss these questions and related topics like: given two families Σ and Σ' such that $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic for all (resp. almost all) values of the parameters σ; what can be said about the relation between Σ and Σ' as families (resp. about $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ for the remaining values of σ).

2. WHY SHOULD ONE STUDY FAMILIES OF SYSTEMS

For the moment we shall keep to the intuitive first approximation of a family of systems as a family of triples of matrices of fixed size depending in some continuous manner on a parameter σ. This is the definition which we also used in the introduction.

2.1 (Singular) Perturbation, Deformation, Approximation

This bit of motivation for studying families of objects, rather than just the objects themselves, is almost as old as mathematics itself. Certainly (singular) perturbations are a familiar topic in the theory of boundary value problems for ordinary and partial differential equations and more recently also in optimal control, cf. e.g. [OMa]. For instance in [OMa], Chapter VI, O'Malley discusses the singularly perturbed regulator problem which consists of the following set of equations, initial conditions and quadratic cost functional which is to be minimized for a control which drives the state

$$x = \begin{pmatrix} y \\ z \end{pmatrix}$$

to zero at time $t = 1$.

$$\begin{aligned} \dot{y} &= A_1(\varepsilon)y + A_2(\varepsilon)z + B_1(\varepsilon)u \quad & y(0,\varepsilon) = y^0(\varepsilon) \\ \varepsilon\dot{z} &= A_3(\varepsilon)y + A_4(\varepsilon)z + B_2(\varepsilon)u \quad & z(0,\varepsilon) = z^0(\varepsilon) \\ J(\varepsilon) &= x^T(1,\varepsilon)\pi(\varepsilon)x(1,\varepsilon) & \\ &\quad + \int_0^1 (x^T(t,\varepsilon)Q(\varepsilon)x(t,\varepsilon) + u^T(t,\varepsilon)R(\varepsilon)u(t,\varepsilon))dt & \end{aligned} \tag{2.1.1}$$

with positive definite $R(\varepsilon)$, and $Q(\varepsilon),\pi(\varepsilon)$ positive semidefinite. Here the upper T denotes transposes. The matrices

$A_i(\varepsilon)$, $i = 1,2,3,4$, $B_i(\varepsilon)$, $i = 1,2$, $\pi(\varepsilon)$, $Q(\varepsilon)$, $R(\varepsilon)$ may also depend on t. For fixed small $\varepsilon > 0$ there is a unique optimal solution. Here one is interested, however, in the asymptotic solution of the problem as ε tends to zero, which is, still quoting from [OMa], a problem of considerable practical importance, in particular in view of an example of Hadlock et al. [HJK] where the asymptotic results are far superior to the physically unacceptable results obtained by setting $\varepsilon = 0$ directly.

Another interesting problem arises maybe when we have a system

$$\dot{x} = Fx + G_1u + G_2v, \qquad y = Hx \tag{2.1.2}$$

where v is noise, and there F, G_1, G_2, H depend on a parameter ε. Suppose we can solve the disturbance decoupling problem for $\varepsilon = 0$. I. e., we can find a feedback matrix L such that in the system with state feedback loop L

$$\dot{x} = (F+GL)x + G_1u + G_2v, \quad y = Hx$$

the disturbances v do not show up any more in the output y, (for $\varepsilon = 0$). Is it possible to find a disturbance discoupler $L(\varepsilon)$ by "perturbation" methods, i.e., as a power series in ε which converges (uniformly) for ε small enough, and such that $L(0) = L$.

In this paper we shall not really pay much more attention to singular perturbation phenomena. For some more systems oriented material on singular perturbations cf. [KKU] and also [Haz 4].

2.2 Systems Over Rings

Let R be an arbitrary commutative ring with unit element. A linear system over R is simply a triple of matrices (F,G,H) of sizes $n \times n$, $n \times m$, $p \times n$ respectively with coefficients in R. Such a triple defines a linear machine

$$x(t+1) = Fx(t) + Gu(t), \quad t = 0,1,2,\ldots, \; x \in R^n, \; u \in R^m$$
$$y(t) = Hx(t), \qquad y \in R^p \tag{2.2.1}$$

which transforms input sequences $(u(0),u(1),u(2),\ldots)$ into output sequences $(y(1),y(2),y(3),\ldots)$ according to the convolution formula (1.2.b).

It is now absolutely standard algebraic geometry to consider these data as a family over Spec(R), the space of all prime ideals of R with the Zariski topology. This goes as follows. For each prime ideal p let $i_p : R \to Q(R/p)$ be the canonical map of R

into the quotient field $Q(R/p)$ of the integral domain R/p. Let $(F(p), G(p), H(p))$ be the triple of matrices over $Q(R/p)$ obtained by applying i_p to the entries of F,G,H. Then $\Sigma(p) = (F(p),G(p),H(p))$ is a family of systems parametrized by Spec(R).

Let me stress that, mathematically, there is no difference between a system over R as in (2.2.1) and the family $\Sigma(p)$. As far as intuition goes there is quite a bit of difference, and the present author e.g. has found it helpful to think about families of systems over Spec(R) rather than single systems over R. Of course such families over Spec(R) do not quite correspond to families as one intuitively thinks about them. For instance if $R = \mathbb{Z}$ = the integers, then Spec($\mathbb{Z}$) consists of (0) and the prime ideals (p), p a prime number, so that a system over $\mathbb{Z}$ gives rise to a certain collection of systems: one over $\mathbb{Q}$ = rational numbers, and one each over every finite field $\mathbb{F}_p = \mathbb{Z}/(p)$. Still the intuition one gleans from thinking about families as families parametrized continuously by real numbers seems to work well also in these cases.

2.3 Delay-Differential Systems

Consider for example the following delay-differential system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t-2) + x_2(t-\alpha) + u(t-1) + u(t) \\ \dot{x}_2(t) &= x_1(t) + x_2(t-1) + u(t-\alpha) \qquad (2.3.1) \\ y(t) &= x_1(t) + x_2(t-2\alpha) \end{aligned}$$

where α is some real number incommensurable with 1. Introduce the delay operators σ_1, σ_2 by $\sigma_1\beta(t) = \beta(t-1)$, $\sigma_2\beta(t) = \beta(t-\alpha)$. Then we can rewrite (2.3.1) formally as

$$\dot{x}(t) = Fx(t) + Gu(t), \quad y(t) = Hx(t) \qquad (2.3.2)$$

with

$$F = \begin{pmatrix} \sigma_1^2 & \sigma_2 \\ 1 & \sigma_1 \end{pmatrix}, \quad G = \begin{pmatrix} 1+\sigma_1 \\ \sigma_2 \end{pmatrix}, \quad H = (1 \quad \sigma_2^2) \qquad (2.3.3)$$

and, forgetting so to speak where (2.3.2), (2.3.3) came from, we can view this set of equations as a linear dynamical system over the ring $\mathbb{R}[\sigma_1,\sigma_2]$, and then using 2.2 above also as a family of systems parametrized by the (complex) parameters σ_1, σ_2, a point of view which has proved fruitful, e.g., in [By]. This idea has been around for some time now [ZW,An,Yo,RMY], though originally the tendency was to consider these systems as systems over the fields $\mathbb{R}(\sigma_1,\ldots,\sigma_r)$; the idea to consider them over the rings $\mathbb{R}[\sigma_1,\ldots,\sigma_r]$ instead is of more recent vintage [Mo,Kam].

There are, as far as I know, no relations between the solutions of (2.3.1) and the solutions of the family of systems (2.3.2), (2.3.3). Still many of the interesting properties and constructions for (2.3.1) have their counterpart for (2.3.2), (2.3.3) and vice versa. For example to construct a stabilizing state feedback loop for the family (2.3.2)-(2.3.3) depending polynomially on the parameters σ_1, σ_2 that is finding a stabilizing state feedback loop for the system over $\mathbb{R}[\sigma_1,\sigma_2]$, means finding an $m\times n$ matrix $L(\sigma_1,\sigma_2)$ with entries in $\mathbb{R}[\sigma_1,\sigma_2]$ such that for all complex σ_1,σ_2 $\det(s-(F+GL))$ has its roots in the left half plane. Reinterpreting σ_1 and σ_2 as delays so that $L(\sigma_1,\sigma_2)$ becomes a feedback matrix with delays one finds a stabilizing feedback loop for (the infinite dimensional) system (2.3.1). (cf. [BC], cf. also [Kam], which works out in some detail some of the relations between (2.3.1) and (2.3.2)-(2.3.3) viewed as a system over the ring $\mathbb{R}[\sigma_1,\sigma_2]$).

As another example a natural notion of isomorphism for systems $\Sigma = (F,G,H)$, $\Sigma' = (F',G',H')$ over a ring R is: Σ and Σ' are isomorphic if there exists an $n\times n$ matrix S over R, which is invertible over R, i.e. such that $\det(S)$ is a unit of R, such that $\Sigma' = \Sigma^S$. Taking $R = \mathbb{R}[\sigma_1,\sigma_2]$ and reinterpreting the σ_i as delays we see that the corresponding notion for the delay-differential systems is coordinate transformations with time delays which is precisely the right notion of isomorphism for studying for instance degeneracy phenomena, cf [Kap].

Finally applying the Laplace transform to (2.3.1) we find a transfer function $T(s,e^{-s},e^{-\alpha s})$, which is rational in s,e^{-s} and $e^{-\alpha s}$. It can also be obtained by taking the family of transfer functions

$$T_{\sigma_1,\sigma_2}(s) = H(\sigma_1,\sigma_2)(s-F(\sigma_1,\sigma_2))^{-1}G(\sigma_1,\sigma_2)$$

and then substituting e^{-s} for σ_1 and $e^{-\alpha s}$ for σ_2. Inversely given a transfer function $T(s)$ which is rational in $s,e^{-s},e^{-\alpha s}$ one may ask whether it can be realized as a system with delays which are multiples of 1 and α. Because the functions s, e^{-s}, $e^{-\alpha s}$ are algebraically independent (if α is incommensurable with 1), there is a unique rational function $\tilde{T}(s,\sigma_1,\sigma_2)$ such that $T(s) = \tilde{T}(s,e^{-s},e^{-\alpha s})$ and the realizability of $T(s)$ by means of a delay system, say a system with transmission lines, is now mathematically equivalent with realizing the two parameter family of transfer functions $T(s,\sigma_1,\sigma_2)$ by a family of systems which depends polynomially on σ_1, σ_2.

2.4 2-d and n-d Systems

Consider a linear discrete time system with direct feed-through term

$$x(t+1) = Fx(t) + Gu(t), \quad y(t) = Hx(t) + Ju(t) \qquad (2.4.1)$$

The associated input/output operator is a convolution operator, viz. (cf. (1.2.b))

$$y(t) = \sum_{i=0}^{t} A_i u(t-i), \; A_0 = J, \; A_i = HF^{i-1}G \qquad (2.4.2)$$

$$\text{for} \quad i = 1,2,\ldots$$

Now there is an obvious (north-east causal) more dimensional generalization of the convolution operator (2.4.2), viz.

$$y(h,k) = \sum_{i=0}^{h} \sum_{j=0}^{k} A_{i,j} u(h-i,k-j), \; h,k = 0,1,2,\ldots \qquad (2.4.3)$$

A (Givone-Roesser) realization of such an operators is a "2-d system"

$$\begin{aligned} x_1(h+1,k) &= F_{11}x_1(h,k) + F_{12}x_2(h,k) + G_1u(h,k) \\ x_2(h,k+1) &= F_{21}x_1(h,k) + F_{22}x_2(h,k) + G_2u(h,k) \qquad (2.4.4) \\ y(h,k) &= H_1x_1(h,k) + H_2x_2(h,k) + Ju(h,k) \end{aligned}$$

which yields an input/output operator of the form (2.4.3) with the $A_{i,j}$ determined by the power series development of the 2-d transfer function $T(s_1,s_2)$

$$\sum_{i,j} A_{i,j} s_1^{-i} s_2^{-j} = T(s_1,s_2) = (H_1 \; H_2)\left(\begin{pmatrix} s_1 I_{n_1} & 0 \\ 0 & s_n I_{n_2} \end{pmatrix} - \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}\right)^{-1} \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + J$$

where I_r is the $r \times r$ unit matrix and n_1 and n_2 are the dimensions of the state vectors x_1 and x_2. There are obvious generalizations to n-d systems, $n \geq 3$. The question now arises whether every proper 2-d matrix transfer function can indeed by so realized. (cf. [Eis] or [So2] for a definition of proper.) A way to approach this is to treat one of the s_i as a parameter, giving us a realization with parameters problem.

More precisely let R_g be the ring of all proper rational functions in s_1. In the 2-d case this is a principal ideal domain which simplifies things considerably. Now consider $T(s_1,s_2)$ as a proper rational function in s_2 with coefficients in R_g. This transfer function can be realized giving us a discrete time system over R_g defined by the quadruple of matrices $(F(s_1), G(s_1), H(s_1), J(s_1))$. Each of these matrices is proper as a function of s_1 and hence can be realized by a quadruple of constant matrices. Suppose that

$$(F_F,G_F,H_F,J_F) \quad \text{realizes} \quad F(s_1)$$

$$(F_G,G_G,H_G,J_G) \quad \text{realizes} \quad G(s_1)$$

$$(F_H,G_H,H_H,J_H) \quad \text{realizes} \quad H(s_1)$$

$$(F_J,G_J,H_J,J_J) \quad \text{realizes} \quad J(s_1)$$

Then, as is easily checked, a realization in the sense of (2.4.4) is defined by

$$F = \begin{pmatrix} F_{11} & F_{12} \\ & \\ F_{21} & F_{22} \end{pmatrix} = \left(\begin{array}{c|cccc} J_F & H_F & H_G & 0 & 0 \\ \hline G_F & F_F & 0 & 0 & 0 \\ 0 & 0 & F_G & 0 & 0 \\ G_H & 0 & F_H & 0 & 0 \\ 0 & 0 & 0 & 0 & F_J \end{array}\right),$$

$$G = \begin{pmatrix} G_1 \\ \\ G_2 \end{pmatrix} = \left(\begin{array}{c} J_G \\ \hline 0 \\ G_G \\ 0 \\ G_J \end{array}\right)$$

$$H = (H_1 \quad H_2) = (J_F \mid 0 \quad 0 \quad H_H \quad H_J), \quad J = J_J$$

his is the procedure followed in [Eis]; a somewhat different pproach, with essentially the same initial step (i.e. realization ith parameters, or realization over a ring) is followed in [So2].

.5 Parameter Uncertainty

Suppose that we have a system $\Sigma = (F,G,H)$ but that we are ncertain about some of its parameters, i.e. we are uncertain bout the precise value of some of the entries of F,G or H. That s, what we really have is a family of systems $\Sigma(\beta)$, where β uns through some set B of parameter values, which we assume ompact. For simplicity assume that we have a one input-one out- ut system. Let the transfer function of $\Sigma(\beta)$ be $T_\beta(s) =$ $f_\beta(s)/g_\beta(s)$. Now suppose we want to stabilize Σ by a dynamic utput feedback loop with transfer function $P(s) = \phi(s)/\psi(s)$, till being uncertain about the value of β. The transfer func- ion of the resulting total system is $T(s)/(1-T(s)P(s))$. So we hall have succeeded if we can find polynomials $\phi(s)$ and $\psi(s)$ uch that for all $\beta \in B$ all roots of

$$g_\beta(s)\psi(s) - f_\beta(s)\phi(s)$$

re in the left halfplane, possibly with the extra requirement hat $P(s)$ be also stable. The same mathematical question arises rom what has been named the blending problem, cf [Ta1]. It can- ot always be solved. In the special but important case where the ncertainty is just a gain factor, i.e. in the case that B is an nterval $[b_1,b_2]$, $b_2 > b_1 > 0$ and $T_\beta(s) = \beta T(s)$, where $T(s)$ s a fixed transferfunction, the problem is solved completely in Ta1].

. THE CLASSIFICATION OF FAMILIES. FINE MODULI SPACES

.1 Introductory and Motivational Remarks

(Why classifying families is essentially more difficult than lassifying systems and why the set of isomorphism classes of single) systems should be topologized.)

Obviously the first thing to do when trying to classify fami- ies up to isomorphism is to obtain a good description of the set f isomorphism classes of (single) systems over a field k, that s to obtain a good description of the sets $L_{m,n,p}(k)/GL_n(k) =$ $M_{m,n,p}(k)$ and of the quotient map $L_{m,n,p}(k) \to M_{m,n,p}(k)$. This ill be done below in section 3.2 for the subset of isomorphism lasses (or sets of orbits) of completely reachable systems. This s not particularly difficult (and also well known) nor is it verly complicated to extend this to a description of all of $M_{m,n,p}(k) = L_{m,n,p}(k)/GL_n(k)$, cf [Haz 6]. Though, as we shall see, there are, for the moment, good mathematical reasons, to limit ourselves to cr systems and families of cr systems, or, dually to limit ourselves to co systems.

Now let us consider the classification problem for families of systems. For definiteness sake suppose we are interested (cf. 2.1 and 2.3 above e.g.) in real families of systems $\Sigma(\sigma) = (F(\sigma), G(\sigma), H(\sigma))$ which depend continuously on a real parameter $\sigma \in \mathbb{R}$. The obvious, straightforward and in fact right thing to do is to proceed as follows. For each $\sigma \in \mathbb{R}$ we have a system $\Sigma(\sigma)$, and hence a point $\phi(\sigma) \in M_{m,n,p}(\mathbb{R}) = L_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$, the set of isomorphism classes or, equivalently, the set of orbits in $L_{m,n,p}$ under the action $(\Sigma,S) \mapsto \Sigma^S$ of $GL_n(\mathbb{R})$ on $L_{m,n,p}(\mathbb{R})$. This defines a map $\phi(\Sigma): R \to M_{m,n,p}(\mathbb{R})$, and one's first guess would be that two families Σ, Σ' are isomorphic iff their associated maps $\phi(\Sigma)$, $\phi(\Sigma')$ are equal. However, things are not that simple as the following example in $L_{1,2,1}(\mathbb{R})$ shows.

Example

$$\Sigma(\sigma) = \left(\begin{pmatrix} 1 & 0 \\ \sigma^2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,2) \right),$$

$$\Sigma'(\sigma) = \left(\begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,2\sigma) \right) \tag{3.1.1}$$

For each $\sigma \in \mathbb{R}$, $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic via $T(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ if $\sigma \neq 0$ and via $T(\sigma) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ if $\sigma = 0$. Yet they are not isomorphic as continuous families, meaning that there exists no continuous map $\mathbb{R} \to GL_2(\mathbb{R})$, $\sigma \mapsto T(\sigma)$, such that $\Sigma'(\sigma) = \Sigma(\sigma)^{T(\sigma)}$ for all $\sigma \in \mathbb{R}$. One might guess that part of the problem is topological. Indeed, it is in any case sort of obvious that one should give $M_{m,n,p}(\mathbb{R})$ as much structure as possible. Otherwise the map $\phi(\Sigma): \mathbb{R} \to M_{m,n,p}(\mathbb{R})$ does not tell us whether it could have come from a continuous family. (Of course if $\Sigma(\sigma)$ is a continuous family over $\mathbb{R}$ giving rise to $\phi(\Sigma)$ and $S \in GL_n(\mathbb{R})$ is such that $\Sigma(0)^S \neq \Sigma(0)$ then the discontinuous family $\Sigma'(\sigma)$, $\Sigma'(\sigma) = \Sigma(\sigma)$ for $\sigma \neq 0$, $\Sigma'(0) = \Sigma(0)^S$ given rise to the same map). Similarly we would like to have $\phi(\Sigma)$ analytic if Σ is an analytic family, polynomial if Σ is polynomial, differentiable if Σ is differentiable, ...

One reason to limit oneself to cr systems is now that the natural topology (which is the quotient topology for $\pi: L_{m,n,p}(\mathbb{R}) \to M_{m,n,p}(\mathbb{R})$) will not be Hausdorff unless we limit ourselves to cr systems. (It is clear that one wants to put in at least all co,cr systems).

There are more reasons to topologize $M_{m,n,p}(\mathbb{R})$ and more generally $M_{m,n,p}(k)$, where k is any field. For one thing it would be nice if $M_{m,n,p}(\mathbb{R})$ had a topology such that the isomorphism classes of two systems Σ and Σ' were close together if and only if their associated input/output maps were close together (in some suitable operator topology; say the weak topology); a requirement which is also relevant to the consistency requirement of maximum likelihood identification of systems, cf. [De, DDH, DH, DS, Han]. Yet topologizing $M_{m,n,p}(\mathbb{R})$ does not remove the problem posed by example (3.1.1). Indeed, giving $M_{m,n,p}(\mathbb{R})$ the quotient topology inherited from $L_{m,n,p}(\mathbb{R})$ the maps defined by the families Σ and Σ' of example (3.1.1) are both continuous.

Restricting ourselves to families consisting of cr systems (or dually to families of co systems), however, will solve the problem posed by example (3.1.1). This same restriction will also see to it that the quotient topology is Hausdorff and it will turn out that $M^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$ is naturally a smooth differentiable manifold. From the algebraic geometric point of view we shall see that the quotient $L^{cr}_{m,n,p}/GL_n$ exists as a smooth scheme defined over $\mathbb{Z}$. It is also pleasant to notice that for pairs of matrices (F,G) the prestable ones (in the sense of [Mu] are precisely the completely reachable ones [Ta2] and they are also the semi-stable points of weight one, [BH].

Ideally it would also be true that every continuous, differentiable, polynomial,... map $\phi : \mathbb{R} \to M^{cr}_{m,n,p}(\mathbb{R})$ comes from a conuous, differentiable, polynomial,... family. This requires assigning to each point of $M^{cr}_{m,n,p}(\mathbb{R})$ a system represented by that point and to do this in an analytic manner. This now really requires a slightly more sophisticated definition of family than we have used up to now, cf. 3.4 below. And indeed to obtain e.g. all continuous map of say the circle into $M_{m,n,p}(\mathbb{R})$ as maps associated to a family one also needs the same more general concept of families of system over the circle.

3.2 <u>Description of the Quotient Set (or Set of Orbits)</u> $L^{cr}_{m,n,p}(k)/GL_n(k)$.

Let k be any field, and fix $n,m,p \in \mathbb{N}$. Let

$$J_{n,m} = \{(0,1),(0,2),\dots,(0,m);\ (1,1),\dots,(1,m);\ \dots\ ;\ (n,1),\dots,(n,m)\}\ , \qquad (3.2.1)$$

lexicographically ordered (which is the order in which we have written down the $(n+1)m$ elements of $J_{n,m}$). We use $J_{n,m}$ to label the columns of the matrix $R(F,G)$, $F \in k^{n\times n}$, $G \in k^{n\times m}$, cf. 1.3 above, by assigning the label (i,j) to the j-th column of the block F^iG.

A subset $\alpha \subset J_{n,m}$ is called *nice* if $(i,j) \in \alpha \Rightarrow (i-1,j) \in \alpha$ or $i = 0$ for all i,j. A nice subset with precisely n elements is called a *nice selection*. Given a nice selection α, a *successor index* of α is an element $(i,j) \in J_{n,m}\backslash\alpha$ such that $\alpha \cup \{(i,j)\}$ is nice. For every $j_0 \in \{1,\dots,m\}$ there is precisely one successor index (i,j) of α with $j = j_0$. This successor index will be denoted $s(\alpha,j_0)$.

Pictorially these definitions look as follows. We write down the elements of $J_{n,m}$ in a square as follows $(m=4,\ n=5)$

```
(0,1)  (1,1)  (2,1)  (3,1)  (4,1)  (5,1)
(0,2)  (1,2)  (2,2)  (3,2)  (4,2)  (5,2)
(0,3)  (1,3)  (2,3)  (3,3)  (4,3)  (5,3)
(0,4)  (1,4)  (2,4)  (3,4)  (4,4)  (5,4)
```

Using dots to represent elements of $J_{n,m}$ and x's to represent elements of α the following pictures represent respectively a nice subset, a not nice subset and a nice selection.

```
. . . . . .    . x x . . .    . . . . . .
x x x . . .    x . . x . .    x x . . . .
x . . . . .    . x . . . .    . . . . . .
. . . . . .    x x . . . .    x x x . . .
```

The successor indices of the nice selection α of the third picture above are indicated by *'s in the picture below

```
* . . . . .
x x * . . .
* . . . . .                 (3.2.2)
x x x * . .
```

We shall use $L_{m,n}(k)$ to denote the set of all pairs of matrices (F,G) over k of sizes $n\times n$ and $n\times m$ respectively;

$L^{cr}_{m,n}(k)$ denotes the subset of completely reachable pairs (cf. 1.3 above). For each subset $\beta \in J_{n,m}$ and each $(F,G) \in L_{m,n}(k)$ we shall use $R(F,G)_\beta$ to denote the matrix obtained from $R(F,G)$ by removing all columns whose index is not in β.

With this terminology and notation we have the following lemma.

3.2.3 Nice Selection Lemma

Let $(F,G) \in L^{cr}_{m,n}(k)$. Then there is a nice selection α such that $\det(R(F,G)_\alpha) \neq 0$.

Proof. Let α be a nice subset of $J_{n,m}$ such that the columns of $R(F,G)_\alpha$ are linearly independent and such that α is maximal with respect to this property. Let

$$\alpha = \{(0,j_1),\ldots,(i_1,j_1); \quad (0,j_2),\ldots,(i_2,j_2); \;\ldots\; ; (0,j_s),\ldots,(i_s,j_s)\}.$$

By the maximality of α we know that the successor indices $s(\alpha,j)$, $j = 1,\ldots,m$ are linearly dependent on the columns of $R(F,G)$. I.e. the columns with indices $(i_1+1,j_1),\ldots,(i_s+1,j_s)$ and $(0,t)$, $t \in \{1,\ldots,m\}\backslash\{j_1,\ldots,j_s\}$ are linearly dependent on the columns of $R(F,G)_\alpha$. Suppose now that with induction we have proved that all columns with indices $(i_r+\ell,j_r)$, $r = 1,\ldots,s$ and $(\ell-1,t)$, $t \in \{1,\ldots,m\}\backslash\{j_1,\ldots,j_s\}$ are linearly dependent on the columns of $R(F,G)_\alpha$, $\ell \geq 1$. This gives us certain relations

$$F^{\ell-1}G_t = \sum_{(i,j)\in\alpha} a(i,j)F^iG_j, \quad F^{i_r+\ell}G_{j_r} = \sum_{(i,j)\in\alpha} b(i,j)F^iG_j$$

(where G_t denotes the t-th column of G). Multiplying on the left with F we find expressions

$$F^{\ell}G_t = \sum_{(i,j)\in\alpha} a(i,j)F^{i+1}G_j,$$

$$F^{i_r+\ell+1}G_{j_r} = \sum_{(i,j)\in\alpha} b(i,j)F^{i+1}G_j$$

expressing $F^\ell G_t$ and $F^{i_r+\ell+1}G_{j_r}$ as linear combination of those columns of $R(F,G)$ whose indices are either in α or a successor index of α. The latter are in turn linear combinations of the columns of $R(F,G)_\alpha$, so that we have proved that all columns of $R(F,G)$ are linear combinations of the columns of $R(F,G)_\alpha$. Now (F,G) is cr so that $\operatorname{rank}(R(F,G)) = n$, so that α must have had n elements, proving the lemma.

For each nice selection α we define

$$U_\alpha(k) = \{(F,G,H) \in L_{m,n,p}(k) \mid \det(R(F,G)_\alpha) \neq 0\} \quad (3.2.4)$$

Recall that $GL_n(k)$ acts on $L_{m,n,p}(k)$ by $(F,G,H)^S = (SFS^{-1},SG,HS^{-1})$.

3.2.5. Lemma. U_α is stable under the action of $GL_n(k)$ on $L_{m,n,p}(k)$. For each $\Sigma \in (F,G,H) \in U_\alpha$ there is precisely one $S \in GL_n(k)$ such that $R(\Sigma^S)_\alpha = R(SFS^{-1},SG)_\alpha = I_n$, the $n\times n$ identity matrix.

Proof. We have

$$R(\Sigma^S) = R(SFS^{-1},SG) = SR(F,G) = S\,R(\Sigma) \quad (3.2.6)$$

It follows that $R(\Sigma^S)_\alpha = SR(\Sigma)_\alpha$, which proves the first statement. It also follows that if we take $S = R(F,G)_\alpha^{-1}$ then $R(\Sigma^S)_\alpha = I_n$ and this is also the only S which does this because in the equation $S\,R(\Sigma)_\alpha = R(\Sigma^S)_\alpha$, $R(\Sigma)_\alpha$ has rank n.

3.2.7. Lemma. Let $x_1,\dots,x_m$ be an arbitrary m-tuple of n-vectors over k and let α be a nice selection. Then there is precisely one pair $(F,G) \in L^{cr}_{m,n}(k)$ such that $R(F,G)_\alpha = I_n$, $R(F,G)_{s(\alpha,j)} = x_j$, $j = 1,\dots,m$.

Proof (by sufficiently complicated example). Suppose $m = 4$, $n = 5$ and that α is the nice selection of (3.2.2) above. Then we can simply read off the desired F,G. In fact we find $G_1 = x_1$, $G_2 = e_1$, $G_3 = x_3$, $G_4 = e_2$, $F_1 = e_3$, $F_2 = e_4$, $F_3 = x_2$, $F_4 = e_5$, $F_5 = x_4$. Writing down a fully general proof is a bit tedious and notationally a bit cumbersome and it should now be trivial exercise.

3.2.8. Corollary. The set of orbits $U_\alpha(k)/GL_n(k)$ is in bijective correspondence with $k^{nm} \times k^{pn}$, and $U_\alpha(k) \cong GL_n(k) \times (k^{nm}\times k^{pn})$ (as sets with $GL_n(k)$-action, where $GL_n(k)$ acts on $GL_n(k) \times (k^{nm} \times k^{pn})$ by multiplication on the left on the first factor).

Proof. This follows immediately from lemma 3.2.5 together with lemma 3.2.7. Indeed given $\Sigma = (F,G,H)_\alpha \in U_\alpha$. Take $S = R(F,G)_\alpha^{-1}$ and let $(F',G',H') = \Sigma^S$. Now define $\phi : U_\alpha(k) \to GL_n(k) \times (k^{nm} \times k^{pn})$ by assigning to (F,G,H) the matrix S^{-1}, the m n-vectors $R(\Sigma^S)_{s(\alpha,j)}$, $j = 1,\dots,m$ and the $p \times n$ matrix H'. Inversely given a $T \in GL_n(k)$, m n-vectors x_j, $j = 1,\dots,m$ and a $p \times n$ matrix y. Let $(F',G') \in L^{cr}_{m,n}(k)$ be the unique pair such that $R(F',G')_\alpha = I_n$, $R(F',G')_{s(\alpha,j)} = x_j$, $j = 1,\dots,m$. Take $H' = y$ and define

$$\psi : GL_n(k) \times (k^{nm} \times k^{pn}) \to U_\alpha(k)$$

by

$$\psi(T,(x,y)) = (F',G',H')^T .$$

It is trivial to check that $\psi\phi = \text{id}$, $\psi\phi = \text{id}$. It is also easy to check that ϕ commutes with the $GL_n(k)$-actions.

3.2.9. The $c_{\#\alpha}$ (local) canonical forms. For each $\Sigma \in U_\alpha(k)$ we denote with $c_{\#\alpha}(\Sigma)$ the triple:

$$c_{\#\alpha}(\Sigma) = \Sigma^S \quad \text{with} \quad S = R(\Sigma)_\alpha^{-1} \tag{3.2.10}$$

i.e. $c_{\#\alpha}(\Sigma)$ is the unique triple Σ' in the orbit of Σ such that $R(\Sigma')_\alpha = I_n$. Further if $z \in k^{mn} \times k^{np}$, then we let $(F_\alpha(z), G_\alpha(z), H_\alpha(z))$ be the triple $\psi(I_n,z)$; that is if $z = ((x_1,\dots,x_m),y)$ $(F_\alpha(z), G_\alpha(z), H_\alpha(z))$ is the unique triple such that:

$$R(F_\alpha(z), G_\alpha(z))_\alpha = I_n, \quad R(F_\alpha(z), G_\alpha(z))_{\sigma(\alpha,j)} = x_j,$$
$$H_\alpha(z) = y \ , \quad z \in ((x_1,\dots,x_m), y) \in k^{nm} \times k^{pn} \tag{3.2.11}$$

3.2.12. Remark. Let $\pi_\alpha : U_\alpha(k) \to k^{nm} \times k^{pn}$ be equal to $\psi : U_\alpha(k) \to GL_n(k) \times (k^{nm} \times k^{pn})$ followed by the projection on the second factor. Then $\tau_\alpha : z \mapsto (F_\alpha(Z), G_\alpha(z), H_\alpha(z))$ is a section of π_α (meaning that $\pi_\alpha \tau_\alpha = \text{id}$), and $c_{\#\alpha}(\tau_\alpha) = \tau_\alpha$. Of course, π_α induces a bijection $U_\alpha(k)/GL_n(k) \to k^{nm} \times k^{pn}$.

3.2.13. Description of the set of orbits. $L^{cr}_{m,n,p}(k)/GL_n(k)$. Order the set of all nice selections from $J_{n,m}$ in some way.

For each $\Sigma \in L^{cr}_{m,n,p}$ let $\alpha(\Sigma)$ be the first nice selection in this ordering such that $R(F,G)_{\alpha(\Sigma)}$ is non-singular. Now assign to Σ the triple $c_{\#\alpha(\Sigma)}(\Sigma)$. This assigned to each $\Sigma \in L^{cr}_{m,n,p}(k)/GL_n(k)$ one particular well defined element in its orbit and this hence gives complete description of the set of orbits

$$L^{cr}_{m,n,p}(k)/GL_n(k).$$

3.3 <u>Topologizing</u> $L^{cr}_{m,n,p}(k)/GL_n(k) = M^{cr}_{m,n,p}(k)$

3.3.1 <u>A more "homogeneous" description of</u> $M^{cr}_{m,n,p}(k)$. The description of the set of orbits of $GL_n(k)$ acting on $L^{cr}_{m,n,p}(k)$ given in 3.2.13 is highly lopsided in the various possible nice selections α. A more symmetric description of $M^{cr}_{m,n,p}(k)$ is obtained as follows. For each nice selection α, let $V_\alpha(k) = k^{nm} \times k^{pn}$ and let for each second nice selection β:

$$V_{\alpha\beta}(k) = \{z \in V_\alpha | \det(R(F_\alpha(z), G_\alpha(z))_\beta) \neq 0\} \qquad (3.3.2)$$

That is, under the section $\tau_\alpha : V_\alpha(k) \to U_\alpha(k)$ of 3.2 above which picks out precisely one element of each orbit in $U_\alpha(k)$ $V_{\alpha\beta}(k)$ corresponds to those orbits which are also in $U_\beta(k)$; or equivalently $V_{\alpha\beta}(k) = \pi_\alpha(U_\alpha(k) \cap U_\beta(k))$. We now glue the $V_\alpha(k)$, α nice, together along the $V_{\alpha\beta}(k)$ by means of the identifications:

$$\phi_{\alpha\beta}: V_{\alpha\beta}(k) \to V_{\beta\alpha}(k),\ \phi_{\alpha\beta}(z) = z' \Leftrightarrow$$

$$(F_\alpha(z), G_\alpha(z), H_\alpha(z))^S = (F_\alpha(z'), G_\alpha(z'), H_\alpha(z'),$$

$$S = R(F_\alpha(z), G_\alpha(z))^{-1}_\beta \qquad (3.3.3)$$

Then, as should be clear from the remarks made just above, $M^{cr}_{m,n,p}(k)$ is the union of the $V_\alpha(k)$ with for each pair of nice selections α, β, $V_{\alpha\beta}(k)$ identified with $V_{\beta\alpha}(k)$ according to (3.3.3).

3.3.4. <u>The analytic varieties</u> $M^{cr}_{m,n,p}(\mathbb{R})$ <u>and</u> $M^{cr}_{m,n,p}(\mathbb{C})$. Now let $k = \mathbb{R}$ or $\mathbb{C}$ and give $V_\alpha(k) = k^{nm} \times k^{pn}$ its usual (real) analytic structure. The subsets $V_{\alpha\beta}(k) \subset V_\alpha(k)$ are then open subsets and the $\phi_{\alpha\beta}(k)$ are analytic diffeomorphisms. It follows that $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{cr}_{m,n,p}(\mathbb{C})$ will be respectively a real analytic (hence certainly C^∞) manifold and a complex analytic manifold, provided we can show that they are Hausdorff.

First notice that if we give $L_{m,n,p}(\mathbb{R})$ and $L_{m,n,p}(\mathbb{C})$ the topology of $\mathbb{R}^{mn+n^2+np}$ and $\mathbb{C}^{n^2+nm+np}$ respectively and the open subsets $U_\alpha(k)$ and $L^{cr}_{m,n,p}(k)$, $k = \mathbb{R}, \mathbb{C}$ the induced topology, then the quotient topology for π_α: $U_\alpha(k) \to V_\alpha(k)$ is precisely the topology resulting from the identification $V_\alpha(k) \simeq k^{nm} \times k^{pn}$. It follows that the topology of $M^{cr}_{m,n,p}(k)$ is the quotient topology of $L^{cr}_{m,n,p}(k) \to L^{cr}_{m,n,p}(k)/GL_n(k) = M^{cr}_{m,n,p}(k)$.

Now let $G_{n,m(n+1)}(k)$ be the Grassmann variety of n-planes in $m(n+1)$-space. For each (F,G), $R(F,G)$ is an $n \times m(n+1)$ matrix of rank n which hence defines a unique point of $G_{n,m(n+1)}(k)$. Because $R(SFS^{-1},SG) = SR(F,G)$ we have that (F,G) and $(F,G)^S$ define the same point in Grassmann space. It follows that by forgetting H we have defined a map:

$$\bar{R}: M^{cr}_{m,n,p}(k) \to G_{n,m(n+1)}(k),\ (F,G) \mapsto \text{subspace spanned by the rows of } R(F,G). \tag{3.3.5}$$

In addition we let h: $M^{cr}_{m,n,p}(k) \to k^{(n+1)^2mp}$ be the map induced by:

$$\tilde{h}(F,G,H) = \begin{pmatrix} A_1 & A_2 & \cdots & A_{n+1} \\ A_2 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{n+1} & & \cdots & A_{2n+1} \end{pmatrix},$$

$$A_i = HF^{i-1}G, \quad i = 1,\dots,2n+1 \tag{3.3.6}$$

It is not particularly difficult to show ([Haz 1-3], cf. also the realization algorithm in 5.2 below) that the combined map $(\bar{R},h): M^{cr}_{m,n,p}(k) \to G_{n,m(n+1)}(k) \times k^{(n+1)^2mp}$ is injective. By the quotient topology remarks above it is then a topological embedding, proving that $M^{cr}_{m,n,p}(k)$ is a Hausdorff topological space. So we have:

3.3.7. Theorem. $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{cr}_{m,n,p}(\mathbb{C})$ are smooth analytic manifolds. The sets $M^{cr,co}_{m,n,p}(\mathbb{R})$ and $M^{cr,co}_{m,n,p}(\mathbb{C})$ are analytic open sub-manifolds. (These are the sets of orbits of the cr and co systems, or equivalently the images of $L^{cr,co}_{m,n,p}(k)$ under $\pi: L^{cr}_{m,n,p}(k) \to M^{cr}_{m,n,p}(k)$, $k = \mathbb{R},\mathbb{C}$).

3.3.8. Remark. A completely different way of showing that the quotient space $M^{cr}_{m,n,p}(\mathbb{R})$ is a differentiable manifold is due to Martin and Krishnaprasad, [MK]. They show that with respect to a suitable invariant metric of $L^{cr,co}_{m,n,p}(k)$, $GL_n(k)$ acts properly discontinuously.

3.3.9. The algebraic varieties $M^{cr}_{m,n,p}(k)$. Now let k be any algebraically closed field. Giving $L_{m,n,p}(k) = k^{n^2+nm+np}$ the Zariski topology and $U_\alpha(k)$ the induced topology for each nice selection α. Then $U_\alpha(k) \simeq GL_n(k) \times V_\alpha(k)$, $V_\alpha(k) = k^{nm+np}$ also as algebraic varieties. The $V_{\alpha\beta}(k)$ are open subvarieties and the $\phi_{\alpha\beta}(k)$: $V_{\alpha\beta}(k) \to V_{\beta\alpha}(k)$ are isomorphisms of algebraic varieties. The map $(\bar{R},h)$ is still injective and it follows that $M^{cr}_{m,n,p}(k)$ has a natural structure of a smooth algebraic variety, with $M^{cr,co}_{m,n,p}(k)$ an open subvariety.

3.3.10. The scheme $M^{cr}_{m,n,p}$. As a matter of fact, the defining pieces of the algebraic varieties $M^{cr}_{m,n,p}(k)$, that is the $V_\alpha(k)$, and the gluing isomorphisms $\phi_{\alpha\beta}(k)$ are all defined over $\mathbb{Z}$. So there exists a scheme $M^{cr}_{m,n,p}$ over $\mathbb{Z}$ such that for all fields k the rational points over k, $M^{cr}_{m,n,p}(k)$, are precisely the orbits of $GL_n(k)$ acting on $L^{cr}_{m,n,p}(k)$. For details cf. section 4 below.

3.4. A universal family of linear dynamical systems

3.4.1. As has been remarked above it would be nice if we could attach in a continuous way to each point of $M_{m,n,p}(k)$ a system over k representing that point. Also it would be pleasant if every appropriate map from a parameter space V to $M^{cr}_{m,n,p}$ came from a family over V. Recalling from 2.2 above that systems over a ring R can be reinterpreted as families over $\mathrm{Spec}(R)$, this would mean that the isomorphism classes of systems over R would correspond bijectively with the R-rational points $M^{cr}_{m,n,p}(R)$ of the scheme $M^{cr}_{m,n,p}$ over $\mathbb{Z}$, cf. 3.3.10.

Both wishes, if they are to be fulfilled require a slightly more general definition of system than we have used up to now. In the case of systems over a ring R the extra generality means that instead of considering three matrices F,G,H over R, that is three homomorphisms $G: R^m \to R^n$, $F: R^n \to R^n$, $H: R^n \to R^p$ we now generalize to the definition: a projective system over R consists of a projective module X as state module together with three homomorphisms $G: R^m \to X$, $F: X \to X$, $H: X \to R^p$. Thus the extra generality sits in the fact that the state R-module X is not required to be free, but only projective. The geometric counterpart of this is a vectorbundle, cf. below in 3.4.2 for the precise definition of a family and the role the vectorbundle plays.

In some circumstances it appears to be natural, in any case as an intermediate step, to consider even more general families. Thus over a ring R it makes perfect sense to consider arbitrary modules as state modules, and indeed these turn up naturally when doing "canonical" realization theory, cf. [Eil, Ch. XVI], which in terms of families means that one may need to consider more general fibrations by vector spaces than locally trivial ones.

3.4.2. <u>Families of linear dynamical systems (over a topological space).</u> Let V be a topological space. A continuous family Σ of real linear dynamical systems over V (or parametrized by V) consists of:

(a) a vectorbundle E over V
(b) a vectorbundle endomorphism $F: E \to E$
(c) a vectorbundle morphism $G: V \times \mathbb{R}^m \to E$
(d) a vectorbundle morphism $H: E \to V \times \mathbb{R}^p$

For each $v \in V$ let $E(v)$ be the fibre of E over v. Then we have homomorphisms of vector spaces $G(v):\{v\} \times \mathbb{R}^m \to E(v)$, $F(v): E(v) \to E(v)$, $H(v): E(v) \to \{v\} \times \mathbb{R}^p$. Thus choosing a basis in $E(v)$, and taking the obvious bases in $\{v\} \times \mathbb{R}^m$ and $\{v\} \times \mathbb{R}^p$ we find a triple of matrices $\bar{F}(v)$, $\bar{G}(v)$, $\bar{H}(v)$. Thus the data listed above do define a family over V in the sense that they assign to each $v \in V$ a linear system. Note however that there is no natural basis for $E(v)$ so that the system is really only defined up to base change, i.e. up to the $GL_n(\mathbb{R})$ action, so that what the data (a)-(d) really do is assign a point of $M_{m,n,p}(\mathbb{R})$ to each point $v \in V$.-

As E is a vectrobundle we can find for each $v \in V$ an open neighborhood W and n-sections $s_1,\dots,s_n$: $W \to E|_W$ such that $s_1(w),\dots,w_n(w) \in E(w)$ are linearly independent for all $w \in W$. Writing out matrices for $F(w)$, $G(w)$, $H(w)$ with respect to the basis $s_1(w),\dots,s_n(w)$ (and the obvious bases in $\{w\} \times \mathbb{R}^m$ and $\{w\} \times \mathbb{R}^p$), we see that over W the family Σ can indeed be described as a triple of matrices depending continuously on parameters. Inversely if $(\bar{F},\bar{G},\bar{H},)$ is a triple of matrices depending continuously on a parameter $v \in V$, then $E = V \times \mathbb{R}^n$, $F(v,x) = (v,\bar{F}(v)x)$, $F(v,u) = (v,\bar{G}(v)u)$, $H(v,x) = (v,\bar{H}(v)x)$ define a family as described above. Thus locally the new definition agrees (up to isomorphism) with the old intuitive one we have been using up to now; globally it does not.

Here the appropriate notion of isomorphism is of course: two families $\Sigma = (E;\ F,\ G,\ H)$ and $\Sigma' = (E';\ F',\ G',\ H')$ over V are isomorphic if there exists a vectorbundle isomorphism $\phi: E \to E'$ such that $F'\phi = \phi F$, $\phi G = G'$, $H = H'\phi$.

3.4.3. Other kinds of families of systems. The appropriate definitions of other kinds of families are obtained from the one above by means of minor and obvious adjustments. For instance, if V is a differentiable (resp. real analytic) manifold then a differentiable (resp. real analytic) family of systems consists of a differentiable vector bundle E with differentiable morphisms F, G, H (resp. an analytic vectorbundle with analytic morphisms F, G, H). And of course isomorphisms are supposed to be differentiable (resp. analytic).

Similarly if V is a scheme (over k) then an algebraic family consists of an algebraic vectorbundle E over V together with morphisms of algebraic vectorbundles $F: E \to E$, $G: V \times \mathbb{A}^m \to E$, $H: E \to V \times \mathbb{A}^p$, where $\mathbb{A}^r$ is the (vectorspace) scheme $\mathbb{A}^r(R) = R^r$ (with the obvious R-module structure).

Still more variations are possible. E.G. a complex analytic family (or holomorphic family) over a complex analytic space V would consist of a complex analytic vectorbundle E with complex analytic vectorbundle homomorphisms $F: E \to E$, $G: V \times \mathbb{C}^m \to E$, $H: E \to V \times \mathbb{C}^p$.

3.4.4. Convention. From now on whenever we speak about a family of systems it will be a family in the sense of (3.4.2) and (3.4.3) above.

3.4.5. The canonical bundle over $G_{n,r}(k)$. Let $G_{n,r}(k)$ be the Grassman manifold of n-planes in r-space ($r > n$). Let $E(k) \to G_{n,r}(k)$ be the fibre bundle whose fibre over $x \in G_{n,r}(k)$ is the n-plane in k^r represented by the point x. If $k = \mathbb{R}$ or $\mathbb{C}$ this is an analytic vector bundle over $G_{n,r}(k)$. More generally this defines an algebraic vectorbundle E over the scheme $G_{n,r}$.

this defines an algebraic vectorbundle E over the scheme $G_{n,r}$.

In terms of trivial pieces and gluing data this bundle can be described as follows. Let $M^{n\times r}_{reg}(k)$ be the space of all $n\times r$ matrices of rank n and let $\pi : M^{n\times r}_{reg}(k) \to G_{n,r}(k)$ be the map which associates to each $n\times r$ matrix of rank n the n-space in k^{nr} spanned by its row vectors. Then the fibre over $E(x)$ of E over $x \in G_{n,r}(k)$ is precisely the vector space of all linear combinations of any element in $\pi^{-1}(x)$. From this there results the following local pieces the gluing data description of $G_{n,r}(k)$ and $E(k)$. For each subset α of size n of $\{1,2,\dots,r\}$ let $U'_\alpha(k)$ be the set of all $n\times r$ matrices A such that A_α is invertible, let $V'_\alpha(k) = k^{n(r-n)}$ and for each $z \in V'_\alpha(k)$, $z = (z_1,\dots,z_{r-n})$, $z_i \in k^n$, let $A_\alpha(z)$ be the unique $n\times r$ matrix such that $(A_\alpha(z))_\alpha = I_n$ and $A_\alpha(z)_{t(j)} = z_j$ where $t(j)$ runs through the elements of $\{1,2,\dots,r\}-\alpha$ in the natural order, $j = 1,\dots,r-n$. Then $G_{n,r}(k)$ consists of the $V'_\alpha(k)$ glued together along the $V'_{\alpha\beta}(k) = \{z \in V'_\alpha(k) \mid A_\alpha(z)_\beta$ is invertible$\}$ by means of the isomorphisms:

$$\phi'_{\alpha\beta}(k)\colon V'_{\alpha\beta}(k) \xrightarrow{\sim} V'_{\beta\alpha}(k),\ z \mapsto z' \Leftrightarrow (A_\alpha(z)_\beta)^{-1}A_\alpha(z) = A_\beta(z') \tag{3.4.7}$$

(Note how very similar this is to the pieces and patching data description of $M^{cr}_{m,n,p}(k)$ given in 3.3.1 above; the reason is understandable if one observes that the map $R\colon L^{cr}_{m,n,p}(k) \to M^{n\times(n+1)m}_{reg}(k)$, induces a map $\bar{R}\colon M^{cr}_{m,n,p}(k) \to G_{n,(n+1)m}(k)$, which is compatible with the local pieces and patching data for the two spaces).

The bundle $E(k)$ over $G_{n,r}(k)$ can now be described as follows. Over each $V'_\alpha(k) \subset G_{n,r}(k)$ we can trivialize $E(k)$ as follows:

$$V'_\alpha(k) \times k^n \xrightarrow{\sim} E(k)|_{V'_\alpha(k)},\ (z,x) \mapsto x^T A_\alpha(z)\ . \tag{3.4.8}$$

It follows that the bundle $E(k)$ over $G_{n,r}(k)$ admits the following local pieces and patching data description which is compatible with the local pieces and patching data description given above for $G_{n,r}(k)$. The bundle $E(k)$ consists of the local pieces $E_\alpha(k) = V'_\alpha(k) \times k^n$ glued together along the $E_{\alpha\beta}(k) = V'_{\alpha\beta}(k) \times k^n$ by means of the isomorphisms:

$$\tilde{\phi}'_{\alpha\beta} : V'_{\alpha\beta}(k) \times k^n \xrightarrow{\sim} V'_{\beta\alpha}(k) \times k^n$$

$$(z,x) \mapsto (\phi'_{\alpha\beta}(z),\ (A_\alpha(z)_\beta)^T x) \tag{3.4.9}$$

The bundle which is really of interest to us is the dual bundle E^d to E described by the local pieces $E^d_\alpha(k) = V'_\alpha(k) \times k^n$ glued together by the patching data:

$$\tilde{\phi}^d_{\alpha\beta}: \; V'_{\alpha\beta}(k) \times k^n \rightrightarrows V'_{\beta\alpha}(k) \times k^n$$

$$(z,x) \mapsto (\phi'_{\alpha\beta}(z), (A_\alpha(z)_\beta)^{-1}x) \qquad (3.4.10)$$

(note that the gluing isomorphisms $\tilde{\phi}^d_{\alpha\beta}$ are compatible with the projections $E^d_\alpha(k) \to V'_\alpha(k)$ and the gluing isomorphisms $\phi'_{\alpha\beta}$ for $G_{n,r}(k)$; note also that all three sets of gluing data $\phi'_{\alpha\beta}$, $\tilde{\phi}'_{\alpha\beta}$, $\tilde{\phi}^d_{\alpha\beta}$ are transitive in the sense that $\tilde{\phi}^d_{\beta\gamma} \circ \tilde{\phi}^d_{\alpha\beta} = \tilde{\phi}^d_{\alpha\gamma}$ are similarly for the $\tilde{\phi}'$ and ϕ').

3.4.11. <u>The underlying vector bundle of the universal family over $M^{cr}_{m,n,p}(k)$.</u> The map $R: L^{cr}_{m,n,p}(k) \to M^{n\times(n+1)m}_{reg}(k)$, $(F,G,H) \mapsto R(F,G)$ induces a map.

$$\bar{R}: M^{cr}_{m,n,p}(k) \to G_{n,(n+1)m}(k) \qquad (3.4.12)$$

(because $R(\Sigma^S) = SR(\Sigma)$, $S \in GL_n(k)$).

If $k = \mathbb{R}$ or $\mathbb{C}$, (3.4.12) is a morphism between analytic manifolds. In general (3.4.12) defines a morphism between the schemes $M^{cr}_{m,n,p}$ and $G_{n,(n+1)m}$. Now let $E^u = \bar{R}^! E^d$, the pull-back by means of $\bar{R}$ of the "canonical" bundle E^d described above in (3.4.5).

Now recall that $M^{cr}_{m,n,p}(k)$ was obtained by gluing the various pieces $V_\alpha(k) = k^{nm} \times k^{pn}$ together, where α runs through all nice selections from $J_{n,m}$. In terms of this description $E^u(k)$ can be described as follows: $E^u(k)$ consists of pieces $E^u_\alpha(k) = V_\alpha(k) \times k^n = k^{nm} \times k^{pn} \times k^n$, one for each nice selection α. For each pair of nice selections $E^u_{\alpha\beta}(k) = V_{\alpha\beta}(k) \times k^n \subset V_\alpha(k) \times k^n$. Now for each pair of nice selections α,β let $\tilde{\phi}_{\alpha\beta}(k)$: $E^u_{\alpha\beta}(k) \to E^u_{\beta\alpha}(k)$ be the isomorphism:

$$\tilde{\phi}_{\alpha\beta}(k)(z,x) = (\phi_{\alpha\beta}(z), (R(F_\alpha(z), G_\alpha(z))_\beta)^{-1} x) \qquad (3.4.13)$$

where $\phi_{\alpha\beta}$: $V_{\alpha\beta}(k) \to V_{\beta\alpha}(k)$ is the isomorphism of 3.3 above (which describes how the $V_\alpha(k)$ should be glued together to give

$M_{m,n,p}(k)$, and $V_\alpha(k) \to U_\alpha(k)$, $z \mapsto (F_\alpha(z), G_\alpha(z), H_\alpha(z))$ is the section τ_α described above in (3.2.12). Then $E^u_\alpha(k)$ is obtained by gluing together the $E^u_\alpha(k)$ along the $E^u_{\alpha\beta}(k)$ by means of the isomorphisms (3.4.13).

3.4.14. Construction of a universal family of cr systems. Let $E^u(k)$ over $M^{cr}_{m,n,p}(k)$ be the bundle described above and view it as obtained via the patching data (3.4.13). Recall also that, cf. (3.3.3) above:

$$\begin{aligned}\phi_{\alpha\beta}(z) = z' \Leftrightarrow (F_\alpha(z), G_\alpha(z), H_\alpha(z))^S &= \\ (F_\beta(z'), G_\beta(z'), H_\beta(z')) & \qquad (3.4.15)\\ \text{with } S = R(F_\alpha(z), G_\alpha(z))^{-1}_\beta &\end{aligned}$$

For each nice selection α we now define a bundle endomorphism $F^u_\alpha(k)$ of $E^u_\alpha(k) = V_\alpha(k) \times k^n$ and bundle morphisms $G^u_\alpha(k)$: $V_\alpha(k) \times k^m \to E^u_\alpha(k)$, $H^u_\alpha(k)$: $E^u_\alpha(k) \to V_\alpha(k) \times k^p$. These are defined as follows:

$$\begin{aligned} F^u_\alpha(k)\ (z,x) &= (z, F_\alpha(z)x) \\ G^u_\alpha(k)\ (z,u) &= (z, G_\alpha(z)x) \qquad (3.4.16)\\ H^u_\alpha(k)\ (z,x) &= (z, H_\alpha(z)x) \end{aligned}$$

We now claim that these bundle morphisms are compatible with the gluing isomorphisms (3.4.13), which means that we must prove the commutativity of the diagram below for each pair of nice selections α, β.

$$\begin{array}{ccccccc} V_{\alpha\beta}\times k^m & \xrightarrow{G^u_\alpha} & E^u_{\alpha\beta} & \xrightarrow{F^u_\alpha} & E^u_{\alpha\beta} & \xrightarrow{H^u_\alpha} & V_{\alpha\beta}\times k^p \\ \downarrow{\scriptstyle \phi_{\alpha\beta}\times id} & & \downarrow{\scriptstyle \tilde\Phi_{\alpha\beta}} & & \downarrow{\scriptstyle \tilde\Phi_{\alpha\beta}} & & \downarrow{\scriptstyle \phi_{\alpha\beta}\times id} \\ V_{\beta\alpha}\times k^m & \xrightarrow{G^u_\beta} & E^u_{\beta\alpha} & \xrightarrow{F^u_\beta} & E^u_{\beta\alpha} & \xrightarrow{H^u_\beta} & V_{\beta\alpha}\times k^p \end{array} \qquad (3.4.17)$$

where we have abbreviated various notations in obvious ways. Now

$$\tilde{\phi}_{\alpha\beta}G^u_\alpha(z,u) = \tilde{\phi}_{\alpha\beta}(z,\ G_\alpha(z)\ u) \qquad \text{by (3.4.16)}$$
$$= (\phi_{\alpha\beta}(z),\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta\ G_\alpha\ (z)\ u) \qquad \text{by (3.4.13)}$$
$$= (\phi_{\alpha\beta}(z),\ G_\beta(z')\ u) \qquad \text{by (3.4.15)}$$
$$= G^u(\phi_{\alpha\beta}\times \mathrm{id}(z,\ u)) \qquad \text{by (3.4.16)}$$

proving the commutativity of the left most square of (3.4.17). Similarly:

$$\tilde{\phi}_{\alpha\beta}F^u_\alpha(z,x) = \tilde{\phi}_{\alpha\beta}(z,\ F_\alpha(z)\ x) \qquad \text{by (3.4.16)}$$
$$= (\phi_{\alpha\beta}(z),\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta\ F_\alpha(z)\ x) \qquad \text{by (3.4.13)}$$
$$= (\phi_{\alpha\beta}(z),\ F_\beta(z')\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta x) \qquad \text{by (3.4.15)}$$
$$= F^u_\beta\ \ \tilde{\phi}_{\alpha\beta}(z,x)$$

proving the commutativity of the middle square of (3.4.17). And finally, and completely analogously:

$$H^u_\beta\tilde{\phi}_{\alpha\beta}(z,x) = H^u_\beta(\phi_{\alpha\beta}(z),\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta\ x) \quad \text{by (3.4.13)}$$
$$= (\phi_{\alpha\beta}(z),\ H_\beta(z')\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta\ x) \qquad \text{by (3.4.16)}$$
$$= (\phi_{\alpha\beta}(z),\ H_\alpha(z)\ x)$$
$$= (\phi_{\alpha\beta}\times \mathrm{id})\ (H^u_\alpha(z,x))$$

proving the commutativity of the last square of (3.4.17).

Thus the F^u_α, G^u_α, H^u_α combine to define bundle morphisms $F^u(k)\colon E^u(k) \to E^u(k)$, $G^u\colon M^{cr}_{m,n,p}(k) \times k^m \to E^u(k)$, $H^u(k)\colon E^u(k) \to M^{cr}_{m,n,p}(k) \times k^p$.

If $k = \mathbb{R}$ or $\mathbb{C}$, $F^u(k)$, $G^u(k)$, $H^u(k)$ are morphisms of analytic vector bundles. Algebraically speaking the $F^u(k)$, $G^u(k)$, $H^u(k)$ for varying k are parts of morphisms of algebraic vector bundles over the scheme $M^{cr}_{m,n,p}$, which are defined over $\mathbb{Z}$.

3.4.18. The pullback construction. Let V be a topological space and $\phi: V \to M^{cr}_{m,n,p}(\mathbb{R})$ a continuous map. Let $\Sigma^u = (E^u; F^u, G^u, H^u)$ be the universal family of systems constructed above. Then associated to ϕ we have an induced family $\phi^!\Sigma^u$ over V (obtained by pullback). The precise formulas are as follows:

- $\phi^!E^u = \{(v,x) \in V \times E^u | \phi(v) = \pi(x)\}$, where $\pi: E^u \to M^{cr}_{m,n,p}(\mathbb{R})$ is the bundle projection; the bundle projection of $\phi^!E^u$ is defined by $(v,x) \to v$;
- $\phi^!F^u: (v,x) \mapsto (v, F^u x) \in \phi^!E^u$
- $\phi^!G^u: (v,u) \mapsto (v, G^u u) \in \phi^!E^u$
- $\phi^!H^u: (v,x) \mapsto (v, H^u x) \in \phi^!(M^{cr}_{m,n,p}(\mathbb{R}) \times R^p) = V \times \mathbb{R}^p$

Obviously $\phi^!E^u$ is (up to isomorphism) the family of systems over V such that the system over $v \in V$ is (up to isomorphism) the system over $\phi(v)$ in the family Σ^u.

If V and ϕ are differentiable (resp. real analytic) there results a differentiable (resp. real analytic) family over V. If $\phi: V \to M^{cr}_{m,n,p}(\mathbb{C})$ is a morphism of complex analytic manifolds there results a complex analytic family and on the algebraic-geometric side of things if $\phi: V \to M^{cr}_{m,n,p}$ is a morphism of schemes one finds thus an algebraic family over the scheme V.

3.4.19. The topological fine moduli theorem. Let V be a topological space and Σ a continuous family of completely reachable systems over V. Then there exists a unique continuous map $\phi: V \to M^{cr}_{m,n,p}(\mathbb{R})$ such that Σ is isomorphic to $\phi^!\Sigma^u$ (as continuous families; i.e. there is a bijective correspondence between continuous maps $V \to M^{cr}_{m,n,p}(\mathbb{R})$ and isomorphism classes of continuous families over V).

3.4.20. The algebraic-geometric fine moduli theorem. Let V be a scheme and Σ an algebraic family of cr systems over V. Then there exists a unique morphism of schemes $\phi: V \to M^{cr}_{m,n,p}$ such that Σ is isomorphic to $\phi^!\Sigma^u$ over V.

3.4.21. On the proof of these theorems. First consider the topological case. The map ϕ associated to Σ is defined as follows. For each $v \in V$ we have a system $\Sigma(v)$, which uniquely determines an isomorphism class of linear dynamical systems (cf. (3.4.2));

that is, it uniquely defines a point $\phi(v)$ of $M^{cr}_{m,n,p}(\mathbb{R})$ which is the space of all isomorphism classes of cr systems (of the dimensions under consideration). This ϕ is obviously continuous. Now $\Sigma^u(z)$ for all $z \in M^{cr}_{m,n,p}(\mathbb{R})$ represents z. So, by 3.4.18, Σ and $\phi^!\Sigma^u$ are two continuous families of cr systems over V such that for all $v \in V$, $\Sigma(v)$ and $\phi^!\Sigma^u(v)$ are isomorphic. It follows that the families Σ and $\Sigma' = \phi^!\Sigma^u$ are isomorphic as continuous families. The reason is the following rigidity property: if $(F, G, H), (F', G', H') \in L^{cr}_{m,n,p}(\mathbb{R})$ are isomorphic then the isomorphism is unique. Indeed, if S is an isomorphism then we must have $SR(F, G) = R(F', G')$ so that if α is a nice selection such that $R(F,G)_\alpha$ is invertible, then $S = R(F', G')_\alpha (R(F,G)_\alpha)^{-1}$. The statement that Σ and Σ' over V are isomorphic if they are pointwise isomorphic results as follows. For every $v \in V$ there is a $V' \ni v$ such that the bundles E and E' of Σ and Σ' are trivial over V' so that over V' the families Σ and Σ' are simply (up to isomorphism) continuously varying triples of matrices $(F(v'), G(v'), H(v')), (F'(v'), G'(v'), H'(v')), v' \in V'$. Let α be a nice selection such that $R(F(v), G(v))_\alpha$ is invertible. Restricting V' a bit more if necessary we can assume that $R(F(v'), G(v'))_\alpha$ is invertible for all $v' \in V'$. Then $S(v') = R(F'(v'),G'(v'))_\alpha$ $(R(F(v'), G(v'))_\alpha)^{-1}$ is a continuous family of invertible matrices taking $\Sigma(v')$ into $\Sigma'(v')$ for all $v' \in V'$. Thus Σ and Σ' are isomorphic over some small neighborhood of every point of V. The isomorphisms in question must agree on the intersections of these neighborhoods, again by the rigidity property. It follows that these local isomorphisms combine to define a global isomorphism over all of V from Σ to Σ'.

A more formal and also more formula based version of this argument can be found in [Haz1]. The scheme theoretic version (theorem 3.4.20) is based on the same rigidity property, cf. section 4 below for some details.

3.4.22. Remark. In [HK] I claimed that the underlying bundle E^u of the universal family Σ^u was the pullback by means of $\bar{R}$ (cf. 3.3.5)) of the bundle E over $G_{n(n+1)m}$ whose fibre over z was the n-plane represented by z. As we have seen it is not; instead E^u is the pullback of the dual bundle E^d of E. Now the determinant bundle of E^d is a very ample line bundle (rather than the determinant bundle of E) so that the argument in [HK] to prove that $M_{m,n}$ is not quasi affine is correct modulo two errors which cancel each other.

4. THE CLASSIFYING "SPACE" $M^{cr}_{m,n,p}$ IS DEFINED OVER $\mathbb{Z}$ AND CLASSIFIES OVER $\mathbb{Z}$.

Mainly for completeness and tutorial reasons I give in this section the algebraic-geometric details of the remarks 3.3.10 and 3.4.20 that there exists a scheme $M^{cr}_{m,n,p}$ over $\mathbb{Z}$ of which the varieties $M^{cr}_{m,n,p}(k)$, cf. 3.3.9, k an algebraically closed field, are obtained by base change and that this scheme is classifying for algebraic families of cr systems, and thus in particular classifying for cr systems over rings (with possibly a projective module as state module).

Those who are not particularly interested in the algebraic-geometric details can skip this section without consequences for their understanding of the remainder of this paper. There is in any case nothing difficult about what follows below and anyone who has once seen, say, the construction of the Grassmann schemes or projective spaces over $\mathbb{Z}$, will have no difficulties in supplying all details for himself from what has been said in section 3 above. All we are really doing below is rewriting a number of formulas of section 3 above using capital letters instead of small ones. This does take a certain number of pages, though. It seemed desirable to include these, as, judging from the audience's remarks during the oral presentation of these lectures, there is, perhaps rightly so, a distinct unwillingness in accepting without further proof a statement on the part of the lecturer like "the algebraic-geometric version of this theorem is proved similarly."

4.1 <u>Definition of the scheme $M^{cr}_{m,n,p}$</u>. For each nice selection $\alpha \subset J_{n,m}$ let

$$V_\alpha = \mathrm{Spec}(\mathbb{Z}\,[X^\alpha_{ij}, Y^\alpha_{rs};\ i = 1,\dots,n,\ j = 1,\dots,m,\ r = 1,\dots,p,\ s = 1,\dots,n]) \qquad (4.1.1)$$

Let $H_\alpha(Y)$ be the $p \times n$ matrix (Y^α_{rs}), and let $(F_\alpha(X), G_\alpha(X))$ be the unique pair of matrices over $\mathbb{Z}\,[X^\alpha_{ij}]$ such that

$$R(F_\alpha(X),G_\alpha(X))_\alpha = I_n,\ R(F_\alpha(X),G_\alpha(X))_{s(\alpha,j)} = \begin{pmatrix} X^\alpha_{1j} \\ \vdots \\ X^\alpha_{nj} \end{pmatrix}, \quad j = 1,\dots,m \qquad (4.1.2)$$

where the $s(\alpha,j)$ are the m successor indices of α, cf. 3.2). Finally for each pair of nice selections α,β let $d_{\alpha\beta}(X) \in \mathbb{Z}[X^\alpha_{ij}]$

be the element

$$d_{\alpha\beta}(X) = \det(R(F_\alpha(X), G_\alpha(X))_\beta) \tag{4.1.3}$$

and let $V_{\alpha\beta}$ be the open subscheme of V_α obtained by localizing with respect to $d_{\alpha\beta}(X)$, i.e.

$$V_{\alpha\beta} = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, d_{\alpha\beta}(X)^{-1}]) \tag{4.1.4}$$

Now for each pair of nice selections α, β write down the formulas

$$\begin{gathered} S_{\alpha\beta}(X)^{-1}F_\alpha(X)S_{\alpha\beta}(X) = F_\beta(X) \\ S_{\alpha\beta}(X)^{-1}G_\alpha(X) = G_\beta(X),\ H_\alpha(Y)S_{\alpha\beta}(X) = H_\beta(Y) \end{gathered} \tag{4.1.5}$$

where

$$S_{\alpha\beta}(X) = R(F_\alpha(X), G_\alpha(X))_\beta \tag{4.1.6}$$

Because the entries of $F_\beta(X)$ and $G_\beta(X)$ are equal to zero, 1 or X^β_{ij} for some i,j and because the (r,s)-th entry of $H_\beta(Y)$ is Y^β_{rs}, the formulae (4.1.5) provide us with certain expressions for the X^β_{ij} and $Y^\beta_{r,s}$ in terms of the X_{ij}, Y_{rs}, which by (4.1.5) and (4.1.3) (and the usual formula for matrix inversion) can be written as polynomials in X^α_{ij}, Y^α_{rs}, $d_{\alpha\beta}(X)^{-1}$, say

$$\begin{aligned} X^\beta_{ij} &= \phi_{\alpha\beta}(i,j)(X_{ij}, d_{\alpha\beta}(X)^{-1}), \\ Y^\beta_{rs} &= \phi_{\alpha\beta}(r,s)(X^\alpha_{ij}, d_{\alpha\beta}(X)^{-1}, Y^\alpha_{rs}) \end{aligned} \tag{4.1.7}$$

Then

$$\phi^*_{\alpha\beta} : X^\beta_{ij} \mapsto \phi_{\alpha\beta}(i,j)(X^\alpha),\ Y^\beta_{rs} \mapsto \phi_{\alpha\beta}(r,s)(X^\alpha, Y^\alpha) \tag{4.1.8}$$

defines an isomorphism of rings.

$$\mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, d_{\beta\alpha}(X)^{-1}] \simeq \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, d_{\alpha\beta}(X)^{-1}]$$

It follows from 4.1.5 that (with the obvious notations)

$$\phi^*_{\alpha\beta}R(F_\beta(X),G_\beta(X)) = S_{\alpha\beta}(X)^{-1}R(F_\alpha(X),G_\alpha(X))$$
$$\phi^*_{\alpha\beta}\, H_\beta(Y) = H_\alpha(Y)S_{\alpha\beta}(X) \tag{4.1.9}$$

and these formulae describe $\phi^*_{\alpha\beta}$ completely. It follows that

$$\phi^*_{\alpha\beta}\, d_{\beta\alpha}(X) = \phi^*_{\alpha\beta}\, \det(R(F_\beta(X),G_\beta(X))_\alpha) = \det(S_{\alpha\beta}(X))^{-1} = d_{\alpha\beta}(X)^{-1}$$

so that $\phi^*_{\alpha\beta}$ does indeed map $d_{\beta\alpha}(X)^{-1}$ into $\mathbb{Z}[X^\alpha_{ij},Y^\alpha_{rs},d_{\alpha\beta}(X)^{-1}]$.

The $\phi^*_{\alpha\beta}$ induce isomorphisms of open subschemes

$$\phi_{\alpha\beta}\colon V_{\alpha\beta} \to V_{\beta\alpha} \tag{4.1.10}$$

and $M^{cr}_{m,n,p}$ is now the scheme obtained by gluing together the schemes V_α, for all nice selections α, by means of the isomorphisms $\phi_{\alpha\beta}$.

As in section 3 above one can now embed $M^{cr}_{m,n,p}$ into a product of a Grassmannian over $\mathbb{Z}$ and an affine space over $\mathbb{Z}$ to see that $M^{cr}_{m,n,p}$ is a separated scheme.

For each nice selection α let V^{co}_α be the open subscheme of V_α defined by

$$V^{co}_\alpha = \bigcup_\gamma \operatorname{Spec}(\mathbb{Z}[X^\alpha_{ij},Y^\alpha_{rs},Q(F_\alpha(X),H_\alpha(Y))^{-1}_\gamma] \tag{4.1.11}$$

where γ runs through all the nice selections of the set of row indices $J_{p,n}$ of $Q(F_\alpha(X),H_\alpha(Y))$. Then the $\phi_{\alpha\beta}$ restrict to give isomorphisms.

$$\phi^{co}_{\alpha\beta}\colon\ V^{co}_{\alpha\beta} \to V^{co}_{\beta\alpha} \tag{4.1.12}$$

where $V^{co}_{\alpha\beta} = V^{co}_\alpha \cap V_{\alpha\beta}$. Gluing together the V^{co}_α by means of the $\phi^{co}_{\alpha\beta}$ we obtain the open subscheme $M^{cr,co}_{m,n,p}$ of $M^{cr}_{m,n,p}$.

To see how all these abstract formulas look in concreto consider the case $m = 2$, $n = 2$, $p = 1$. In this case, there are three nice selections $\alpha,\beta,\gamma \subset J_{2,2}$, viz.

$$\alpha = \{(0,1),(0,2)\},\quad \beta = \{(0,1),(1,1)\}\ ,$$
$$\gamma = \{(0,2),(1,2)\} \tag{4.1.13}$$

We have

$$F_\alpha(X) = \begin{pmatrix} X_{11}^\alpha & X_{12}^\alpha \\ X_{21}^\alpha & X_{22}^\alpha \end{pmatrix}, \quad G_\alpha(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_\alpha(Y) = (Y_1^\alpha, Y_2^\alpha)$$

$$F_\beta(X) = \begin{pmatrix} 0 & X_{11}^\beta \\ 1 & X_{21}^\beta \end{pmatrix}, \quad G_\beta(X) = \begin{pmatrix} 1 & X_{12}^\beta \\ 0 & X_{22}^\beta \end{pmatrix}, \quad H_\beta(Y) = (Y_1^\beta, Y_2^\beta)$$

$$F_\gamma(X) = \begin{pmatrix} 0 & X_{12}^\gamma \\ 1 & X_{22}^\gamma \end{pmatrix}, \quad G_\gamma(X) = \begin{pmatrix} X_{11}^\gamma & 1 \\ X_{21}^\gamma & 0 \end{pmatrix}, \quad H_\gamma(Y) = (Y_1^\gamma, Y_2^\gamma)$$

Thus

$$d_{\alpha\beta}(X) = X_{21}^\alpha, \quad d_{\alpha\gamma}(X) = -X_{12}^\alpha, \quad d_{\beta\gamma}(X) = X_{12}^\beta X_{12}^\beta + X_{12}^\beta X_{21}^\beta X_{22}^\beta - X_{11}^\beta X_{22}^\beta X_{22}^\beta$$

$$d_{\beta\alpha}(X) = X_{22}^\beta, \quad d_{\gamma\alpha}(X) = -X_{21}^\gamma, \quad d_{\gamma\beta}(X) = X_{11}^\gamma X_{11}^\gamma + X_{11}^\gamma X_{22}^\gamma X_{21}^\gamma - X_{21}^\gamma X_{21}^\gamma X_{12}^\gamma$$

$$S_{\alpha\beta}(X) = \begin{pmatrix} 1 & X_{11}^\alpha \\ 0 & X_{21}^\alpha \end{pmatrix}, \quad S_{\beta\alpha}(X) = \begin{pmatrix} 1 & X_{12}^\beta \\ 0 & X_{22}^\beta \end{pmatrix},$$

$$S_{\alpha\gamma}(X) = \begin{pmatrix} 0 & X_{21}^\alpha \\ 1 & X_{22}^\alpha \end{pmatrix}, \quad S_{\gamma\alpha}(X) = \begin{pmatrix} X_{11}^\gamma & 1 \\ X_{21}^\gamma & 0 \end{pmatrix},$$

$$S_{\beta\gamma}(X) = \begin{pmatrix} X_{12}^\beta & X_{11}^\beta X_{22}^\beta \\ X_{22}^\beta & X_{12}^\beta + X_{21}^\beta X_{22}^\beta \end{pmatrix}, \quad S_{\gamma\beta}(X) = \begin{pmatrix} X_{11}^\gamma & X_{12}^\gamma X_{21}^\gamma \\ X_{21}^\gamma & X_{11}^\gamma + X_{22}^\gamma X_{21}^\gamma \end{pmatrix}$$

Thus for example the two isomorphisms $\phi^*_{\alpha\beta}$ and $\phi^*_{\beta\alpha}$ are given by

$$\phi^*_{\alpha\beta} : \mathbb{Z}[X^\beta_{ij}, Y^\beta_r, (X^\beta_{22})^{-1}] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_r, (X^\alpha_{21})^{-1}]$$
$$X^\beta_{12} \mapsto -(X^\alpha_{21})^{-1}X^\alpha_{11},\ X^\beta_{22} \mapsto (X^\alpha_{21})^{-1}$$
$$X^\beta_{11} \mapsto X^\alpha_{12}X^\alpha_{21} - X^\alpha_{11}X^\alpha_{22},\ X^\beta_{21} \mapsto X^\alpha_{11} + X^\alpha_{22}$$
$$Y^\beta_1 \mapsto Y^\alpha_1,\ Y^\beta_2 \mapsto (X^\alpha_{21})^{-1}Y^\alpha_2 - (X^\alpha_{21})^{-1}X^\alpha_{11}Y^\alpha_1$$

$$\phi^*_{\alpha\beta} : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_r, (X^\alpha_{21})^{-1}] \to \mathbb{Z}[X^\beta_{ij}, Y^\beta_r, (X^\beta_{22})^{-1}]$$
$$X^\alpha_{11} \mapsto -(X^\beta_{22})^{-1}X^\beta_{12},\ X^\alpha_{21} \mapsto (X^\beta_{22})^{-1}$$
$$X^\alpha_{12} \mapsto X^\beta_{11}X^\beta_{22} - X^\beta_{12}X^\beta_{21} - (X^\beta_{22})^{-1}X^\beta_{12}X^\beta_{12}$$
$$X^\alpha_{22} \mapsto (X^\beta_{22})^{-1}X^\beta_{12} + X^\beta_{21}$$
$$Y^\alpha_1 \mapsto Y^\beta_1,\ Y^\alpha_2 \mapsto (X^\beta_{22})^{-1}Y^\beta_2 - (X^\beta_{22})^{-1}Y^\beta_1X^\beta_{12}$$

and one checks without trouble that indeed $d_{\beta\alpha}(X)^{-1} = (X^\beta_{22})^{-1}$ gets mapped into $\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_r, (X^\alpha_{21})^{-1}]$ and $d_{\alpha\beta}(X)^{-1} = (X^\alpha_{21})^{-1}$ into $\mathbb{Z}[X^\beta_{ij}, Y^\beta_r, (X^\beta_{22})^{-1}]$ and that indeed $\phi^*_{\alpha\beta} \circ \phi^*_{\beta\alpha} = \mathrm{id}$, $\phi^*_{\beta\alpha} \circ \phi^*_{\alpha\beta} = \mathrm{id}$. (The formulas are not always so simple; for instance the formulas for $\phi^*_{\beta\gamma}$ and $\phi^*_{\gamma\beta}$ are a good deal more complicated.

4.2. Small Intermezzo: Completely reachable systems over a ring.

A system $\Sigma = (F,G,H)$ over a ring R is said to be completely reachable if $R(F,G)\colon R^r \to R^n$, $r = (n+1)m$ is a surjective map, cf. e.g. [Sol] or [Rou]. This is equivalent to each element of the family $\Sigma(p) = (F(p),G(p),H(p))$, $p \in \mathrm{Spec}(R)$ being completely reachable. Indeed $R(F,G)\colon R^r \to R^m$ is surjective if it is surjective mod every maximal ideal [Bou, Ch. II, §3.3, Prop. 11] and the statement follows.

4.3. The Algebraic Geometric Version of the Nice Selection Lemma.

The next thing to do is to discuss the algebraic-geometric version of the nice selection lemma, 3.2.3. Recall that this lemma says that if the system (F,G,H) over a field k is cr then there is a nice selection α such that $R(F,G)_\alpha$ is invertible. Now let (F,G,H) be a cr system over a ring R, which per definition means that $R(F,G)\colon R^r \to R^n$, r $(n+1)m$, is surjective, which in turn is equivalent to condition that the systems $\Sigma(p) = (F(p),G(p),H(p)$ over $k(p)$, the quotient field of R/p, are cr for all prime ideals p. Then of course one does not expect the existence of a nice selection α such that $R(F,G)_\alpha$ is an invertible matrix over R; after all $\Sigma = (F,G,H)$ should be interpreted as a family and not as a single system.

For a continuous topological family $\Sigma(\sigma)$ over a topological space M the nice selection lemma implies that there is a finite covering $M = \cup U_\alpha$ such that for all $\sigma \in U_\alpha$, $R(F(\sigma),G(\sigma))_\alpha$ is invertible. And this property generalizes nicely.

4.3.1. Lemma. Let $\Sigma = (F,G,H)$ be a cr system over a ring R. For each nice selection α let $d_\alpha = \det(R(F,G)_\alpha)$. Then the ideal generated by the d_α is the whole ring R. (This means of course that the $U_\alpha = \mathrm{Spec}(R[d_\alpha^{-1}])$ cover all of $\mathrm{Spec}(R)$).

Proof. Let I be the ideal generated by the d_α, α nice. Suppose that $I \neq R$. Then there is a maximal ideal m such that $I \subset m$. Consider $\Sigma(m) = (F(m),G(m),H(m))$. Then $\det(R(\Sigma(m))_\alpha) = 0$ in R/m for all α, showing that $\Sigma(m)$ is not cr (by the old nice selection lemma 3.2.3 over the field R/m) which contradicts the assumption that Σ was cr.

To state the more global version of this lemma we need a bit of notation. Let Σ be a family of cr systems over a scheme V. For each nice selection α we define

$$U_\alpha = \{v \in V \mid \det(R(\Sigma(v))_\alpha) \neq 0\} \tag{4.3.2}$$

This definition seems a bit ambiguous at first because $R(\Sigma(v))$ depends on what basis we choose in the state space of $\Sigma(v)$ and hence is only defined up to multiplication on the left by an $n \times n$ invertible matrix with coefficients in $k(v)$. This matrix being invertible, however, means that the whole symbol group $\det(R(\Sigma(v))_\alpha) \neq 0$ makes perfectly good sense so that U_α is well defined. Of course U_α is an open subscheme of V.

4.3.3. Lemma. Let Σ be a family of cr systems over a scheme V. For each nice selection α let U_α be as in (4.3.2). Then $\bigcup_{\alpha\,\mathrm{nice}} U_\alpha = V$.

This follows immediately from lemma 4.3.1 because V can be covered with affine schemes $\mathrm{Spec}(R_i)$ (such that moreover the underlying bundle of Σ is trivial over each $\mathrm{Spec}(R_i)$).

4.4. <u>The Universal Bundle</u> E^u <u>over</u> $M^{cr}_{m,n,p}$. The universal bundle E^u over $M^{cr}_{m,n,p}$ is constructed just as in 3.4.11 above. Writing things out in relentness detail one obtains the following algebraic-geometric local pieces and patching data description.

For each nice selection α let

$$E_{\alpha} = \mathrm{Spec}(\mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{rs}] \otimes \mathbb{Z}[Z^{\alpha}_1,\dots,Z^{\alpha}_n]) = V_{\alpha} \times \mathbb{A}^n \tag{4.4.1}$$

where $\mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{rs}]$ is as in 4.1.1; i.e. Spec $\mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{rs}] = V_{\alpha}$. Let

$$\pi_{\alpha} : E_{\alpha} \to V_{\alpha} \tag{4.4.2}$$

be the projection induced by the natural inclusion

$$\pi^{*}_{\alpha} : \mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{r,s}] \subset \mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{r,s},Z^{\alpha}_{t}] .$$

Define for each pair of nice selections α,β.

$$E_{\alpha\beta} = \mathrm{Spec}\ \mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{r,s},Z^{\alpha}_{t},d_{\alpha\beta}(X)^{-1}]) = V_{\alpha\beta}\times \mathbb{A}^n \tag{4.4.3}$$

and let

$$\tilde{\Phi}_{\alpha\beta} : E_{\alpha\beta} \to E_{\beta\alpha} \tag{4.4.4}$$

be the isomorphism given by the ring isomorphism

$$\tilde{\Phi}^{*}_{\alpha\beta} : \mathbb{Z}[X^{\beta}_{ij},Y^{\beta}_{rs},Z^{\beta}_{t},d_{\beta\alpha}(X)^{-1}] \to \mathbb{Z}[X^{\alpha}_{ij},Y^{\alpha}_{rs},Z^{\alpha}_{t},d_{\alpha\beta}(X)^{-1}] \tag{4.4.5}$$

given by

$$X^{\beta}_{ij} \to \phi_{\alpha\beta}(i,j)(X^{\alpha}), Y^{\beta}_{r,s} \to \phi_{\alpha\beta}(r,s)(X\ ,Y\ ),$$

$$Z^{\beta}_{t} \to \tilde{\Phi}_{\alpha\beta}(t)(X^{\alpha},Z^{\alpha}) \tag{4.4.6}$$

where the $\tilde{\Phi}_{\alpha\beta}(t)(X^{\alpha},Z^{\alpha})$ are defined by the equality

$$\begin{pmatrix} \tilde{\Phi}_{\alpha\beta}(1)(X^{\alpha},Z^{\alpha}) \\ \vdots \\ \tilde{\Phi}_{\alpha\beta}(n)(X^{\alpha},Z^{\alpha}) \end{pmatrix} = S_{\alpha\beta}(X)^{-1} \begin{pmatrix} Z^{\alpha}_1 \\ \vdots \\ Z^{\alpha}_n \end{pmatrix}$$

The $\tilde{\Phi}_{\alpha\beta}$ are compatible (by their definition) with the $\phi_{\alpha\beta}$ in that the following diagram commutes for each pair of nice selections α,β.

$$\begin{array}{ccc} E_{\alpha\beta} & \xrightarrow{\tilde{\Phi}_{\alpha\beta}} & E_{\beta\alpha} \\ \downarrow \pi_\alpha & & \downarrow \pi_\beta \\ V_{\alpha\beta} & \xrightarrow[\tilde{\phi}_{\alpha\beta}]{} & V_{\beta\alpha} \end{array} \tag{4.4.8}$$

It follows that by gluing the E_α together by means of the $\tilde{\Phi}_{\alpha\beta}$ we obtain a vector bundle E^u.

$$\pi : E^u \to M^{cr}_{m,n,p} \tag{4.4.9}$$

4.5. <u>The Morphism into $M^{cr}_{m,n,p}$ Associated to an Algebraic Family of cr Systems</u>. We start with the case that the underlying vector bundle E of the family Σ is trivial and that the parametrizing scheme V is affine. Σ is then described by a ring R, $V = \mathrm{Spec}(R)$, $E = \mathrm{Spec}(R[Z_1,\dots,Z_n]$, $\pi : E \to V$ induced by the natural inclusion $R \to R[Z_1,\dots,Z_n]$, and vector bundle homomorphisms $F: E \to E$, $G: \mathrm{Spec}(R[U_1,\dots,U_m]) \to E$, $H: E \to \mathrm{Spec}(R[Y_1,\dots,Y_m])$. The fact that these morphisms are vector bundle homomorphisms is reflected by the fact that the associated homomorphisms of rings

$$F^* : R[Z_1,\dots,A_n] \to R[Z_1,\dots,Z_n],\quad G^* : R[Z_1,\dots,Z_n] \to R[U_1,\dots,U_m],\quad H^* : R[Y_1,\dots,Y] \to R[Z_1,\dots,Z_n]$$

are firstly R-algebra homomorphisms and further of the form

$$F^*(Z_i) = \sum_{j=1}^{n} f_{ij}Z_j,\quad G^*(Z_i) = \sum_{j=1}^{m} g_{ij}U_j,\quad H^*(Y_i) = \sum_{j=1}^{n} h_{ij}Z_j \tag{4.5.1}$$

where the f_{ij}, g_{ij}, h_{ij} are elements of R. This defines a triple of matrices $\bar{F} = (f_{ij})$, $\bar{G} = (g_{ij})$, $\bar{H} = (h_{ij})$. For each nice selection α let $S_\alpha = R(\bar{F},\bar{G})_\alpha$, $d_\alpha = \det(S_\alpha) \in R$, let $U_\alpha = \mathrm{Spec}(R[d_\alpha^{-1}])$, and let $V_\alpha = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}])$ be "the nice-selection-α-piece of $M^{cr}_{m,n,p}$" of 4.1 above. Now define

$$\psi_\alpha : U_\alpha \to V_\alpha \tag{4.5.2}$$

by the morphism of rings

$$\psi_\alpha^* : \mathbb{Z}[X_{ij}^\alpha, Y_{rs}^\alpha] \to R[d_\alpha^{-1}] \tag{4.5.3}$$

given by

$$X_{ij}^\alpha \mapsto \text{i-th entry of the column vector } S_\alpha^{-1}R(\bar F,\bar G)_{s(\alpha,j)}$$

$$Y_{rs}^\alpha \mapsto \text{r-th entry of the column } s \text{ of the matrix } \bar H S_\alpha \tag{4.5.4}$$

where $s(\alpha,j)$ is the j-th successor index of the nice selection α, cf. 3.2 above. Or, using the obvious notation, ψ_α^* is defined by

$$\psi_\alpha^*(R(F_\alpha(X),G_\alpha(X)) = S_\alpha^{-1}R(\bar F,\bar G),\quad \psi_\alpha^* H_\alpha(Y) = \bar H S_\alpha \tag{4.5.5}$$

Now let β be a second nice selection. We claim that the ψ_α and ψ_β agree on $U_\alpha \cap U_\beta = \mathrm{Spec}(R[d_\alpha^{-1},d_\beta^{-1}])$. In view of how the V_α, V_β are glued together to obtain $M^{cr}_{m,n,p}$ this means that we must prove the commutativity of the diagram

$$\begin{tikzcd} \mathbb{Z}[X_{ij}^\alpha, Y_{rs}^\alpha, d_{\alpha\beta}(X)^{-1}] \arrow[dr, "\psi_\alpha^*"] & \\ & R[d_\alpha^{-1},d_\beta^{-1}] \\ \mathbb{Z}[X_{ij}^\beta, Y_{rs}^\beta, d_{\beta\alpha}(X)^{-1}] \arrow[uu, "\phi_{\alpha\beta}^*"] \arrow[ur, "\psi_\beta^*"'] & \end{tikzcd} \tag{4.5.6}$$

Note first that

$$\psi_\alpha^*(S_{\alpha\beta}(X)) = \psi_\alpha^*(R(F_\alpha(X),G_\alpha(X)_\beta) = S_\alpha^{-1}R(\bar F,\bar G)_\beta = S_\alpha^{-1}S_\beta \tag{4.5.7}$$

so that ψ_α^* does indeed map $d_{\alpha\beta}(X)^{-1}$ into $R[d_\alpha^{-1},d_\beta^{-1}]$. Now ψ_β^* is described by

$$\psi_\beta^*(R(F_\beta(X),G_\beta(X)) = S_\beta^{-1}R(\bar F,\bar G),\ \psi_\beta^* H_\beta(Y) = \bar H S_\beta \tag{4.5.8}$$

and on the other hand

$$\psi_\alpha^*\phi_{\alpha\beta}^* R(F_\beta(X),G_\beta(X)) = \psi_\alpha^* (S_{\alpha\beta}(X)^{-1}R(F_\alpha(X),G_\alpha(X)))$$

(by (4.1.9))

$$= S_\beta^{-1} S_\alpha S_\alpha^{-1} R(\bar{F},\bar{G})$$

(by (4.5.7) and (4.5.5))

$$= S_\beta^{-1} R(\bar{F},\bar{G})$$

which fits perfectly with (4.5.8). Similarly $\psi_\alpha^* \phi_{\alpha\beta}^* H_\beta(X) = \psi_\alpha^* H_\alpha(Y) S_{\alpha\beta}(X) = \bar{H} S_\alpha S_\alpha^{-1} S_\beta = \bar{H} S_\beta = \psi_\beta^* H_\beta(Y)$, so that (4.5.6) is indeed commutative. Thus the $\psi_\alpha : U_\alpha \to V_\alpha$ are compatible, and because $\bigcup_{\alpha \text{ nice}} U_\alpha = \mathrm{Spec}(R)$ we obtain a morphism of schemes

$$\psi_\Sigma : V = \mathrm{Spec}(R) \to M^{cr}_{m,n,p}$$

4.5.9. Lemma. The morphism ψ_Σ depends only on the isomorphism class of Σ (so in particular ψ_Σ does not depend on how E is trivialized).

Proof. Let Σ' be a second family of cr systems over $V = \mathrm{Spec}(R)$ with trivial underlying vectorbundle $E' = \mathrm{Spec}(R[Z_1', Z_2', \ldots, Z_n'])$. Suppose Σ' is isomorphic to Σ and let the isomorphism be $\mu : E \to E'$. Because μ is a morphism of vectorbundles over $V = \mathrm{Spec}(R)$ its ring homomorphism

$$\mu^* : R[Z_1', Z_2', \ldots, Z_n'] \to R[Z_1, Z_2, \ldots, Z_n]$$

is an R-algebra homomorphism of the form

$$\mu^*(Z_i') = \sum_{j=1}^n s_{ij} Z_j, \qquad s_{ij} \in R$$

Let S be the matrix (s_{ij}). Then S is invertible (over R) because μ is an isomorphism. Now because μ defines an isomorphism $\Sigma' \simeq \Sigma$ we have $F'\mu = \mu F$, $\mu G = G'$, $H = H'\mu$ which in terms of the matrices $\bar{F},\bar{G},\bar{H}$ associated to Σ (cf. (4.5.1) above) and the analogous matrices $\bar{F}',\bar{G}',\bar{H}'$ of Σ' means that

$$S\bar{F} = \bar{F}'S, \; S\bar{G} = \bar{G}', \; \bar{H} = \bar{H}'S$$

It follows that if d_α', S_α', U_α' are defined analogously to d_α, S_α, U_α then $S_\alpha' = SS_\alpha$. $d_\alpha' = \det(S) d_\alpha$ so that $U_\alpha' = U_\alpha$ and $\psi_\alpha' = \psi_\alpha$ all because $SR(\bar{F},\bar{G}) = R(\bar{F}',\bar{G}')$, $\bar{H}S^{-1} = \bar{H}'$, which proves the lemma.

4.5.10. Construction of ψ_Σ for families whose underlying bundle is not necessarily trivial.

Now let $\Sigma = (E; F,G,H)$ be a family of cr systems over a scheme V. We can cover V with affine pieces $U_i = \mathrm{Spec}(R_i)$ such that E is trivializable over U_i. By the construction above and lemma 4.5.9 this gives us morphisms (independent of the trivialization chosen)

$$\psi_i : U_i \to M^{cr}_{m,n,p}$$

Now on $U_i \cap U_j$ the ψ_i and ψ_j must agree, because by lemma 4.5.9 again ψ_i and ψ_j agree on all affine pieces $\mathrm{Spec}(R) \subset U_i \cap U_j$. Hence the ψ_i combine to define a morphism

$$\psi_\Sigma : V \to M^{cr}_{m,n,p}$$

which, again by lemma 4.5.9 depends only on the isomorphism class of Σ.

4.6. The universal family Σ^u of cr systems over $M^{cr}_{m,n,p}$. Let E^u be the vectorbundle over $M^{cr}_{m,n,p}$ constructed in 4.4 above. In this section I describe a (universal) family of cr systems over $M^{cr}_{m,n,p}$ whose underlying bundle is E^u. (That this family is indeed universal will be proved in 4.9 below).

Recall that E^u was constructed out of affine pieces $E_\alpha = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t])$ glued together by means of certain isomorphisms $\tilde{\Phi}_{\alpha\beta}$, cf. 4.4. Let $\mathbb{A}^r = \mathrm{Spec}(\mathbb{Z}[U_1,\dots,U_r])$. To define $\Sigma^u = (E^u;F^u,G^u,H^u)$ it suffices to define vectorbundle homomorphisms

$$F_\alpha : E_\alpha \to E_\alpha,\ G_\alpha : V_\alpha \times \mathbb{A}^m \to E_\alpha,\ H_\alpha : E_\alpha \to V_\alpha \times \mathbb{A}^p \tag{4.6.1}$$

which are compatible with the identifications

$$\tilde{\Phi}_{\alpha\beta}: E_{\alpha\beta} \to E_{\beta\alpha},\ \phi_{\alpha\beta} \times \mathrm{id}: V_{\alpha\beta} \times \mathbb{A}^m \to V_{\beta\alpha} \times \mathbb{A}^m,$$

$$\phi_{\alpha\beta} \times \mathrm{id}: V_{\alpha\beta} \times \mathbb{A}^p \to V_{\beta\alpha} \times \mathbb{A}^p$$

in the sense that the following diagram must be commutative

$$\begin{tikzcd}
V_{\alpha\beta} \times \mathbb{A}^m \arrow[r, "G_\alpha"] \arrow[d, "\phi_{\alpha\beta}\times \mathrm{id}"] & E_{\alpha\beta} \arrow[r, "F_\alpha"] \arrow[d, "\tilde{\phi}_{\alpha\beta}"] & E_{\alpha\beta} \arrow[r, "H_\alpha"] \arrow[d, "\tilde{\phi}_{\alpha\beta}"] & V_{\alpha\beta} \times \mathbb{A}^p \arrow[d, "\phi_{\alpha\beta}\times \mathrm{id}"] \\
V_{\beta\alpha} \times \mathbb{A}^m \arrow[r, "G_\beta"] & E_{\beta\alpha} \arrow[r, "F_\beta"] & E_{\beta\alpha} \arrow[r, "H_\beta"] & V_{\beta\alpha} \times \mathbb{A}^p
\end{tikzcd} \tag{4.6.2}$$

(cf. also (3.4.17)). We now describe $F_\alpha, G_\alpha, H_\alpha$ as those morphisms which on the ring level are given by the $\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}]$ - algebra homomorphisms

$$F^*_\alpha\colon \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t], \quad Z^\alpha \mapsto F_\alpha(X)Z^\alpha \tag{4.6.3}$$

$$G^*_\alpha\colon \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, U_1, \ldots, U_m], \quad Z^\alpha \mapsto G_\alpha(X)U \tag{4.6.4}$$

$$H^*_\alpha\colon \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, V_1, \ldots, V_p] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t], \quad V \mapsto H_\alpha(Y)Z^\alpha \tag{4.6.5}$$

where Z^α, U, V are respectively the column vectors $(Z^\alpha_1, \ldots, Z^\alpha_n)^T$, $(U_1, \ldots, U_m)^T$, $(V_1, \ldots, V_p)^T$.

It remains to check that the diagram (4.6.2) is indeed commutative, which is done by checking that the dual diagram of rings homomorphisms is commutative.

This comes down to precisely the same calculations as in 3.4.14. As an example we check that the diagram

$$\begin{tikzcd}
\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t, d_{\alpha\beta}(X)^{-1}] & \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t, d_{\alpha\beta}(X)^{-1}] \arrow[l, "F^*_\alpha"'] \\
\mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, Z^\beta_t, d_{\beta\alpha}(X)^{-1}] \arrow[u, "\tilde{\phi}^*_{\alpha\beta}"] & \mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, Z^\beta_t, d_{\beta\alpha}(X)^{-1}] \arrow[l, "F^*_\beta"'] \arrow[u, "\tilde{\phi}^*_{\alpha\beta}"']
\end{tikzcd}$$

is commutative. Because $\tilde{\phi}^{*}_{\alpha\beta}$ maps $\mathbb{Z}[X^{\beta}_{ij}, Y^{\beta}_{rs}, d_{\beta\alpha}(X)^{-1}]$ into $\mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{rs}, d_{\alpha\beta}(X)^{-1}]$ and because F^{*}_{α} and F^{*}_{β} are respectively $\mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{rs}]$-algebra and $\mathbb{Z}[X^{\beta}_{ij}, Y^{\beta}_{rs}]$-algebra homomorphisms it suffices to check that

$$\tilde{\phi}^{*}_{\alpha\beta}\, F^{*}_{\beta}(Z^{\beta}_{t}) = F^{*}_{\alpha}\phi^{*}_{\alpha\beta}(Z^{\beta}_{t}), \quad t = 1,\ldots,n$$

By the definitions (4.4.7), (4.6.3) and using the definition of $\phi^{*}_{\alpha\beta}$, cf. 4.1, we have

$$F^{*}_{\alpha}\tilde{\phi}^{*}_{\alpha\beta}(Z^{\beta}) = F^{*}_{\alpha}(S_{\alpha\beta}(X)^{-1}Z^{\alpha}) = S_{\alpha\beta}(X)^{-1}F_{\alpha}(X)Z^{\alpha}$$

$$\begin{aligned}\phi^{*}_{\alpha\beta}F^{*}_{\beta}(Z^{\beta}) &= \tilde{\phi}^{*}_{\alpha\beta}(F_{\beta}(X)Z^{\beta}) = \phi^{*}_{\alpha\beta}(F_{\beta}(X))S_{\alpha\beta}(X)^{-1}Z^{\alpha} \\ &= S_{\alpha\beta}(X)^{-1}F_{\alpha}(X)S_{\alpha\beta}(X)S_{\alpha\beta}(X)^{-1}Z^{\alpha} \\ &= S_{\alpha\beta}(X)^{-1}F_{\alpha}(X)Z^{\alpha} \quad .\end{aligned}$$

The remaining two squares of diagram (4.6.2) are similarly shown to be commutative.

4.7. A rigidity lemma.

The key to the proof of theorem 3.4.20 (the algebraic-geometric classifying theorem) is (as was remarked before) a rigidity property which in this context takes the following form.

4.7.1. Proposition. Let Σ, Σ' be two families of cr systems over a scheme V. Suppose that there is a covering by open subschemes (U_i) of V such that the two families Σ and Σ' restricted to U_i are isomorphic for all i. Then Σ and Σ' are isomorphic as algebraic families over V.

We note that no such proposition holds for arbitrary families of systems cf. [HP] for a counterexample.

Proof. We can assume that the underlying vectorbundles E and E' have been obtained by gluing together trivial pieces over affine subschemes of V. Refining the covering (U_i) if necessary (this does not change the validity of the hypothesis of the proposition) we can therefore assume that E and E' have been obtained by gluing together trivial bundles $U_i \times A^n$ over affine schemes U_i.

Our data are then as follows. We have for each i an affine scheme $U_i = \mathrm{Spec}(R_i)$ for each i,j isomorphisms of (trivial) bundles

$$\phi_{ij}, \phi'_{ij} : (U_i \cap U_j) \times \mathbb{A}^n \to (U_j \cap U_i) \times \mathbb{A}^n$$

which respectively define the bundles E and E'. The remaining ingredients of the two families of systems Σ and Σ' are then given by vectorbundle homomorphisms

$$F_i, F'_i : U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^n, \quad G_i, G'_i : U_i \times \mathbb{A}^m \to U_i \times \mathbb{A}^n$$

$$H_i, H'_i : U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^p \tag{4.7.1}$$

such that the following diagrams are commutative for all i,j (where U_{ij} is short for $U_i \cap U_j$)

$$\begin{tikzcd}
 & U_{ij} \times \mathbb{A}^n \arrow[r, "{F_i, F'_i}"] \arrow[d, "{\phi_{ij}, \phi'_{ij}}"] & U_{ij} \times \mathbb{A}^n \arrow[dr, "{H_i, H'_i}"] \arrow[d, "{\phi_{ij}, \phi'_{ij}}"] & \\
U_{ij} \times \mathbb{A}^m \arrow[ur, "{G_i, G'_i}"] \arrow[dr, "{G_j, G'_j}"'] & & & U_{ij} \times \mathbb{A}^p \\
 & U_{ij} \times \mathbb{A}^n \arrow[r, "{F_j, F'_j}"] & U_{ij} \times \mathbb{A}^n \arrow[ur, "{H_j, H'_j}"'] &
\end{tikzcd} \tag{4.7.2}$$

Finally the fact that Σ and Σ' are isomorphic over each U_i means that there are vectorbundle isomorphisms $\phi_i : U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^n$ such that the following diagram is commutative for all i

$$\begin{tikzcd}
 & U_i \times \mathbb{A}^n \arrow[r, "F_i"] \arrow[d, "\phi_i"] & U_i \times \mathbb{A}^n \arrow[dr, "H_i"] \arrow[d, "\phi_i"] & \\
U_i \times \mathbb{A}^m \arrow[ur, "G_i"] \arrow[dr, "G'_i"'] & & & U_i \times \mathbb{A}^p \\
 & U_i \times \mathbb{A}^n \arrow[r, "F'_i"] & U_i \times \mathbb{A}^n \arrow[ur, "H'_i"'] &
\end{tikzcd} \tag{4.7.3}$$

We now claim that the ϕ_i are compatible and combine to define an isomorphism $\phi: E \to E'$ (it then follows, because this is locally true, that $\phi F = F'\phi$, $\phi G = G'$, $H'\phi = H$). To prove this we must show that for each $\mathrm{Spec}(R) = U \subset U_{ij} = U_i \cap U_j$ the following diagram commutes

$$\begin{array}{ccc} U\times\mathbb{A}^n & \xrightarrow{\phi_{ij}} & U\times\mathbb{A}^n \\ \downarrow{\scriptstyle\phi_i} & & \downarrow{\scriptstyle\phi_j} \\ U\times\mathbb{A}^n & \xrightarrow{\phi'_{ij}} & U\times\mathbb{A}^n \end{array} \qquad (4.7.4)$$

Now vectorbundle homomorphisms of trivial vectorbundles over an affine scheme $U = \mathrm{Spec}(R)$ are given by matrices with coefficients in R as we explained *en passant* in the first few paragraphs of 4.5 above. Let $\bar{G}_i$, $\bar{G}'_i$, $\bar{F}_i$, $\bar{F}'_i$, $\bar{H}_i$, $\bar{H}'_i$, S_{ij}, S'_{ij}, S_i, S_j be the matrices of the morphisms of vectorbundles G_i, G'_i, F_i, F'_i, H_i, H'_i, ϕ_{ij}, ϕ'_{ij}, ϕ_i, ϕ_j restricted to U. The commutativity relations (4.7.2) and (4.7.3) then imply for these matrices with coefficients in R that

$$S_{ij}\bar{G}_i = \bar{G}_j,\ S_{ij}\bar{G}'_i = \bar{G}'_j,\ S_{ij}\bar{F}_i = \bar{F}_jS_{ij},\ \ S_{ij}\bar{F}'_i = \bar{F}'_jS'_{ij}$$

$$S_{ij}\bar{H}_i = \bar{H}_j,\ S'_{ij}\bar{H}'_i = \bar{H}'_j,\ S_i\bar{G}_i = \bar{G}'_i,\ \ S_i\bar{F}_i = \bar{F}'_iS_i$$

$$\bar{H}'_iS_i = \bar{H}_i, \qquad (4.7.5)$$

$$S_j\bar{G}_j = \bar{G}'_j,\ S_j\bar{F}_j = \bar{F}'_jS_j,\ \bar{H}'_jS_j = \bar{H}_j$$

and the matrices S_i, S_j, S_{ij}, S'_{ij} are all invertible because they come from vectorbundle isomorphisms.

It follows that

$$\begin{aligned} S_jS_{ij}R(\bar{F}_i,\bar{G}_i) &= S_jR(\bar{F}_j,\bar{G}_j) = R(\bar{F}'_j,\bar{G}'_j) \\ &= S'_{ij}R(\bar{F}'_i,\bar{G}'_i) = S'_{ij}S_iR(\bar{F}_i,\bar{G}_i) \end{aligned} \qquad (4.7.6)$$

Now Σ is a family of cr systems and hence so is its restriction to $U = \mathrm{Spec}(R)$. It follows (cf. 4.2 above) that $R(\bar{F}_i,\bar{G}_i):R^r \to R^n$ $r = (n+1)m$, is a surjective map. Hence, (4.7.6) implies that $S_jS_{ij} = S'_{ij}S_i$ proving the commutativity of (4.7.4) and hence the proposition.

4.8. <u>On the pullback construction.</u> Let $\Sigma = (E; F,G,H)$ be a family of systems over a scheme M and let $\psi : V' \to M$ be a morphism of schemes. Assume that everything is given in terms of local affine pieces and patching data; i.e. Σ is given by trivial bundles $U_i \times \mathbb{A}^n \to U_i = \mathrm{Spec}(R_i) \subset M$ with vectorbundle isomorphisms $\phi_{ij}: U_{ij} \times \mathbb{A}^n \to U_{ij} \times \mathbb{A}^n$ and vector bundle mormorphisms $F_i: U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^n$, $G_i: U_i \times \mathbb{A}^m \to U_i \times \mathbb{A}^n$, $H_i: U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^p$ such the nonprime diagram (4.7.2) is commutative, and ψ is given by affine morphisms $\psi_i: U_i' \to U_i, U_i' = \mathrm{Spec}(R_i')$. Let $\psi_i^*: R_i \to R_i'$ be the ring homomorphism of ψ_i. Let, as before, $\bar{F}_i, \bar{G}_i, \bar{H}_i$ be the matrices of the vectorbundle morphisms F_i, G_i, H_i.

Then the local pieces of the pullback family $\psi^!\Sigma = \Sigma'$ are the trivial bundles $U_i' \times \mathbb{A}^n \to U_i'$ with the vectorbundle homomorphisms $F_i': U_i' \times \mathbb{A}^n \to U_i' \times \mathbb{A}^n$, $G_i': U_i' \times \mathbb{A}^m \to U_i' \times \mathbb{A}^n$, $H_i': U_i' \times \mathbb{A}^n \to U_i' \times \mathbb{A}^p$ given by the matrices $\bar{F}_i' = \psi_i^*\bar{F}_i$, $\bar{G}_i' = \psi_i^*\bar{G}_i$, $\bar{H}_i' = \psi_i^*\bar{H}_i$. The patching data are defined as follows. If $U' = \mathrm{Spec}(R) \subset U_i' \cap U_j'$ maps into $U = \mathrm{Spec}(R) \subset U_i \cap U_j$ under ψ and $\psi_1^*: R \to R'$ is the associated homomorphism of rings, then over Spec(R') the isomorphim $\phi_{ij}': U' \times \mathbb{A}^n \to U' \times \mathbb{A}^n$ is given by the matrix $S_{ij}' = \psi_1^* S_{ij}$ if S_{ij} is the matrix of $\phi_{ij}: U \times \mathbb{A}^n \to U \times \mathbb{A}^n$.

This can be taken as the definition of the pullback family $\psi^!\Sigma$. It agrees of course with the more informal description given in section 3 above.

4.9. <u>The classifying theorem for algebraic families of cr systems over schemes</u> ($M^{cr}_{m,n,p}$ <u>is classifying over</u> $\mathbb{Z}$).

We can now prove the algebraic-geometric classifying theorem for families of cr systems, i.e. theorem 3.4.20. Stated more precisely this theorem says

<u>4.9.1 Theorem.</u> Let Σ be an algebraic family of cr systems over a scheme V. Then there exists a unique morphism of schemes $\psi_\Sigma: V \to M^{cr}_{m,n,p}$ (defined in 4.5 above) such that $\psi_\Sigma^!\Sigma^u \simeq \Sigma$ where Σ^u is the universal family constructed in section 4.6 above. That is the map $\Sigma \mapsto \psi_\Sigma$ and the map $\psi \mapsto \psi^!\Sigma^u$ (of 4.8 above) set up a bijective correspondence between the set of scheme morphisms

$V \to M^{cr}_{m,n,p}$ and isomorphism classes of families of cr systems over V. Moreover this isomorphism is functorial.

Proof. First let $\psi: V \to M^{cr}_{m,n,p}$ be a morphism of schemes; let $\Sigma = \psi^! \Sigma^u$. Then we must show that $\psi_\Sigma = \psi$. To do this it suffices to show that ψ_Σ and ψ agree on all elements of some affine covering (U_i) of V. We can take this covering to be finer than the covering $(\psi^{-1}(V_\alpha), \alpha \text{ nice})$ where $V_\alpha \subset M^{cr}_{m,n,p}$ is the piece belonging to the nice selection α, cf. 4.1. Let therefore $U = \mathrm{Spec}(R)$ be such that $\psi(U) \subset V_\alpha$, and let

$$\psi^* : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}] \to R$$

be the associated ring homomorphism. Then according to 4.8 above and the definition of Σ^u, cf. 4.6, the family Σ over U is described by the three matrices

$$\bar{F} = \psi^* F_\alpha(X),\ \bar{G} = \psi^* G_\alpha(X),\ \bar{H} = \psi^* H_\alpha(Y)\ . \qquad (4.9.2)$$

By 4.5 above the morphism $\psi^*_\Sigma : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}] \to R$ associated to this family is characterized by

$$\psi^*_\Sigma(R(F_\alpha(X),\ G_\alpha(X)) = S^{-1}_\alpha R(\bar{F},\bar{G}),\ \psi^*_\alpha H_\alpha(Y) = \bar{H}S_\alpha \quad (4.9.3)$$

where $S_\alpha = R(\bar{F},\bar{G})_\alpha$. Because $R(F_\alpha(X),\ G_\alpha(X))_\alpha = I_n$, $S_\alpha = I_n$ in this case (cf. (4.9.2)) so that indeed (comparing (4.9.2) and (4.9.3)) $\psi^*_\Sigma = \psi^*$.

Now let Σ over V be a family of cr systems and let $\psi_\Sigma: V \to M^{cr}_{m,n,p}$ be the associated morphism as defined in 4.5. We have to show that $\psi^!_\Sigma \Sigma^u$ is isomorphic to Σ. By the rigidity result 4.7.1 it suffices to show that $\psi^!_\Sigma \Sigma^u$ and Σ are isomorphic over each element of some affine covering (U_i) of V, which we can take fine enough so that the underlying bundle E of Σ is trivial over each U_i. Let therefore $U = \mathrm{Spec}(R)$ be such that Σ over U is described by the triple of matrices $\bar{F},\bar{G},\bar{H}$. Let $d_\alpha = \det(\bar{R}(\bar{F},\bar{G})_\alpha)$ for each nice selection α. Then U in turn is covered by the $U_\alpha = \mathrm{Spec}(R[d^{-1}_\alpha])$ (by the nice selection lemma). So taking a still finer covering (if necessary) we can assume that $U = \mathrm{Spec}(R)$ is such that for a certain nice selection α we have that $S_\alpha = R(\bar{F},\bar{G})_\alpha$ is invertible over R. Then by 4.5 ψ_Σ is given on U by the ring homomorphism

$$\psi^* : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{r,s}] \to R$$

characterized by

$$\psi^* R(F_\alpha(X), G_\alpha(X)) = S_\alpha^{-1} R(\bar{F},\bar{G}),\ \psi^* H_\alpha(Y) = \bar{H}S_\alpha \qquad (4.9.4)$$

By 4.8 the family of cr systems $\psi_\Sigma^! \Sigma^u$ is defined by the matrices

$$\bar{F}' = \psi^* F_\alpha(X),\ \bar{G}' = \psi^* G_\alpha(X),\ \bar{H}' = \psi^* H_\alpha(Y) \qquad (4.9.5)$$

Comparing (4.9.4) and (4.9.5) we see that over U the families defined by $\bar{F},\bar{G},\bar{H}$ and by $\bar{F}',\bar{G}',\bar{H}'$ are indeed isomorphic with the isomorphism being defined by S_α (which is invertible over R). This concludes the proof of the theorem.

4.10. On cr systems over rings. The classifying theorem 4.9.1 of course also applies to systems over rings R. Such a system (with finitely generated projective state module X) gives rise to a family of cr systems over R iff $R(F,G)$: $R^r \to X$, $r = (n+1)m$, is surjective (cf. 4.2). If R is such that all finitely generated projective modules are free (which happens e.g. if R is a ring of polynomials over a field by the Quillen-Suslin theorem [Qu,Sus], then theorem 4.9.1 says that the R-rational points of $M^{cr}_{m,n,p}$ are precisely the $GL_n(R)$ orbits in $L^{cr}_{m,n,p}(R)$, i.e.

$$M^{cr}_{m,n,p}(R) \simeq L^{cr}_{m,n,p}(R)/GL_n(R) \quad \text{(if } R \text{ is projective free).}$$

In general the theorem gives a canonical injection

$$L^{cr}_{m,n,p}(R)/GL_n(R) \to M^{cr}_{m,n,p}(R)$$

with the remaining points of $M^{cr}_{m,n,p}(R)$ corresponding to systems over R whose state module is projective but not free.

4.11. A few final remarks. There is a completely dual theory from the co instead of cr point of view. Also the open subscheme $M^{cr,co}_{m,n,p}$ is of course classifying for families of co and cr systems. This scheme is embeddable (over $\mathbb{Z}$) in an affine scheme $\mathbb{A}^{(n+1)mp}$ as a locally closed subscheme.

5. EXISTENCE AND NONEXISTENCE OF GLOBAL CONTINUOUS CANONICAL FORMS

As a first application of the fine moduli spaces of sections 3 and 4 above we discuss existence and nonexistence of global continuous canonical forms for linear dynamical systems.

5.1. The topological case. Let L' be a $GL_n(\mathbb{R})$-invariant subspace of $L_{m,n,p}(\mathbb{R})$. A canonical form for $GL_n(\mathbb{R})$ acting on L' is a mapping $c: L' \to L'$ such that the following three properties hold

$$c(\Sigma^S) = c(\Sigma) \quad \text{for all } \Sigma \in L',\ S \in GL_n(\mathbb{R}); \tag{5.1.1}$$

$$\text{for all } \Sigma \in L' \text{ there is an } S \in GL_n(\mathbb{R}) \text{ such that } c(\Sigma) = \Sigma^S ; \tag{5.1.2}$$

$$c(\Sigma) = c(\Sigma') \Rightarrow \exists S \in GL_n(\mathbb{R}) \text{ such that } \Sigma' = \Sigma^S . \tag{5.1.3}$$

(Note that (5.1.3) is implied by (5.1.2).)

Thus a canonical form selects precisely one element out of each order of $GL_n(\mathbb{R})$ acting on L'. We speak of a continuous canonical form if c is continuous.

Of course there exist (many) canonical forms. E.G. order the set of all nice selection α in $J_{n,m}$ in some way. For each $\Sigma \in L^{cr}_{m,n,p}(\mathbb{R})$ let $\alpha(\Sigma)$ be the first α such that $R(\Sigma)_\alpha$ is nonsingular. Then

$$\Sigma \mapsto c_{\alpha(\Sigma)}(\Sigma) = \Sigma^S, \quad S = R(\Sigma)^{-1}_{\alpha(\Sigma)} \tag{5.1.4}$$

is a canonical form on $L^{cr}_{m,n,p}(\mathbb{R})$ (Luenberger canonical forms à la Bryson). This mapping is not continuous, however, except when $m = 1$ (in which case there is only one nice selection), which entails a number of drawbacks, e.g. in numerical calculations and in identification procedures, cf [GWi] for a discussion in the similar case of Jordan canonical forms.

5.1.5. Theorem. There is a continuous canonical form on $L^{cr,co}_{m,n,p}(\mathbb{R})$ if and only if $p = 1$ or $m = 1$.

Proof. If $m = 1$ let $\alpha \subset J_{1,n} = \{(0,1),(1,1),\ldots,(n,1)\}$ be the unique nice selection $(0,1),\ldots,(n-1,1)$. Then

$$c_{\#\alpha}: \Sigma \mapsto c_{\#\alpha}(\Sigma) = \Sigma^S, \quad S = R(\Sigma)_\alpha^{-1} \tag{5.1.6}$$

is a continuous canonical form, because $R(\Sigma)_\alpha$ is always invertible for Σ cr.

Similarly if $p = 1$, let $\beta \subset J_{n,1}$, be the unique nice row selection. Then $\Sigma \mapsto \Sigma$, $S = Q(\Sigma)_\beta$ is a continuous form because $Q(\Sigma)_\beta$ is invertible for all co Σ (if $p = 1$).

It remains to show that there cannot be a continuous canonical form c on all of $L^{cr,co}_{m,n,p}(\mathbb{R})$ if both $m > 1$, $p > 1$.

To do this we construct two families of linear dynamical systems as follows for all $a \in \mathbb{R}$, $b \in \mathbb{R}$ (We assume $n \geq 2$; if $n = 1$ the examples must be modified somewhat).

$$G_1(a) = \left(\begin{array}{cc|ccc} a & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \hline 2 & 1 & & & \\ \vdots & \vdots & & B & \\ 2 & 1 & & & \end{array}\right),$$

$$G_2(b) = \left(\begin{array}{cc|ccc} 1 & b & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \hline 2 & 1 & & & \\ \vdots & \vdots & & B & \\ 2 & 1 & & & \end{array}\right)$$

where B is some (constant) $(n-2) \times (m-2)$ matrix with coefficients in $\mathbb{R}$,

$$F_1(a) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & n \end{pmatrix} = F_2(b)$$

$$H_1(a) = \left(\begin{array}{cc|ccc} y_1(a) & 1 & 2 & \cdots & 2 \\ y_2(a) & 1 & 1 & \cdots & 1 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C & \\ 0 & 0 & & & \end{array}\right),$$

$$H_2(b) = \left(\begin{array}{cc|ccc} x_1(b) & 1 & 2 & \cdots & 2 \\ x_2(b) & 1 & 1 & \cdots & 1 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C & \\ 0 & 0 & & & \end{array}\right)$$

where C is some (constant) real $(p-2)\times(n-2)$ matrix. Here the continuous functions $y_1(a)$, $y_2(a)$, $x_1(b)$; $x_2(b)$ are e.g. $y_1(a) = a$ for $|a| \le 1$, $y_1(a) = a^{-1}$ for $|a| \ge 1$, $y_2(a) = \exp(-a^2)$, $x_1(b) = 1$ for $|b| \le 1$, $x_1(b) = b^{-2}$ for $|b| \ge 1$, $x_2(b) = b^{-1}\exp(-b^{-2})$ for $b \ne 0$, $x_2(0) = 0$. The precise form of these functions is not important. What is important is that they are continuous, that $x_1(b) = b^{-1}y_1(b^{-1})$, $x_2(b) = b^{-1}y_2(b^{-1})$ for all $b \ne 0$ and that $y_2(a) \ne 0$ for all a and $x_1(b) \ne 0$ for all b.

For all $b \ne 0$ let $T(b)$ be the matrix

$$T(b) = \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \qquad (5.1.7)$$

Let $\Sigma_1(a) = (F_1(a), G_1(a), H_1(a))$, $\Sigma_2(b) = (F_2(b), G_2(b), H_2(b))$. Then one easily checks that

$$ab = 1 \Rightarrow \Sigma_1(a)^{T(b)} = \Sigma_2(b) \quad . \qquad (5.1.8)$$

Note also that $\Sigma_1(a)$, $\Sigma_2(b) \in L^{co,cr}_{m,n,p}(\mathbb{R})$ for all $a,b \in \mathbb{R}$; in fact

$$\Sigma_1(a) \in U_\alpha,\quad \alpha = ((0,2),(1,2),\ldots,(n-1,2)) \text{ for all } a \in \mathbb{R} \tag{5.1.9}$$

$$\Sigma_2(b) \in U_\beta,\quad \beta = ((0,1),(1,1),\ldots,(n-1,1)) \text{ for all } b \in \mathbb{R} \tag{5.1.10}$$

which proves the complete reachability. The complete observability is seen similarly.

Now suppose that c is a continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$. Let $c(\Sigma_1(a)) = (\bar{F}_1(a),\bar{G}_1(a),\bar{H}_1(a))$, $c(\Sigma_2(b)) = (\bar{F}_2(b),\bar{G}_2(b),\bar{H}_2(b))$. Let $S(a)$ be such that $c(\Sigma_1(a)) = \Sigma_1(a)^{S(a)}$ and let $\bar{S}(b)$ be such that $c(\Sigma_2(b)) = \Sigma_2(b)^{\bar{S}(b)}$.

It follows from (5.1.9) and (5.1.10) that

$$\begin{aligned} S(a) &= R(\bar{F}_1(a),\bar{G}_1(a))_\alpha R(F_1(a),G_1(a))_\alpha^{-1} \\ \bar{S}(b) &= R(F_2(b),G_2(b))_\beta R(F_2(b),G_2(b))_\beta^{-1} \end{aligned} \tag{5.1.11}$$

Consequently $S(a)$ and $S(b)$ are (unique and are) continuous functions of a and b.

Now take $a = b = 1$. Then $ab = 1$ and $T(b) = I_n$ so that (cf. (5.1.7), (5.1.8) and (5.1.11)) $S(1) = \bar{S}(1)$. It follows from this and the continuity of $S(a)$ and $\bar{S}(b)$ that we must have

$$\operatorname{sign}(\det S(a)) = \operatorname{sign}(\det \bar{S}(b)) \text{ for all } a,b \in \mathbb{R}\ . \tag{5.1.12}$$

Now take $a = b = -1$. Then $ab = 1$ and we have, using (5.1.8),

$$\begin{aligned} \Sigma_1(-1)^{(\bar{S}(-1)T(-1))} &= (\Sigma_1(-1)^{T(-1)})^{\bar{S}(-1)} \\ &= \Sigma_2(-1)^{\bar{S}(-1)} = c(\Sigma_2(-1)) \\ &= c(\Sigma_1(-1)) = \Sigma_1(-1)^{S(-1)}\ . \end{aligned}$$

It follows that $S(-1) = \bar{S}(-1)T(-1)$, and hence by (5.1.7), that

$$\det(S(-1)) = -\det(\bar{S}(-1))$$

which contradicts (5.1.12). This proves that there does not exist a continuous canonical form $L^{co,cr}_{m,n,p}(\mathbb{R})$ if $m \geq 2$, and $p \geq 2$.

5.1.13. Remark. By choosing the matrices B, C in $G_1(a)$, $G_2(b)$, $H_1(a)$, $H_2(b)$ judiciously we can also see to it that rank $G_1(a) = m =$ rank $G_2(b)$, rank $H_1(a) = p =$ rank $H_2(b)$ if $p < n$ and $m < n$. Note also that F in the example above has n distinct real eigenvalues so that a restriction like "F must be semi-simple" also does not help much.

5.1.14. Discussion of the proof of theorem 5.1.5. The proof given above, though definitely a proof, is perhaps not very enlightening. What is behind it is the following. Consider the natural projection

$$\pi : L^{cr,co}_{m,n,p}(\mathbb{R}) \to M^{cr,co}_{m,n,p}(\mathbb{R}) \quad . \tag{5.1.15}$$

Let c be a continuous canonical form. Because c is constant on all orbits c induces a continuous map $\tau : M^{cr,co}_{m,n,p}(\mathbb{R}) \to L^{cr,co}_{m,n,p}(\mathbb{R})$ which clearly is a section of π, (cf. (5.1.1) - (5.1.3)). Inversely if τ is a continuous section of π then $\tau \cdot \pi : L^{co,cr}_{m,n,p}(\mathbb{R}) \to L^{co,cr}_{m,n,p}(\mathbb{R})$ is a continuous canonical form.

Now (5.1.15) is (fairly easily at this stage, cf [Haz 1]), seen to be a principal $GL_n(\mathbb{R})$ fibre bundle. Such a bundle is trivial iff it admits a continuous section. The mappings

$$a \mapsto \Sigma_1(a), \quad b \mapsto \Sigma_2(b)$$

of the proof above now combine to define a continuous map of $P^1(\mathbb{R})$ = circle into $M^{cr,co}_{m,n,p}(\mathbb{R})$ such that the pullback of the fibre bundle (5.1.15) is nontrivial. In fact the associated determinant $GL_1(\mathbb{R})$ fibre bundle is the Möbius band (minus zero section) over the circle.

5.2. The algebraic-geometric case. The result corresponding to theorem 5.1.5 in the algebraic-geometric case is the following. For simplicity we state it for varieties (over algebraically closed fields).

5.2.1. Theorem. Let k be an algebraically closed field. Then there exists a canonical form $c: L^{cr,co}_{m,n,p}(k) \to L^{cr,co}_{m,n,p}(k)$ which is a morphism of algebraic varieties if and only if $m = 1$ or $p = 1$.

Here of course a canonical form is defined just as in 5.1 above; simply replace $\mathbb{R}$ with k everywhere in (5.1.1)-(5.1.3) and replace the word "continuous" with "morphism of algebraic varieties," which means that locally c is given by rational expressions in the coordinates.

The proof is rather similar to the one briefly indicated in 5.1.14 above. In this case $L^{cr,co}_{m,n,p} \to M^{cr,co}_{m,n,p}(k)$ is an algebraic prinicpal $GL_n(k)$ bundle and one again shows that it is trivial if and only if $m = 1$ or $p = 1$. The only difference is the example used to prove nontriviality. The map used in 5.1.14 is non-algebraic, nor is there an algebraic injective morphism $\mathbb{P}^1(k) \to M^{co,cr}_{m,n,p}(k)$. Instead one defines a three dimensional manifold much related to the families $\Sigma_1(a)$, $\Sigma_2(b)$ together with an injection into $M^{cr,co}_{m,n,p}(k)$ such that the pullback of this principal bundle is easily seen to be nontrivial. Cf. [Haz 2] for details.

6. REALIZATION WITH PARAMETERS AND REALIZING DELAY-DIFFERENTIAL SYSTEMS

As a second application of the existence of fine moduli spaces for cr systems we discuss realization with parameters (cf. also [By1]) and realization of delay-differential systems. A preliminary step for this is the following bit of realization theory.

6.1. Resumé of some realization theory. Let $T(s)$ be a proper rational matrix-valued function of s with the (formal) power series expansion (around $s = \infty$)

$$T(s) = A_1 s^{-1} + A_2 s^{-2} + \dots, \quad A_i \in k^{p\times m} . \qquad (6.1.1)$$

One says that $T(s)$ is realizable by a linear system of dimension $\leq n$, if $T(s)$ is the Laplace transform (resp. z-transform) of a linear differentiable (resp. difference) system $\Sigma = (F,G,H) \in L_{m,n,p}(k)$. This means that

$$T(s) = H(sI_n - F)^{-1}G \qquad (6.1.2)$$

, equivalently

$$A_i = HF^{i-1}G, \quad i = 1,2,3,\ldots \tag{6.1.3}$$

ne cessary and sufficient condition that $\Gamma(s)$ be realizable a system of dimension n is that the associated Hankel matrix $\mathscr{A})$ of the sequence $\mathscr{A} = (A_1, A_2, A_3, \ldots)$ be of rank $\leq n$. Here $\mathscr{A})$ is the block Hankel matrix.

$$h(\mathscr{A}) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & & \\ A_3 & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \end{pmatrix} .$$

e precisely we have the partial realization result which says t there eixst $F,G,H \in L^{co,cr}_{m,n,p}(k)$ such that $A_i = HF^{i-1}G$ iff k $h_n(\mathscr{A}) = \text{rank}\ h_{n+1}(\mathscr{A}) = n$, where $h_i(\mathscr{A})$ is the block rix consisting of the first i block rows and the first i ck-columns of $h(\mathscr{A})$.

Now suppose that rank $h(\mathscr{A})$ is precisely n, and let ,H realize $\mathscr{A}$.

We have

$$h(\mathscr{A}) = \begin{pmatrix} H \\ HF \\ HF^2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} (G \mid FG \mid F^2G \mid \ldots)$$

it follows by the Cayley-Hamilton theorem that $R(F,G)$ and ,H) are both of rank n so that $\Sigma = (F,G,H)$ is in this e both cr and co.

Finally we recall that if Σ and Σ' are both cr and co both realize $\mathscr{A}$, then Σ and Σ' are isomorphic, i.e., re is an $S \in GL_n(k)$ such that $\Sigma' = \Sigma^S$.

For all these facts, cf e.g. [KFA] or [Haz 3].

6.2. <u>A realization algorithm</u>. Now let $\mathcal{A}$ be such that rank $h(\mathcal{A}) = n$. We describe a method for calculating a

$$\Sigma = (F,G,H) \in L^{cr,co}_{m,n,p}(k)$$

which realizes $\mathcal{A}$. By the above we know that there exist a nice selection $\alpha_c \subset J_{m,n}$ the set of column indices of

$$h_{n+1}(\mathcal{A}) = \begin{pmatrix} A_1 & A_2 & \cdots & A_{n+1} \\ A_2 & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ A_{n+1} & & \cdots & A_{2n+1} \end{pmatrix} \qquad (6.2.1)$$

and a nice selection $\alpha_r \subset J_{p,n}$, the set of row indices of $h_{n+1}(\mathcal{A})$, such that the $n \times n$ matrix $h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c}$ has rank n. Here $h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c}$ is the matrix obtained from $h_{n+1}(\mathcal{A})$ by removing all rows whose index is not in α_r and all columns whose index is not in α_c. We now describe a method for finding a $\Sigma = (F,G,H) \in L^{cr,co}_{m,n,p}(k)$ such that Σ realizes $\mathcal{A}$ and such that $R(F,G)_{\alpha_c} = I_n$. (Such a Σ is unique).

Let γ_r be the subset of $J_{p,n}$ of the first p row indices, so that $h_{n+1}(\mathcal{A})_{\gamma_r}$ consists of the first row of blocks in (6.2.1). Now let

$$H = h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c} \quad . \qquad (6.2.2)$$

Now let

$$S = h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c} \qquad (6.2.3)$$

and define $R' = S^{-1}(h_{n+1}(\mathscr{A})_{\alpha_r})$. Then $(R')_{\alpha_c} = I_n$ and we let F,G be the unique $n \times n$ and $n \times m$ matrices such that

$$R(F,G) = R' \tag{6.2.4}$$

Recall, cf 3.2.7 above, that the columns of F and G can be simply read from the columns of R', being equal to either a standard basis vector or equal to a column of R'.

For every field k and each pair of nice selections $\alpha_c \subset J_{m,n}$, $\alpha_r \subset J_{p,n}$ let $W(\alpha_r,\alpha_c)(k)$ be the space of all sequences of $p \times m$ matrices $\mathscr{A} = (A_1,\dots,A_{2n+1})$ such that $\operatorname{rank}(h_{n+1}(\mathscr{A})) = n$ and $\operatorname{rank}(h_{n+1}(\mathscr{A})_{\alpha_r,\alpha_c}) = n$. Then the above defines a map

$$\tau(\alpha_r,\alpha_c)\colon W(\alpha_r,\alpha_c)(k) \to L^{cr,co}_{m,n,p}(k) \ . \tag{6.2.5}$$

6.2.6 Lemma. If $k = \mathbb{R}$ or $\mathbb{C}$ the map $\tau(\alpha_r,\alpha_c)$ is analytic, and algebraic-geometrically speaking the $\tau(\alpha_r,\alpha_c)$ define a morphism of schemes from the affine scheme $W(\alpha_r,\alpha_c)$ into the quasi affine scheme $L^{cr,co}_{m,n,p}$.

6.2.7 Lemma. Let $W(k)$ be the space of all sequences of $p \times m$ matrices $\mathscr{A} = (A_1,A_2,\dots,A_{2n+1})$ such that $\operatorname{rank}(h_{n+1}(\mathscr{A})) = n = \operatorname{rank} h_n(\mathscr{A})$. Let $h\colon L^{cr,co}_{m,n,p}(k) \to W(k)$ be the map $h(F,G,H) = (HG,HFG,\dots,HF^{2n}G)$. Then $h \cdot \tau(\alpha_r,\alpha_c)$ is equal to the natural embedding of $W(\alpha_r,\alpha_c)(k)$ in $W(k)$. (I.e. $h \cdot \tau(\alpha_r,\alpha_c)$ is the identity of $W(\alpha_r,\alpha_c)(k)$.)

Proof. Let $\mathscr{A} \in W(\alpha_r,\alpha_c)(k)$. By partial realization theory (cf. 6.1 above) we know that $\mathscr{A}$ is realizable, say by $\Sigma' = (F',G',H')$. Then because $\mathscr{A} \in W(\alpha_r,\alpha_c)(k)$ we have that $S = R(F',G')_{\alpha_c}$ is invertible. Let

$$\Sigma = (F,G,H) = \Sigma'^{S^{-1}} = (S^{-1}F'S, S^{-1}G', H'S).$$

Then Σ also realizes $\mathscr{A}$ and $R(F,G)_{\alpha_c} = I_n$. Now observe that the realization algorithm described above simply recalculates precisely these F,G,H from $\mathscr{A}$.

6.2.8. <u>Corollary</u>. Let $k = \mathbb{R}$ or $\mathbb{C}$ and let $h\colon M^{cr,co}_{m,n,p}(k) \to W(k)$ be the map induced by $h\colon L_{m,n,p}(k) \to W(k)$. Then h is an isomorphism of analytic manifolds.

6.2.9. <u>Corollary</u>. More generally $h\colon L^{co,cr}_{m,n,p} \to W$ induces an isomorphism of schemes $M_{m,n,p} \to W$. In particular if k is an algebraically closed field then we have an isomorphism of the algebraic varieties $M_{m,n,p}(k)$ and $W(k)$.

6.3. <u>Realization with parameters.</u>

6.3.1. <u>The topological case</u>. Let $T_a(s)$, $a \in V$ be a family of transfer functions depending continuously on a parameter $a \in V$. For each $a \in V$ write $T_a(s) = A_1(a)s^{-1} + A_2(a)s^{-2} + \dots$ and for each a let $n(a)$ be the rank of the block Hankel matrix of $\mathcal{A}(a) = (A_1(a), A_2(a), \dots)$. The question we ask is: does there exist a continuous family of systems $\Sigma(a) = (F(a),G(a),H(a))$ such that the transfer function of $\Sigma(a)$ is $T_a(s)$ for all a. The answer to this is definitely "yes" provided $n(a)$ is bounded as a function of a. Simply take a long enough chunk of the $\mathcal{A}(a)$ of all a and do the usual realization construction by means of block companion matrices and observe that this is continuous in the $A_i(a)$. [True if V is paracompact and normal, one needs partitions of unity (in any case, I do) to find <u>continuous</u> $T_i(a)$ such that $B_{n+1} = T_1B_n + \dots + T_nB_1$ where B_i is the i-th block column of $h(\mathcal{A})$.] The question becomes much more delicate if we ask for a continuous family of realizations which are all cr and co. This obviously requires that $n(a)$ is constant and provided that the space V is such that all $n = n(a)$ dimensional bundles are trivial this condition is also sufficient. Indeed if $n(a)$ is constant then the $\mathcal{A}(a)$ determine a continuous map $V \to W(\mathbb{R})$ and hence by Corollary 5.2.8 a continuous map $V \to M^{cr,co}_{m,n,p}(\mathbb{R})$. Pulling back the universal family over $M^{cr,co}_{m,n,p}(\mathbb{R})$ to a family over V gives us a family $(E;F,G,H)$ over V such that the transfer function of the system over $a \in V$ is $T_a(s)$ for all a. The bundle E is trivial by hypothesis, so there are continuous sections $e_1,\dots,e_n\colon V \to E$ such that $\{e_1(a),\dots,e_n(a)\}$ is a basis for $E(a)$ for all $a \in V$. Now write out the matrices of F,G,H with respect to these bases to find a continuous family $\Sigma(a)$, which realizes $T_a(s)$ and such that $\Sigma(a)$ is cr and co for all a.

6.3.2. The polynomial case. Let k be a field and $\bar{k}$ its algebraic closure, e.g. $k = \mathbb{R}$ and $\bar{k} = \mathbb{C}$. Let $T_x(s)$ be a transfer function with coefficients in $k[x_1,\dots,x_q]$, where $x_1,\dots,x_q$ are indeterminates. We ask whether there exists a realization of $T(s)$ over $k[x_1,\dots,x_q]$, that is a triple of matrices (F,G,H) with coefficients in $k[x_1,\dots,x_q]$ such that $T_x(s) = H(sI-F)^{-1}G$. Again the answer is obviously "yes" if we do not require any minimality conditions on the realization (provided $n(x_1,\dots,x_q)$ the degree of the Hankel matrix of $T(s)$ is bounded for all $(x_1,\dots,x_q) \in \bar{k}^q$

Now assume that $n(x_1,\dots,x_q)$ is constant for all $(x_1,\dots,x_q) \to \bar{k}^q$. Then $(x_1,\dots,x_q) \mapsto \mathscr{A}(x_1,\dots,x_q)$ defines a morphism of algebraic varieties $\bar{k}^q \to M^{cr,co}_{m,n,p}(k)$. Pulling back the universal family by means of this morphism we find a family $(E;F,G,H)$ over $\bar{k}^q$ which is defined over k because the morphism $\bar{k}^q \to W(k)$ and the isomorphism with $M^{cr,co}_{m,n,p}(k)$ are defined over k. Thus Σ is defined over k and by the Quillen-Suslin theorem E is trivializable over k. Taking the corresponding sections and writing out the matrices of F,G,H with respect to the resulting bases we find an F,G,H with coefficients in $k[x_1,\dots,x_q]$ which realize $T_x(s)$ for all $x \in \bar{k}^q$, i.e. such that $T_x(s) = H(sI-F)^{-1}G$. Moreover this system (F,G,H) is cr over $k[x_1,\dots,x_q]$ meaning that $R(F,G)$: $k[x_1,\dots,x_q]^{(n+1)m} \to k[x_1,\dots,x_q]^n$ is surjective; it is also co and even stronger its dual system is also cr (i.e. (F,G,H) is split in the terminology of [So 31]).

6.3.3. Realization by means of delay-differentiable systems. Let $\Sigma = (F(\sigma_1,\dots,\sigma_q), G(\sigma_1,\dots,\sigma_q), H(\sigma_1,\dots,\sigma_q))$ be a delay differential system with q incommensurable delays. Here σ_i stands for the delay operator $\sigma_i F(t) = F(t-a_i)$, cf. 2.3 above for this notation. The transfer function of Σ is then

$$T(s) = G(e^{-a_1 s},\dots,e^{-a_q s})(sI-F(e^{-a_1 s},\dots,e^{-a_q s}))^{-1} H(e^{-a_1 s},\dots,e^{-a_q s}) \qquad (6.3.4)$$

which is a rational function in s whose coefficients are polynomials in

$$e^{-a_1 s}, \ldots, e^{-a_q s} .$$

Now inversely suppose we have a transfer function $T(s)$ like (6.3.4) and we ask whether it can be realized by means of a delay-differential system $\Sigma(\sigma)$. Now if the a_i are incommensurable then the functions

$$s, e^{-a_1(s)}, \ldots, e^{-a_q s} .$$

are algebraically independent and there is precisely one transfer function $T(s;\sigma_1,\ldots,\sigma_q)$ whose coefficients are polynomials in $\sigma_1,\ldots,\sigma_q$ such that $T(s) = T'(s,e^{-a_1 s},\ldots,e^{-a_q s})$. Thus the problem is mathematically identical with the one treated just above 6.3.2. In passing let us remark that complete reachability for delay-systems in the sense of that the associated system over the ring $\mathbb{R}[\sigma_1,\ldots,\sigma_q]$ is required to be cr seems often a reasonable requirement, e.g. in connection with pole placement, cf. [So 1] and [Mo].

7. THE "CANONICAL" COMPLETELY REACHABLE SUBSYSTEM.

7.1. Σ^{cr} <u>for systems over fields</u>. Let $\Sigma = (F,G,H)$ be a system over a field k. Let X^{cr} be the image of $R(F,G):k^r \to k^n$, $r = m(n+1)$. Then obviously $F(X^{cr}) \subset X^{cr}$, $G(k^m) \subset X^{cr}$, so that there is an induced subsystem $\Sigma^{cr} = (X^{cr};F',G',H')$ which is called the canonical cr subsystem of Σ. In terms of matrices this means that there is an $S \in GL_n(k)$ such that Σ^S has the form

$$\Sigma^S = \left(\begin{pmatrix} G_1 \\ 0 \end{pmatrix}, \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} H_1 & H_2 \end{pmatrix} \right) \qquad (7.1.1)$$

with $(F_{11},G_1,H_1) = \Sigma^{cr}$, the "canonical" cr subsystem. The words Kalman "decomposition" are also used in this context. There is a dual construction relating to co and combining these two constructions "decomposes" the system into four parts.

In this section we examine whether this construction can be globalized, i.e. we ask whether this construction is continuous, and we ask whether something similar can be done for time varying linear dynamical systems.

7.2. Σ^{cr} <u>for time varying systems.</u> Now let $\Sigma = (F,G,H)$ be a time varying system, i.e. the coefficients of the matrices F,G,H are allowed to vary, say differentiably, with time. For time varying systems the controllability matrix $R(\Sigma) = R(F,G)$ must be redefined as follows

$$R(F,G) = (G(0) \mid G(1) \mid \ldots \mid G(n)) \tag{7.2.1}$$

where

$$G(0) = G;\ G(i) = FG(i-1) - \dot{G}(i-1) \tag{7.2.2}$$

where the $\dot{}$ denotes the differentiation with respect to time, as usual. Note that this gives back the old $R(F,G)$ if F,G do not depend on time. The system is said to be cr if this matrix $R(\Sigma)$ has full rank. These seem to be the appropriate notions for time varying systems; cf. e.g. [We, Haz 5] for some supporting results for this claim.

A time variable base change $x' = Sx$ (with $S = S(t)$ invertible for all t) changes Σ to Σ^S with

$$\Sigma^S = (SFS^{-1} + \dot{S}S^{-1}, SG, HS^{-1}) . \tag{7.2.3}$$

Note that $R(\Sigma)$ hence transforms as

$$R(\Sigma^S) = SR(\Sigma) . \tag{7.2.4}$$

7.2.5. <u>Theorem.</u> Let Σ be a time varying system with differentiably varying parameters. Suppose that rank $R(\Sigma)$ is constant as a function of t. Then there exists a differentiable time varying matrix S, invertible for all t, such that Σ^S has the form (7.1.1) with (F_{11},G_1,H_1) cr.

<u>Proof.</u> Consider the subbundle of the trivial $(n+1)m$ dimensional bundle over the real line generated by the rows of $R(\Sigma)$. This is a vectorbundle because of the rank assumption. This bundle is trivial. It follows that there exist r sections of the bundle, where $r = \text{rank } R(\Sigma)$, which are linearly independent everywhere. The continuous sections of the bundle are of the form $\Sigma a_i(t)z_i(t)$, where $z_i(t),\ldots,z_n(t)$ are the rows of $R(\Sigma)$ and the $a_i(t)$ are continuous functions of t. Let $b_1(t),\ldots,$ $b_r(t)$ be the r everywhere linear independent sections and let $b_j(t) = \Sigma a_{ji}(t)z_i(t)$, $j = 1,\ldots,r;$ $i = 1,\ldots,n$.

Let E' be the r dimensional subbundle of the trivial bundle E of dimension n over the real line generated by the r row vectors $a_j(t) = (a_{j1}(t),\ldots,a_{jn}(t))$. Because the quotient bundle E/E' is trivial we can complete the r vectors $a_1(t),\ldots,a_r(t)$ be a set of n vectors $a_1(t),\ldots,a_n(t)$ such that the determinant of the matrix formed by these vectors is nonzero for all t. Let $S_1(t)$ be the matrix formed by these vectors, then $S_1R(\Sigma)$ has the property that for all t its first r rows are linearly independent and that it is of rank r for all t. It follows that there are unique continuous functions $c_{ki}(t)$, $k = r+1,\ldots,n$; $i = 1,\ldots,r$ such that $z_k'(t) = \Sigma c_{ki}(t)z_i'(t)$, where $z_j'(t)$ is the j-th row of $S_1R(\Sigma)$. Now let

$$S_2(t) = \begin{pmatrix} I_r & 0 \\ -C(t) & I_{n-r} \end{pmatrix}$$

Then $S(t) = S_2(t)S_1(t)$ is the desired transformation matrix (as follows from the transformation formula (7.2.4)).

Virtually the same arguments give a smoothly varying $S(t)$ if the coefficients of Σ vary smoothly in time, and give a polynomial $S(t)$ if the coefficients of Σ are polynomials in t (where in the latter case we need the constancy of the rank also for all complex values of t and use that projective modules over a principal ideal ring are free).

7.3. Σ^{cr} for families. For families of systems these techniques give

7.3.1. Theorem. Let Σ be a continuous family parametrized by a contractible topological space (resp. a differentiable family parameterized by a contractible manifold; resp. a polynomial family). Suppose that the rank of $R(\Sigma)$ is constant as a function of the parameters. Then there exists a continuous (resp. differentiable; resp. polynomial) family of invertible matrices S such that Σ^S has the form (7.1.1) with (F_{11},G_1,H_1) a family of cr systems.

The proof is virtually the same as the one given above of theorem 7.2.5; in the polynomial case one, of course, relies on the Quillen-Suslin theorem [Qu; Sus] to conclude that the appropriate bundles are trivial. Note also that, inversely, the existence of an S as in the theorem implies that the rank of $R(\Sigma)$ is constant.

For delay-differential systems this gives a "Kalman decomposition" provided the relevant, obviously necessary, rank condition is met.

Another way of proving theorem 7.3.1 for systems over certain rings rests on the following lemma which is also a basic tool in the study of isomorphisms of families in [HP] and which implies a generalization of the main lemma of [OS] concerning the solvability of sets of linear equations over rings.

7.3.2. Lemma. Let R be a reduced ring (i.e. there are no nilpotents $\neq 0$) and let A be a matrix over R. Suppose that the rank of $A(p)$ over the quotient field of R/p is constant as a function of p for all prime ideals p. Then $\mathrm{Im}(A)$ and $\mathrm{Coker}(A)$ are projective modules.

Now let Σ over R be such that rank $R(\Sigma(p))$ is constant, and let R be projective free (i.e. all finitely generated projective modules over R are free). Then $\mathrm{Im}\, R(\Sigma) \subset R^n$ is projective and hence free. Taking a basis of $\mathrm{Im}\, R(\Sigma)$ and extending it to a basis of all of R^n, which can be done because $R^n/\mathrm{Im}\, R(\Sigma) = \mathrm{Coker}\, R(\Sigma)$ is projective and hence free, now gives the desired matrix S.

There is a complete set of dual theorems concerning co.

7.4. Σ^{cr} for delay differential systems. Now let $\Sigma(\sigma) = (F(\sigma), G(\sigma), H(\sigma))$ be a delay differential system. Then, of course, we can interpret Σ as a polynomial system over $\mathbb{R}[\sigma] = [\sigma_1,\ldots,\sigma_r]$ and apply theorem 7.3.1. The hypothesis that rank $R(\Sigma(\sigma))$ be constant as a function of $\sigma_1,\ldots,\sigma_r$ (including complex and negative values of the delays) is rather strong though.

Now if we assume that all functions involved in

$$\dot{x}(t) = F(\sigma)x(t) + G(\sigma)u(t),\ y(t) = H(\sigma)x(t) \qquad (7.4.1)$$

are zero sufficiently far in the past, an assumption which is not unreasonable and even customary in this context, then it makes perfect sense to talk about base changes of the form

$$x' = S(\sigma)x \qquad (7.4.2)$$

where $S(\sigma)$ is matrix whose coefficients are power series in the delays $\sigma_1,\ldots,\sigma_r$ and which is invertible over the ring of power series $\mathbb{R}[[\sigma_1,\ldots,\sigma_r]]$. Indeed if $\sigma_1\alpha(t) = \alpha(t-a_1)$, $a_1 > 0$ and the function $\beta(t)$ is zero for $t < -Na_1$ then

$$\left(\sum_{i=0}^{\infty} b_i\sigma_1^i\right) \ \beta(t) = \sum_{i=0}^{N+N'} b_i\beta(t-ia_i)$$

where N' is such that $t < N'a_i$.

Allowing such basis changes one has

7.4.3. Theorem. Let $\Sigma(\sigma)$ be a delay-differential system. Suppose that rank $R(\Sigma(\sigma))$ considered as a matrix over the quotient field $k(\sigma_1,\ldots,\sigma_r)$ is equal to rank $R(\Sigma(0))$ (over $\mathbb{R}$) where $\Sigma(0)$ is the system obtained from $\Sigma(\sigma)$ by setting all σ_i equal to zero. Then there exists a power series base change matrix $S \in GL_n(\mathbb{R}[[\sigma]])$ such that Σ^S has the form (7.1.1) with (F_{11},G_1,H_1) a cr system (over $\mathbb{R}[[\sigma]]$).

The proof is again similar where now, of course, one uses that a projective module over a local ring is free.

Note that $\Sigma(0)$ is not the system obtained from $\Sigma(\sigma)$ by setting all delays equal to zero. For example if $\Sigma(\sigma)$ is the one dimensional, one delay system $\dot{x}(t) = x(t) + 2x(t-1) + u(t) + u(t-2)$, $y(t) = 2x(t) - x(t-1)$, then $\Sigma(0)$ is the system $\dot{x}(t) = x(t) + u(t)$, $y(t) = 2x(t)$ obtained by removing all delay terms.

8. CONCLUDING REMARKS ON FAMILIES OF SYSTEMS AS OPPOSED TO SINGLE SYSTEMS

8.1. Non extendability of moduli spaces $M^{cr}_{m,n,p}$ and $M^{co}_{m,n,p}$. One aspect of the study of families of systems rather than single systems is the systematic investigation of which of the many constructions and algorithms of systems and control theory are continuous in the system parameters (or more precisely to determine, so to speak, the domains of continuity of these constructions). This is obviously important if one wants e.g. to execute these algorithms numerically.

Intimately (and obviously) related to this continuity problem is the question of how a given single system can sit in a family of systems (deformation (perturbation) theory). The fine moduli spaces $M^{cr}_{m,n,p}$ and $M^{co}_{m,n,p}$ answer precisely this question (for a system which is cr or co): for a given cr (resp. co) system the local structure of $M^{cr}_{m,n,p}(\mathbb{R})$ (resp. $M^{co}_{m,n,p}(\mathbb{R})$) around the point represented by the given system describes exactly the most complicated family in which the given system can occur (all other families can up to isomorphism be uniquely obtained from this one by a change of parameters). Thus one may well be interested to see whether these moduli spaces can be extended a bit. In particular one could expect that $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{co}_{m,n,p}(\mathbb{R})$ could be combined in some way to give a moduli space for all systems which are cr or co. The following example shows that this is a bit optimistic.

8.1.1. Example. Let Σ and Σ' be the two families over $\mathbb{C}$ (or $\mathbb{R}$) given by the triples of matrices

$$\Sigma = \left(\begin{pmatrix} 1 & 1 \\ \sigma & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,0) \right),$$

$$\Sigma' = \left(\begin{pmatrix} 1 & \sigma \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,0) \right).$$

Σ is co everywhere and cr everywhere but in $\sigma = 0$, and Σ' is cr everywhere and co everywhere but in $\sigma = 0$. The systems $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic for all $\sigma \neq 0$, but $\Sigma(0)$ and $\Sigma'(0)$ are definitely not isomorphic. This kills all chances of having a fine moduli space for families which consist of systems which are co or cr. There cannot even be a coarse moduli space for such families.

Indeed let $\mathcal{F}$ be the functor which assigns to every space the set of all isomorphism classes of families of cr or co systems. Then a coarse moduli space for $\mathcal{F}$ (cf. [Mu] for a precise definition) consists of a space M together with a functor transformation $\mathcal{F}(-) \to \mathrm{Mor}(-,M)$ which is an isomorphism if $- = pt$ and which also enjoys an additional universality property. Now consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(\mathbb{C}\setminus\{0\}) \arrow[r] & \mathrm{Mor}(\mathbb{C}\setminus\{0\},M) \\
\mathcal{F}(\mathbb{C}) \arrow[u] \arrow[d] \arrow[r, "\alpha"] & \mathrm{Mor}(\mathbb{C},M) \arrow[u] \arrow[d] \\
\mathcal{F}(\{0\}) \arrow[r, "\simeq"] & \mathrm{Mor}(\{0\},M)
\end{tikzcd}.$$

Consider the elements of $\mathcal{F}(\mathbb{C})$ represented by Σ and Σ'. Because Σ and Σ' are isomorphic as families restricted to $\mathbb{C}\setminus\{0\}$ we see by continuity (of the elements of $\mathrm{Mor}(\mathbb{C},M)$) that $\alpha(\Sigma) = \alpha(\Sigma')$. Because $\Sigma(0)$ and $\Sigma'(0)$ are not isomorphic this gives a contradiction with the injectivity of $\mathscr{F}(\{o\}) \to \mathrm{Mor}(\{o\},M)$.

Coarse moduli spaces represent one possible weakening of the fine moduli space property. Another, better adapted to the idea of studying families by studying a maximally complicated example, is that of a versal deformation. Roughly a versal holomorphic deformation of a system Σ over $\mathbb{C}$ is a family of systems $\Sigma(\sigma)$ over a small neighborhood U of 0 (in some parameter space) such that $\Sigma(0) = \Sigma$ and such that for every family Σ' over V such that $\Sigma'(0) = \Sigma$ there is some (not necessarily unique) holomorphic map ϕ (i.e., a holomorphic change in parameters) such that $\phi^!\Sigma \simeq \Sigma'$ in a neighborhood of 0.

For square matrices depending holomorphically on parameters (with similarity as isomorphism) Arnol'd, [Ar], has constructed versal deformations and the same ideas work for systems (in any case for pairs of matrices (F,G), cf. [Ta 2]).

8.2. <u>On the geometry of $M^{cr,co}_{m,n,p}$</u>. From the identification of systems point of view not only the local structure of $M^{co,cr}_{m,n,p}(\mathbb{R})$ is important but also its global structure cr. also [BrK] and [Haz 8]. Thus, for example, if $m = 1 = p$, $M^{co,cr}_{m,n,p}(\mathbb{R}) = \mathrm{Rat}(n)$ decomposes into $(n+1)$ components, and some of these components are of rather complicated topological types, [Br], which argues ill for the linearization tricks which are at the back of many identification procedures. One way to view identification is as finding a sequence of points in $M^{co,cr}_{m,n,p}(\mathbb{R})$ as more and more data come in. Ideally this sequence of points will then converge to something. Thus the question comes up of whether $M^{co,cr}_{m,n,p}(\mathbb{R})$ is compact, or compactifiable in such a way that the extra points can be interpreted as some kind of systems. Now $M^{co,cr}_{m,n,p}(\mathbb{R})$ is never compact. As to the compactification question, there does exist a partial compactification $\bar{M}_{m,n,p}$ such that the extra points, i.e. the points of $\bar{M}_{m,n,p} \backslash M^{cr,co}_{m,n,p}$ correspond to systems of the form

$$\dot{x} = Fx + Bu, \quad y = Hx + J(D)u \tag{8.2.1}$$

where D is the differentiation operator and J is a polynomial in D. This seems to give still more motivation for studying systems more general than $\dot{x} = Fx + Gu$, $y = Hx$ [Ros]. This partial compactification is also maximal in the sense that if a family of systems converges in the sense that the associated family of input/

output operators converges (in the weak topology) then the limit input/output operator is the input/output operator of a system of the form (8.2.1). Cf. [Haz 4] for details.

8.3. Pointwise-local-global isomorphism theorems. One perennial question which always turns up when one studies families rather than single objects is: to what extent does the pointwise or local structure of a family determine its global properties. Thus for square matrices one has e.g. the question studied by Wasov [Wa], cf. also [OS]: given two families of matrices $A(z)$, $A'(z)$ depending holomorphically on some parameters z. Suppose that for each separate value of z, $A(z)$ and $A'(z)$ are similar; does it follow that $A(z)$ and $A'(z)$ are similar as holomorphic families?

For families of systems the corresponding question is: let $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ be two families of systems and suppose that $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic for all values of σ. Does it follow that Σ and Σ' are isomorphic as families (globally or locally in a neighborhood of every parameters value σ).

Here there are (exactly as in the holomorphic-matrices-under-similarity-case) positive results provided the dimension of the stabilization subgroups $\{S \in GL_n(\mathbb{R}) | \Sigma(\sigma)^S = \Sigma(\sigma)\}$ is constant as a function of σ, cf. [HP].

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