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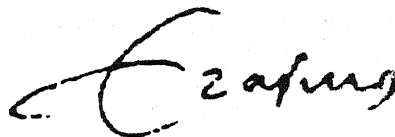
# ON THE THEORY OF FAMILIES OF LINEAR SYSTEMS

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# ON THE THEORY OF FAMILIES OF LINEAR SYSTEMS.

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## Abstract

Let $\Sigma$ and $\Sigma'$ be two families of linear dynamical systems, or, almost equivalently, let $\Sigma$ and $\Sigma'$ be two systems over a ring. This paper addresses itself to the question, what, if anything, can be said about the relations between $\Sigma$ and $\Sigma'$ if it is known that $\Sigma$ and $\Sigma'$ are pointwise isomorphic for all or almost all of the parameter values.

## 1. INTRODUCTION

inear dynamical system is a system of differen-
l equations

$$\dot{x} = Fx + Gu, \quad y = Hx \qquad (1.1)$$

$\Sigma \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, i.e. we have state space
mension n, m inputs and p outputs. Now let Q
a topological space. Roughly a family of
near dynamical systems over Q consists of a
llection of such equations (1.1), one for each
$\in Q$, such that the matrices F,G,H depend
ntinuously on the parameter q. More generally
and also more properly) a family over Q consists
f a vectorbundle E over Q (of dimension n),
vectorbundle endomorphism F: E → E and two
ectorbundle homomorphisms G: Q × $\mathbb{R}^m$ → E,
I: E → Q × $\mathbb{R}^p$. The two definitions agree locally
(i.e) over small enough open subsets of Q and for
the purposes of this paper the first definition
mostly suffices.

Analogously one considers systems of equations

$$x(t+1) = Fx(t) + Gu(t), \quad y(t) = Hx(t) \qquad (1.2)$$

where now the matrices F,G,H can have their
coefficients in any ring R (and t = 0,1,2,...,say).
For each prime ideal $\mathfrak{p}$ of R let $R(\mathfrak{p})$ be the
quotient field of the integral domain $R/\mathfrak{p}$. This

gives us a family of systems

$$x(t+1) = F(\mathfrak{p})x(t) + G(\mathfrak{p})u(t), \quad y(t) = H(\mathfrak{p})x(t) \qquad (1.3)$$

which is the local algebraic-geometric analogue
of the topological concept of a family introduced
above. The main goal of the theory of families of
systems is now to develop techniques and prove
theorems which do for families all the nice
things one can do for a single linear dynamical
system, as for example - realization theory for
a family of input/output maps (cf. also [3,4] )
- pole placement and stabilization by feedback
(cf. also [4,14]
- decomposition (e.g. completely reachable sub-
systems)
- Controllability subspaces and theirapplication
In view of the reinterpretation (sketched above)
of a system(1.2) over a ring R as an algebraic-
geometric family of systems over Spec(R), the
general project encompasses trying to do all
these things for systems over rings, and this
constitutes an important bit of motivation for
studying families of systems.

A related, and important, bit of motivatic
comes from linear delay differential dynamical

systems as e.g.

$$\dot{x}_1(t) = x_1(t) + x_2(t-1) + u(t-1)$$
$$\dot{x}_2(t) = x_1(t-1) + u(t) \qquad\qquad (1.4)$$
$$y(t) = x_1(t) + x_2(t-2)$$

Introducing the delay operator $\sigma$, $\sigma x(t) = x(t-1)$, we can write (1.4) formally as a linear system over the ring $R[\sigma]$, viz.

$$\dot{x}(t) = F(\sigma)x(t) + G(\sigma)u(t)$$
$$y(t) = H(\sigma)x(t) \qquad\qquad (1.5)$$

where $F(\sigma)$, $G(\sigma)$, $H(\sigma)$ are the following matrices with coefficients in the ring of polynomials $R[\sigma]$

$$F(\sigma) = \begin{pmatrix} 1 & \sigma \\ \sigma & 0 \end{pmatrix} \;,\; G(\sigma) = \begin{pmatrix} \sigma \\ 1 \end{pmatrix} \;,\; H(\sigma) = (1,\sigma^2).$$

As it turns out this rather formal looking procedure is most useful, [9]. For instance in a very nice paper [8], Ed Kamen has worked out some of the relationship between the spectral properties of (1.4) and the commutative algebra which goes into the study of (1.5). And, using this, and the reinterpretation of (1.5) as a family of systems, Chrys Byrnes [4] has been able to do things about the feedback stabilization theory of (1.4).

her bits of motivation for studying families
e.g. from identification theory, [7] and study of high-gainfeedback systems, [10].
both these cases it is important to know in at ways a family of systems can suddenly
generate. Ideally one would like to write down ocal (uni)versal deformations for each system, s Arnol'd did for matrices in [1]. For complete-y reachable or completely observable systems universal deformations result from the fine moduli spaces of [5,6]. And in fact the original starting point for this paper was the far too optimistic idea that these moduli spaces might quite well be extendable to some extent. Thus the main problem considered in this paper became: Given two families of linear dynamical systems $\Sigma$, $\Sigma'$ over a manifold $Q$. Suppose that pointwise the systems $\Sigma_q$, $\Sigma'_q$ are isomorphic for all or almost all $q \in Q$. What can be said

about the relation between $\Sigma$ and $\Sigma'$ as families and what can be said about the relations between $\Sigma_q$ and $\Sigma'_q$ at the remaining points of $Q$.

The first question is of course entirely analogous to the one studied by Wasow [13], and later in an algebraic setting by Ohm and Schneider [11], with respect to similarity of families of matrices which depend (holomorphically) on a parameter.

## 2. ALMOST EVERYWHERE ISOMORPHIC FAMILIES OF SYSTEMS.

We use the abbreviations cr for completely reachable and co for completely observable. Recall that the system (1.1) is cr iff the matrix

$$R(F,G) = (G \quad FG \quad \ldots \quad F^n G) \qquad \text{(\!\!\!\!\!\bullet\!\!\!)}$$

is of full rank n, and that (1.1) is co iff the matrix $Q(F,H)$ is of full rank n. Here $Q(F,H)$ is defined as

$$Q(F,H)^T = (H^T \quad F^T H^T \quad \ldots \quad F^{nT} H^T) \qquad (2.2)$$

where the symbol $T$ means "transposes".
If $\Sigma = (F,G,H)$ is a family of linear dynamical systems o er a topological space $Q$ we denote with $\Sigma(q)$ the system $(F(q),G(q),H(q))$. Completely analogously if $\Sigma = (F,G,H)$ is a (discrete time) system over a ring $R$ then $\Sigma(p) = (F(p), G(p), H(p))$ is the induced system over $R(p)$, The quotient field of $R/p$.

2.3. THEOREM. Let $\Sigma$ and $\Sigma'$ be two families over a topological space $Q$. Let $U_1 = \{q \in Q: \Sigma(q) \text{ and } \Sigma'(q) \text{ are both cr}\}$ and $U_2 = \{q \in Q: \Sigma(q) \text{ and } \Sigma'(q) \text{ are both co}\}$. Suppose that $U_1 \cup U_2 = Q$ and suppose that $\Sigma(q)$ and $\Sigma'(q)$ are pointwise isomorphic for a dense set $Z$ of points $q$ in $Q$. Then $\Sigma$ and $\Sigma'$ are isomorphic as families over $Q$, (which, by definition, means that there is a <u>continuous</u> map $Q \to GL_n(\mathbb{R})$, $q \mapsto S(q)$, such that $F'(q) = S(q)F(q)S(q)^{-1}$, $G'(q) = S(q)G(q)$, $H'(q) = H(q)S(q)^{-1}$ for all $q \in Q$).

It follows in particular that $\Sigma(q)$ and $\Sigma'(q)$ are also isomorphic in all the points of $Q \smallsetminus Z$.

The (local) algebraic geometric version of this theorem is

2.4. THEOREM. Let $\Sigma$ and $\Sigma'$ be two systems over a ring R. Let $U_1 = \{\mathfrak{p} \in \text{Spec}(R) | \Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are both cr$\}$, $U_2 = \{\mathfrak{p} \in \text{Spec}(R) | \Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are both co$\}$. Suppose that $U_1 \cup U_2 = \text{Spec}(R)$ and that there is a dense subset $Z \subset \text{Spec}(R)$ such that $\Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are isomorphic for all $\mathfrak{p} \in Z$. Then $\Sigma$ and $\Sigma'$ are isomorphic as systems over R.

This means in particular that if R is an integral domain and $\Sigma = (F,G,H)$, $\Sigma' = (F',G',H')$ are two n-dimensional systems over R which are isomorphic over K, the quotient field of R, and if moreover for all maximal ideals $\mathfrak{m} \subset R$ we have that the rank of both $R(F,G)$, $R(F',G')$ or of both $Q(F,H)$, $Q(F',H')$ stays n mod $\mathfrak{m}$, then $\Sigma$ and $\Sigma'$ are also isomorphic as systems over R.

Both theorems 2.3 and 2.4 are almost trivial consequences of the existence of fine moduli spaces for cr families and for co families. These exist both in the topological case (cf.[5]) and the algebraic geometric case (cf.[6]). The proofs of 2.3 and 2.4 now go roughly as follows. By the existence of the fine moduli space $M^{cr}$ for cr families, such families over Q correspond (up to isomorphism) bijectively to continuous maps $Q \to M^{cr}$. It follows that $\Sigma$ and $\Sigma'$ are isomorphic over $U_1$. Similarly using the fine moduli space $M^{co}$ they are isomorphic over $U_2$. On $U_1 \cap U_2$ finally these isomorphisms agree because two cr or co systems can have at most one isomorphism between them.

The trouble with theorems 2.3 and 2.4 is that, unless one demands something like pointwise isomorphism everywhere, or cr everywhere, or co everywhere, the condition $U_1 \cup U_2 = Q$ cannot be stated in terms of the separate families $\Sigma$ and $\Sigma'$. So one is lead to ask whether not a condition like everywhere co or cr would be sufficient. It is not, as is more or less predictable from the wellknown fact that as a rule it is perfectly possible for two nonisomorphic systems $\Sigma$ and $\Sigma'$ over an integral domain R to become isomorphic

over the quotient field, [12]. The simplest such example is undoubtedly the following one dimensional one over $\mathbb{R}[\sigma]$.

$$\Sigma : F = 1, G = \sigma, H = 1$$
$$\Sigma': F'= 1, G'= 1, H'= \sigma \quad (2.5)$$

Considered as families over $Q = \mathbb{R}$, parametrized by $\sigma$, we have that $\Sigma$ is co everywhere and cr everywhere except in 0, while $\Sigma'$ is cr ev erywhere and co everywhere except in 0. Thus $U_1 = U_2 = \mathbb{R} \smallsetminus \{0\}$. Also $\Sigma(q)$ and $\Sigma'(q)$ are isomorphic for all $q \neq 0$. But of course $\Sigma$ and $\Sigma'$ are not isomorphic as families nor as systems over the ring $\mathbb{R}[\sigma]$.

Another example, which is slightly more illustrative of what goes on is given by the families

$$\Sigma = ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}1&1\\\sigma&b\end{smallmatrix}), (1.0))$$
$$\Sigma'= ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}1&\sigma\\1&b\end{smallmatrix}), (1,0)) \quad (2.6)$$

which have essentially the same properties as the families (2.5). And here we note that though $\Sigma(0)$ and $\Sigma'(0)$ are of course not isomorphic, they are also not totally unrelated. In fact they agree on the completely reachable subsystem of $\Sigma(0)$. (For a more precise description of what this means, cf. below). Note also that these examples largely destroy all hope about extending the fine moduli spaces $M^{cr}_{m,n,p}$ and $M^{co}_{m,n,p}$ a bit.

2.7. MORPHISMS. Let $\Sigma$ and $\Sigma'$ be two families over Q. A morphism $\Sigma \to \Sigma'$ over Q then consist of a continuous map $\psi : Q \to M^{n \times n}$ the space of n x n matrices such that for all $q \in Q$, $\psi(q)G(q) = G'(q)$, $F'(q)\psi(q) = \psi(q)F(q)$, $H'(q)\psi(q) = H(q)$. Completely analogously a morphism $\Sigma \to \Sigma'$ between two systems over a ring R is an n x n matrix T such that $TG = G'$, $F'T = TF$, $H'T = H$. Using this notion one can now state the two following (dual) "mildness of degeneracy" results.

2.8. THEOREM. Let $\Sigma$ and $\Sigma'$ be two families over Q. Suppose that $\Sigma(q)$ is cr for all $q \in Q$. Suppose moreover that $\Sigma'(q)$ and $\Sigma(q)$ are isomorphic for all q in a dense subset Z of Q. Then there is a

morphism T: $\Sigma \to \Sigma'$ over Q such that
T(q): $\Sigma(q) \to \Sigma'(q)$ is an isomorphism for all
$q \in Z$ and such that T(q): $\Sigma(q) \to \Sigma'(q)$ maps the
state space of $\Sigma(q)$ onto the completely
reachable subspace of the state space of $\Sigma'(q)$
for all $q \in Q$.

2.9. THEOREM. Let $\Sigma$ and $\Sigma'$ be two families over
Q. Suppose that $\Sigma(q)$ is co for all $q \in Q$. Suppose
moreover that $\Sigma'(q)$ and $\Sigma(q)$ are isomorphic for
all q in a dense subset Z of Q. Then there is a
morphism T: $\Sigma' \to \Sigma$ over Q such that T(q): $\Sigma(q) \to$
$\Sigma'(q)$ is an isomorphism for all $q \in Z$ and such
that for all $q \in Q \smallsetminus Z$ two states x,x' in state
space of $\Sigma'(q)$ are indistinguishable (by means
of observations) if and only if their difference
$x - x'$ is in Ker(T (q)).

There are of course the obvious analogous results
for systems over rings. In this case 2.8 says,
among other things, that the system over a ring
R which is cr everywhere is maximal in the
lattice of all realizations over R which realize
the same input/output behaviour; similarly 2.9
says that the everywhere co realization is the
minimal element of this lattice.

2.10. ON THE PROOFS OF 2.8 AND 2.9.
Let $q \in Q$. Because $\Sigma$ is cr in q, there are a
nice selection $\alpha$ (cf.[5]) and an open subset U
of q such that $R(F(q'),G(q'))_\alpha$ is invertible
for all $q' \in U$. Now let $z_1, z_2, \ldots$ be a sequence
of points of $Z \cap U$ converging to q.
We define the matrix T(q) as the limit

$$T(q) = \lim_{i \to \infty} R(F(z_i),G(z_i))_\alpha^{-1} R(F'(z_i),G'(z_i))_\alpha$$

It is not difficult to check that T(q) does not
depend on the choice of $\alpha$ or on the choice of
$z_1, z_2, \ldots$ and to check that the T(q) combine
to define a continuous map T: $Q \to M^{n \times n}$. If
$q \in Z$, then T(q) is of course the unique
isomorphism $\Sigma(q) \to \Sigma'(q)$. It follows that T
induces a morphism $\Sigma \to \Sigma'$ over Z and by
continuity it follows that T is a morphism over
Q. For each $q \in Q$ we then have

$$T(q)R(F(q),G(q)) = R(F'(q),G'(q)) \qquad (2.11).$$

The last statement of the theorem now follows by
a rank consideration. The proof of 2.9 is similar
(or use duality).

2.12. EXAMPLE. Let $\Sigma$ and $\Sigma'$ be two families over
Q, which are pointwise isomorphic over a dense
subset Z of Q. Then, without any further
assumptions, we know of course that for all $q \in Q$,
$\Sigma(q)$ and $\Sigma'(q)$ are related in the sense that their
cr and co subquotients are isomorphic. This
follows from the continuity of the Laplace
transform. Beyond this there seems little one can
say (without making some sort of stableness
hypothesis as in 2.8 and 2.9 above), as the
following example shows.

$$\Sigma = ((\begin{smallmatrix}1\\0\end{smallmatrix})), (\begin{smallmatrix}1 & a\\\sigma & 2\end{smallmatrix}), (o,1)) \qquad (2.13)$$

$$\Sigma' = ((\begin{smallmatrix}\sigma\\1\end{smallmatrix}), (\begin{smallmatrix}1-\sigma a & \sigma a\\- a & \sigma a+2\end{smallmatrix}), (0\ \sigma))$$

These families are pointwise isomorphic for all
$\sigma \neq 0$. But for $\sigma = 0$ there is not even a morphism
$\Sigma(0) \to \Sigma'(0)$, in fact there is not a morphism
between the input parts of the completely
reachable subsystems of $\Sigma(0)$ and $\Sigma'(0)$.

3. EVERYWHERE POINTWISE ISOMORPHIC FAMILIES OF
SYSTEMS.

Now let $\Sigma$ and $\Sigma'$ be families of systems over Q
(resp. Spec(R)) which are pointwise isomorphic
everywhere. Then it does          not necessarily
follow that $\Sigma$ and $\Sigma'$ are isomorphic as families
over Q (resp. are isomorphic as systems over R),
as the following example shows.

3.1. EXAMPLE. Consider the two families over R
(or the two systems over $\mathbb{R}[\sigma]$) defined by

$$\Sigma = ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}1 & 0\\\sigma 2 & 1\end{smallmatrix}), (1,2))$$

$$\Sigma' = ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}1 & 0\\\sigma & 1\end{smallmatrix}), (1,2\sigma))$$

These two families are pointwise isomorphic for
all $\sigma$(resp. the systems $\Sigma(p)$, $\Sigma'(p)$ are
isomorphic for all prime ideals $p \subset \mathbb{R}[\sigma]$) but
they are not isomorphic as families over $\mathbb{R}$
(resp. as systems over $\mathbb{R}[\sigma]$); indeed $\Sigma$ and $\Sigma'$
are not isomorphic in any neighbourhood of 0
(resp. not isomorphic over any localization
$\mathbb{R}[\sigma]_f$ of $\mathbb{R}[\sigma]$ for which $f(0) \neq 0$).

So we shall need some sort of extra condition to insure that pointwise isomorphism implies isomorphism as families.

3.2. STABILIZER SUBGROUPS. Let $\Sigma$ be a family over $Q$. Then for each $q \in Q$ we define

$$N(q) = \{S \in GL_n(\mathbb{R}): SF(q) = F(q)S, SG(q) = *G(q), H(q)S = H(q)\}.$$

This is the stabilizer subgroup in $GL_n(\mathbb{R})$ of the system $\Sigma(q)$. The Lie algebra of $N(q)$ is

$$L(q) = \{T \in M^{n \times n} | TF(q) = F(q)T, TG(q) = 0, H(q)T = 0\}$$

We use $r(q)$ to denote the dimension of $N(q)$ which is of course equal to the dimension of $L(q)$. Completely analogously one defines in the case of a system $\Sigma = (F,G,H)$ over a ring $R$ the subgroup $N(\mathfrak{p})$ of $GL_n(R(\mathfrak{p}))$ consisting of all invertible matrices $S$ over the field $R(\mathfrak{p})$ (= quotient field of $R/\mathfrak{p}$), such that $SF(\mathfrak{p}) = F(\mathfrak{p})S$, $SG(\mathfrak{p}) = G(\mathfrak{p})$, $H(\mathfrak{p})S = H(\mathfrak{p})$, and $L(\mathfrak{p})$ as the Lie algebra of all $n \times n$ matrices $T$ with coefficients in $R(\mathfrak{p})$ such that $TF(\mathfrak{p}) = F(\mathfrak{p})T$, $TG(\mathfrak{p}) = 0$, $H(\mathfrak{p})T = 0$.

3.3. THEOREM. Let $\Sigma$ and $\Sigma'$ be two differentiable families over the differentiable manifold $Q$. Suppose that $\Sigma$ and $\Sigma'$ are pointwise isomorphic everywhere. Suppose moreover that $r(q) = \dim N(q)$ (= dim $L(q)$) is constant in some neighbourhood $U$ of $q_0 \in Q$. Then there is a (possibly smaller) neighbourhood $V$ of $q_0$ such that $\Sigma$ and $\Sigma'$ are isomorphic as differentiable families over $V$.

Here a family is differentiable if the map $q \mapsto (F(q),G(q),H(q))$ is differentiable. and an isomorphism of families $V \to GL_n(\mathbb{R})$ is differentiable if this map is differentiable. For the proof at least, some sort of differentiability restriction is necessary. There are analogous theorems for holomorphic families and real analytic families. The corresponding theorem for systems over rings is

3.4. THEOREM. Let $\Sigma$ and $\Sigma'$ be two systems over a ring $R$. Suppose that $\Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are isomorphic for all prime ideals $\mathfrak{p}$ contained in

some open subset $U$ of $Spec(R)$. Suppose moreover that $r(\mathfrak{p}) = \dim N(\mathfrak{p})$ is constant for some neighbourhood $U'$ of $\mathfrak{p}_0 \in U$. Then there exists an open neighbourhood $V = Spec(R_f)$, $f \in R$, of $\mathfrak{p}_0$ such that $\Sigma$ and $\Sigma'$ are isomorphic as systems over $R_f$ (or, equivalently, as families over $V$).

For both these theorems it is in general not true that $\Sigma$ and $\Sigma'$ are necessarily isomorphic over all of $Q$ (resp. isomorphic as systems over $R$) as the following example shows.

3.5. EXAMPLE. Consider the following two systems, either as families over $\mathbb{R}$ or as systems over the ring $\mathbb{R}[\sigma]$

$$\Sigma = ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}1 & \sigma_2\\0 & \sigma\end{smallmatrix}), (\sigma^2-1,-\sigma))$$
$$\Sigma' = ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}1 & \sigma+2\\0 & \sigma^2\end{smallmatrix}), (\sigma^2-1,-\sigma-2))$$

These two families are pointwise isomorphic everywhere; the dimension of the stabilizer subgroups is 1 everywhere; in addition one has that rank $R(F(\sigma), G(\sigma))$ and rank $Q(F(\sigma),H(\sigma))$ are also equal to 1 everywhere. As families the two systems are isomorphic over $\mathbb{R} \smallsetminus \{-1\}$ and also over $\mathbb{R} \smallsetminus \{1\}$. As systems over rings they are isomorphic over $\mathbb{R}[\sigma]_{\sigma-1}$ and $\mathbb{R}[\sigma]_{\sigma+1}$, but not, as is easily checked, as systems over $\mathbb{R}[\sigma]$ itself. The systems $\Sigma$ and $\Sigma'$ are not even isomorphic as differentiable (or topological) families. Indeed such an isomorphism must necessarily be of the form $\sigma \mapsto \begin{pmatrix} 1 & c_{12} \\ 0 & c_{22} \end{pmatrix}$, where $c_{12}$ and $c_{22}$ are continuous functions, such that $c_{22}$ is nowhere zero on $\mathbb{R}$. One calculates that $c_{12}, c_{22}$ must then satisfy that

$$c_{12}(\sigma) = (\sigma^2-1)^{-1}c_{22}(\sigma)(\sigma+2) - \sigma$$

For this to remain finite in $\sigma = 1$ and $-1$, we must have $3c_{22}(1) - 1 = 0$ and $c_{22}(-1) + 1 = 0$. i.e. $c_{22}(1) = 3^{-1}$, $c_{22}(-1) = -1$ and there is no real continuous function assuming these values in 1 and $-1$ and which is also everywhere nonzero.

3.6. ON THE PROOF OF THEOREM 3.3. To prove theorem 3.3 one considers the map $Q \times GL_n \to Q \times L_{m,p,p}$, given by $\phi : (q,S) \mapsto (q,(SG(q), SF(q)S^{-1},H(q)S^{-1}))$. The constant dimension

assumption means that this map has constant rank, so that the image is locally a differentiable submanifold of $Q \times L_{m,n,p}$. Note that the fibre of $\phi$ at $(q, \Sigma'(q))$ is precisely the set of all possible isomorphisms $\Sigma(q) \to \Sigma'(q)$. Let $Q'$ be the submanifold of $Q \times L_{m,n,p}$ defined by $q \mapsto (q, \Sigma'(q))$. Then $Q' \subset \text{Im}\phi$ by the everywhere pointwise isomorphic hypothesis. Using that $\phi$ is a submersion onto its image it now follows that $\phi^{-1}(Q') \to Q'$ admits local sections, proving the theorem.

To prove the local algebraic geometric version of theorem 3.3, that is theorem 3.4, we use a somewhat different idea. The main ingredient is the following generalization of the central lemma of [11].

3.7. LEMMA. Let R be a ring without nilpotents, let A be an $m \times n$ matrix with coefficients in R and let $a \in R^m$. Consider the equation $Ax = a$. Suppose that the equation $A(\mathfrak{p})y = a(\mathfrak{p})$ over the field $R(\mathfrak{p})$ can be solved for all prime ideals $\mathfrak{p}$. Suppose moreover that $r(\mathfrak{p}) = \text{rank } A(\mathfrak{p})$ is constant (as a function of $\mathfrak{p}$). Then $Ax = a$ is solvable over R. Moreover if $\mathfrak{m}$ is a maximal ideal of R and $y(\mathfrak{m})$ is any pregiven solution of $A(\mathfrak{m})y = a(\mathfrak{m})$, then there is a solution x of $Ax = a$ over R such that $x = y(\mathfrak{m}) \mod \mathfrak{m}$. Finally if $\mathfrak{p}$ is a prime ideal and $y(\mathfrak{p})$ is any given solution of $A(\mathfrak{p})y = a(\mathfrak{p})$ then there is an $f \in R \smallsetminus \mathfrak{p}$ and a solution of $Ax = a$ over $R_f$ such that $x = y(\mathfrak{p}) \mod \mathfrak{p} R_f$.

3.8. ON THE PROOF OF LEMMA 3.7. Let $P = \text{Im}(A)$, $Q = R^m/\text{Im}(A)$. Then it readily follows from the rank hypothesis and the fact that R has no nilpotents that for all prime ideals $\mathfrak{p}$ the localization $Q_{\mathfrak{p}} = \text{Coker}(R_{\mathfrak{p}}^n \to R_{\mathfrak{p}}^m)$ is free of rank $m - r$ (where $r = r(\mathfrak{p})$). It follows that Q is a projective R-module ([2], Ch. II, §5) and hence a direct summand of a free module. Now consider the image $\bar{a}$ of a in $Q = R^m/\text{Im}(A)$. The solvability of $A(\mathfrak{p})y = a(\mathfrak{p})$ means that $\bar{a}$ maps to zero under $Q \to Q(\mathfrak{p}) = Q \otimes R(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$. Because Q is projective and R is reduced it follows that $\bar{a} = 0$ proving that $Ax = a$ is solvable over R. Now let $y(\mathfrak{m})$ be any pregiven

solution of $.(\mathfrak{m})y = a(\mathfrak{m})$ where $\mathfrak{m}$ is a maximal ideal of R. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \to & C & \to & R^n & \overset{A}{\to} & P & \to & 0 \\
& & \downarrow j' & & \downarrow j & & \downarrow & & \\
0 & \to & C(\mathfrak{m}) & \to & R(\mathfrak{m})^n & \to & P(\mathfrak{m}) & \to & 0
\end{array}
$$

where C is the kernel of $A: R^n \to R^m$. The module P is also projective as the kernel of $R \to Q$. It follows that the lower sequence is also exact. Some diagram chasing, using that $j'$ is surjective now readily proves the second assertion of the lemma. If $\mathfrak{p} \subset R$ is prime, one argues exactly the same. The only extra difficulty is that $j': C \to C(\mathfrak{p})$ is not necessarily surjective. However, if $z \in C(\mathfrak{p})$ is any element, then there always is an $f \in R \smallsetminus \mathfrak{p}$ such that z is in the image of $C_f \to C(\mathfrak{p})$.

3.9. ON THE PROOF OF THEOREM 3.4. Given the lemma, the proof of theorem 3.4 is entirely straightforward. Indeed one considers the linear map $A: R^k \to R^\ell$ given by $X \mapsto (XF - F'X, XG, H'X)$ where $k = n^2$ and X is a k-vector written as an $n \times n$ matrix. Here $\ell = n^2 + nm + np$. Now let $a \in R^\ell$ be the vector $(0, G', H)$. The constancy of dim $N(\mathfrak{p}) = $ dim $L(\mathfrak{p})$ means that rank $A(\mathfrak{p}) = $ constant. Now let $\mathfrak{p}_0$ be any prime ideal and $S(\mathfrak{p}_0)$ an invertible matrix over $R(\mathfrak{p}_0)$ taking $\Sigma(\mathfrak{p}_0)$ into $\Sigma'(\mathfrak{p}_0)$. Then $S(\mathfrak{p}_0)$ solves $A(\mathfrak{p}_0)y = a(\mathfrak{p}_0)$. So by the lemma there is a solution S over $R_f$ for some $f \in R \smallsetminus \mathfrak{p}_0$ of $Ax = a$ which moreover agrees with $S(\mathfrak{p}_0) \mod \mathfrak{p}_0$. Because $S(\mathfrak{p}_0)$ is invertible S is invertible over $R_{ff'}$, for some suitable $f' \in R \smallsetminus \mathfrak{p}_0$.

3.10. EXAMPLES. It does not appear that the condition that the dimension of the stabilizer subgroups $N(q)$ remains constant as q varies has much to do with conditions which seem system-theoretically more natural like rank $R(F(q), G(q))$ is constant. Consider for example the family

$$
\Sigma = ((\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 0 & \sigma \end{smallmatrix}), (0,2))
$$

For this family over $\mathbb{R}$ one has rank$(R(F(q)), G(q))$ = 1 = rank$(Q(F(\sigma)), H(\sigma))$ for all $\sigma \in \mathbb{R}$, but dim $N(\sigma) = 1$ if $\sigma = 1$ and dim $N(\sigma) = 0$ otherwise. On the other hand the family

160

$$\Sigma = ((\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}0 & 1\\ \sigma & 1\end{smallmatrix}), (1,0))$$

has dim $N(\sigma) = 0$ everywhere but $\text{rank}(R(F(\sigma),G(\sigma)))$

$= 2$ if $\sigma \neq 0$ and $= 1$ if $\sigma = 0$ (and $\text{rank}(Q(F(\sigma),$

$H(\sigma)) = 2$ everywhere.

## 4. CONCLUSIONS.

The main questions studied in this paper were:

(1) Given two families of system $\Sigma$ and $\Sigma'$ which
are pointwise isomorphic. Are they then also
isomorphic as families?

(2) Given two families of systems $\Sigma$ and $\Sigma'$ over $Q$
which are pointwise isomorphic over $Q$ or some dense
subset $Z$ of $Q$. What can be said about the relations
between $\Sigma(q)$ and $\Sigma'(q)$ at the points of $Q \smallsetminus Z$.
Question (1) received a positive answer which
specializes to a theorem of Wasow's [13] for
holomorphic families of matrices under similarity
It seems also likely that the theorem is best
possible in the sense that if $\Sigma$ is a family such
that dim $N(q)$ is not constant then there is a
family $\Sigma'$ which is pointwise isomorphic to $\Sigma$
everywhere but not isomorphic as families in any
neighbourhood of a point $q$ where dim $N(q)$ sudden-
ly increases. As to question (2), they are
definite relations between $\Sigma(q)$ and $\Sigma'(q)$ if
either $\Sigma$ or $\Sigma'$ is cr or co in a neighbourhood of
$q$. If not than a number of examples show that the
ways in which a family of systems can degenerate
do not depend only on the isomorphism classes
of the systems involved but also on the systems
themselves (apart from the subquotients which
are recoverable from the transferfunctions
(cf. also [7]). Thus one has here the usual scaling
and singular perturbation phenomena. It remains
to construct local versal deformations of non cr
and non co systems.

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