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ASSOCIATION SCHEMES AND THE SHANNON CAPACITY: EBERLEIN POLYNOMIALS AND THE ERDŐS – KO – RADO THEOREM

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ABSTRACT

We use some results of [16] (where Delsarte's linear programming bound for cliques in association schemes is set against Lovász's ϑ-bound for the Shannon capacity of a graph) to prove that the Shannon capacity of the graph $K_{\nu,n,t}$ equals $\binom{\nu - t}{n - t}$ if ν is large with respect to n ($K_{\nu,n,t}$ is the graph with vertices all n-subsets of a fixed ν-set, two of them being adjacent iff their intersection contains less than t elements).

0. INTRODUCTION

In [16] (and, independently, in [14] (cf. [13]), and also by A.M. Odlyzko & L. Shepp) relations between two upper bound functions are described: Lovász's ϑ-function [12], an upper bound for the "Shannon capacity" $\Theta(G)$ of a graph G, and Delsarte's linear programming bound [5], an upper bound for the size of "cliques in association schemes". In the present paper, after a review of these two bounds and their interrelations, we apply results of [16] to prove that the Shannon capacity

$\Theta(K_{v,n,t})$ of $K_{v,n,t}$ equals $\binom{v-t}{n-t}$ if v is large enough (with respect to n); here $K_{v,n,t}$ denotes the graph with vertices all n-subsets of a fixed v-set, two of them being adjacent iff their intersection contains at most $t-1$ elements. This extends theorems of Erdős, Ko & Rado [7] (the independence number $\alpha(K_{v,n,t})$ equals $\binom{v-t}{n-t}$ if v is large enough) and Lovász [12] ($\Theta(K_{v,n,1}) = \binom{v-1}{n-1}$ if v is large enough). (Alas we were not able to derive a good estimate how large v must be.) The proof method may be viewed as an algebraic extension (using Eberlein polynomials) of Katona's method [10] (using t-designs) to show (partially) the Erdős – Ko – Rado theorem.

1. ASSOCIATION SCHEMES AND DELSARTE'S LINEAR PROGRAMMING BOUND

(See Delsarte [5] or MacWilliams & Sloane [15].) A pair $(X, \mathscr{R})$, where $\mathscr{R} = (R_0, \ldots, R_n)$ is a partition of $X \times X$, is called a *(symmetric) association scheme* (introduced by Bose & Shimamoto [4]), with *intersection numbers* p_{ij}^k $(i, j, k = 0, \ldots, n)$, if

(1) $R_0 = \{(x, x) \mid x \in X\}$;

(2) $(x, y) \in R_k$ iff $(y, x) \in R_k$, for $k = 0, \ldots, n$;

(3) for all $i, j, k = 0, \ldots, n$, and $(x, y) \in R_k$:
$|\{z \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}| = p_{ij}^k$.

So $p_{ij}^k = p_{ji}^k$. We may consider the pairs (X, R_k) as undirected graphs $(k = 1, \ldots, n)$; (X, R_k) is regular of valency $v_k = p_{kk}^0$ $(v_0 = 1)$. Therefore $p_{ij}^0 = \delta_{ij} v_i$.

It is easily seen that a symmetric association scheme is equivalent to a labeling of the edges of a complete graph K_m with labels $1, \ldots, n$ such that, for each $k = 1, \ldots, n$, the edges with label k build up a regular graph, and, for each $i, j, k = 1, \ldots, n$, any edge $\{x, y\}$ with label k is in exactly p_{ij}^k triangles xyz with $\{x, z\}$ labelled i and $\{y, z\}$ labelled j (this number p_{ij}^k only being dependent on i, j and k and not on the particular choice of $\{x, y\}$).

We give three families of examples of association schemes; the first two examples are of interest for coding theory.

(a) Let n and q be natural numbers and let X be the set of vectors of length n, with entries in $\{0, \ldots, q-1\}$. Moreover, let, for $k = 0, \ldots, n$:

(4) $$R_k = \{(x, y) \in X \times X \mid d_H(x, y) = k\},$$

where $d_H(x, y)$ denotes the *Hamming distance* of the vectors x and y, i.e. the number of coordinate places in which x and y differ. Let $\mathscr{R} = (R_0, \ldots, R_n)$. As can be checked easily $(X, \mathscr{R})$ is an association scheme. Schemes obtained in this way are called *Hamming schemes.*

(b) The second family is obtained as follows. Let v and n be natural numbers and let X be the set of $0, 1$-vectors of length v with exactly n ones. Moreover, let, for $k = 0, \ldots, n$:

(5) $$R_k = \{(x, y) \in X \times X \mid d_J(x, y) = k\},$$

where $d_J(x, y) = \frac{1}{2} d_H(x, y)$ is the *Johnson distance* between x and y. Let $\mathscr{R} = (R_0, \ldots, R_n)$. Then $(X, \mathscr{R})$ is an association scheme; schemes constructed in this way are called *Johnson schemes.*

(c) A third family of association schemes is formed by *strongly regular graphs* (introduced by Bose [3], cf. Seidel [17]). These are exactly those graphs (X, R_1) such that $(X, \mathscr{R})$ is an association scheme, where $\mathscr{R} = (R_0, R_1, R_2)$ and $R_2 = (X \times X) \setminus (R_0 \cup R_1)$. It follows that the complementary graph of a strongly regular graph is strongly regular again.

Let D_i denote the adjacency matrix of the graph (X, R_i) $(i = 0, \ldots, n)$.

Since, by (3), one has $D_i D_j = \sum_k p_{ij}^k D_k$ (for $i, j = 0, \ldots, n$) the matrices $D_0, \ldots, D_n$ generate a commutative algebra of symmetric matrices of dimension $n + 1$, called the *Rose – Mesner algebra* of the scheme. So $D_0, \ldots, D_n$ can be diagonalized simultaneously (i.e., there is a nonsingular matrix S such that $SD_0S^T, \ldots, SD_nS^T$ are diagonal matrices) and there is a matrix $P = (P_k^u)_{k,u=0}^n$ so that

(6) $\qquad P_k^0, \ldots, P_k^n$ are the eigenvalues of D_k $(k = 0, \ldots, n)$,

and

(7) $\qquad P_0^u, \ldots, P_n^u$ belong to a common eigenvector x of $D_0, \ldots, D_n$, respectively, (i.e., $D_0x = P_0^u x, \ldots, D_nx = P_n^u x$) $(u = 0, \ldots, n)$.

We may assume that $P_k^0 = \nu_k$, for $k = 0, \ldots, n$. Set

$$Q_k^u = \frac{\mu_u}{\nu_k} P_k^u, \tag{8}$$

where μ_u is the dimension of the common eigenspace of $D_0, \ldots, D_n$ belonging to $P_0^u, \ldots, P_n^u$, respectively $(u = 0, \ldots, n)$. It can be shown that

$$\sum_{u=0}^n P_k^u Q_h^u = m\delta_{kh} \quad \text{and} \quad \sum_{k=0}^n P_k^u Q_k^v = m\delta_{uv}, \tag{9}$$

where $m = |X|$, i.e. P and $\frac{1}{m} Q^T$ represent inverse matrices.

For Haming schemes (example (a)) the values of ν_k, μ_u and P_k^u are given by:

$$\nu_k = \binom{n}{k}(q-1)^k, \quad \mu_u = \binom{n}{u}(q-1)^u, \tag{10}$$

$$\begin{aligned} P_k^u = K_k(u) &= \sum_{j=0}^k (-1)^j (q-1)^{k-j} \binom{u}{j}\binom{n-u}{k-j} = \\ &= \sum_{j=0}^k (-q)^j (q-1)^{k-j} \binom{n-j}{k-j}\binom{u}{j}, \end{aligned} \tag{11}$$

for $k, u = 0, \ldots, n$ ($K_k(u)$ is the *Krawtchouk polynomial* of degree k in the variable u).

The parameters of Johnson schemes (example (b)) are:

(12) $$v_k = \binom{n}{k}\binom{v-n}{k}, \quad \mu_u = \binom{v}{u} - \binom{v}{u-1} = \frac{v-2u+1}{v-u+1}\binom{v}{u},$$

(13) $$P_k^u = E_k(u) = \sum_{j=0}^{k} (-1)^{k-j}\binom{n-j}{k-j}\binom{n-u}{j}\binom{v-n+j-u}{j} = \sum_{j=0}^{k} (-1)^j\binom{u}{j}\binom{n-u}{k-j}\binom{v-n-u}{k-j},$$

for $k, u = 0, \ldots, n$ ($E_k(u)$ is the *Eberlein polynomial* of degree $2k$ in the variable u).

The main problem of combinatorial coding theory is to estimate the maximum size of any subset C (a "code") of (the set X in) Hamming and Johnson schemes such that no two elements in C have (Hamming or Johnson) distance less than a given value d. A generalized translation of this problem in the language of association schemes needs the notion of an M-clique; given $0 \in M \subset \{0, \ldots, n\}$, a subset Y of X is an *M-clique* if $(x, y) \in \bigcup_{k \in M} R_k$ for all $x, y \in Y$. So the coding problem is to determine the maximum cardinality of $\{0, d, d+1, \ldots, n\}$-cliques in Hamming and Johnson schemes.

To obtain an upper bound for the size of cliques in an association scheme $(X, \mathscr{R})$, define, for $Y \subset X$, the *inner distribution* $(a_0, \ldots, a_n)$ of Y by

(14) $$a_k = \frac{|R_k \cap (Y \times Y)|}{|Y|},$$

for $k = 0, \ldots, n$; so $a_0 = 1$ and $\sum_{k=0}^{n} a_k = |Y|$. Moreover, if Y is an M-clique then $a_k = 0$ whenever $k \notin M$. The number a_k may be seen as the average number, over $x \in Y$, of elements $y \in Y$ with $(x, y) \in R_k$.

Delsarte showed that, for the inner distribution of any subset Y of X, one has

(15) $$\sum_{k=0}^{n} a_k Q_k^u \geqslant 0,$$

for $u = 0, \ldots, n$. Therefore, for M-cliques Y one has

$$
(16)\quad \begin{aligned}
|Y| \leqslant \max \{ \sum_{k=0}^{n} a_k \mid a_0, \ldots, a_n \geqslant 0;\ a_0 = 1;\ a_k = 0 \\
\text{for } k \notin M;\ \sum_{k=0}^{n} a_k Q_k^u \geqslant 0 \text{ for } u = 0, \ldots, n \} = \\
= \min \{ \sum_{u=0}^{n} b_u \mid b_0, \ldots, b_n \geqslant 0;\ b_0 = 1; \\
\sum_{u=0}^{n} b_u P_k^u \leqslant 0 \text{ for } k \in M \setminus \{0\} \}.
\end{aligned}
$$

The equality in (16) follows from the duality theorem of linear programming. This bound on the size of cliques is called *Delsarte's linear programming bound.* One may apply linear programming techniques to calculate its value – see [2] for applications in coding theory.

Clearly, Delsarte's bound for M-cliques is a bound for the independence number $\alpha(G)$ of the graph G with vertex set X, two vertices x and y being adjacent iff (x, y) is not contained in a class R_k with $k \in M$.

The following result of Delsarte shows that the linear programming bound is a sharpening of the Hamming bound in coding theory. Let $(X, \mathscr{R})$ be an association scheme, with $\mathscr{R} = (R_0, \ldots, R_n)$, and let $0 \in M \subset \subset \{0, \ldots, n\}$ and $\bar{M} = \{0\} \cup (\{0, \ldots, n\} \setminus M)$. Then

(17) the product of the linear programming bound for M-cliques and the linear programming bound for $\bar{M}$-cliques is at most $|X|$.

Hence $|Y| \cdot |Z| \leqslant |X|$ for M-cliques Y and $\bar{M}$-cliques Z. Taking $M = \{0, d, d+1, \ldots, n\}$ in a Hamming scheme the Hamming bound follows.

2. THE SHANNON CAPACITY AND LOVÁSZ'S BOUND

Lovász [12] introduced, for any graph G, the number $\vartheta(G)$, being an upper bound for the "Shannon capacity" $\Theta(G)$.

Let $\alpha(G)$ be the maximum number of independent (i.e., non-adjacent) points in a graph G, and let $G \cdot H$ denote the (normal) product of graphs

G and H, i.e., the point set of $G \cdot H$ is the cartesian product of the point sets of G and H, whereas two distinct points of $G \cdot H$ are adjacent iff in both coordinate places the elements are adjacent or equal. G^k denotes the product of k copies of G. Shannon [18] introduced the following number for graphs G:

$$(18) \qquad \Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)},$$

which number is called the *Shannon capacity* of G.

If one considers the points of G as letters in an alphabet, two points being adjacent iff they are "confoundable", then $\alpha(G^k)$ may be interpreted as the maximum number of k-letter messages such that any two of them are inconfoundable in at least one coordinate place.

Since $\alpha(G)^k \leqslant \alpha(G^k)$, it follows that $\alpha(G) \leqslant \Theta(G)$. Equality does not hold in general; e.g., $\alpha(C_5) = 2$, whereas $\alpha(C_5^2) = 5 \leqslant \Theta(C_5)^2$. Lovász showed that, in fact, $\Theta(C_5) = \sqrt{5}$. Actually, he gave a general upper bound for $\Theta(G)$ as follows.

Let $G = (V, E)$ be a graph, with vertex set $V = \{1, \dots, n\}$, and define

$$(19) \qquad \begin{aligned} \vartheta(G) = \min \{ &\operatorname{lev} A \mid A = (a_{ij}) \text{ is a symmetric } n \times n\text{-matrix} \\ &\text{such that } a_{ij} = 1 \text{ whenever } \{i, j\} \notin E \}, \end{aligned}$$

where $\operatorname{lev} A$ denotes the largest eigenvalue of A. Now, if $\alpha(G) = k$, each matrix A satisfying the conditions mentioned in (19) has a $k \times k$ all-one principal submatrix (with largest eigenvalue k), hence $\operatorname{lev} A \geqslant k$. Therefore $\alpha(G) \leqslant \vartheta(G)$. Since, as Lovász proved, $\vartheta(G \cdot H) = \vartheta(G) \cdot \vartheta(H)$ for all graphs G and H, one has $\alpha(G^k) \leqslant \vartheta(G)^k$, which yields the stronger inequality $\Theta(G) \leqslant \vartheta(G)$ (Haemers [9] showed the existence of graphs G with $\Theta(G) < \vartheta(G)$).

Moreover Lovász showed

$$(20) \qquad \begin{aligned} \vartheta(G) = \max \Big\{ &\sum_{i,j} b_{ij} \mid B = (b_{ij}) \text{ is an } n \times n \text{ positive semi-definite} \\ &\text{matrix, with } \operatorname{Tr} B = 1, \text{ and } b_{ij} = 0 \text{ whenever } \{i, j\} \in E \Big\}. \end{aligned}$$

So $\vartheta(G)$ may be considered as both a maximum and a minimum, which makes the function ϑ easier to handle. Lovász found, inter alia, for graphs G (with n points):

(21) $\vartheta(G) \cdot \vartheta(\bar{G}) \geq n$ (where $\bar{G}$ denotes the complementary graph), with equality if G is vertex-transitive; and

(22) $\vartheta(G) \leq \dfrac{-n\lambda_n}{\lambda_1 - \lambda_n}$ if G is regular (λ_1 and λ_n being the largest eigenvalues of the adjacency matrix of G), with equality if G is edge-transitive.

A consequence of (22) is: let $\nu \geq 2n$ and let $K(\nu, n)$ be the graph whose vertices are the n-subset of some fixed ν-set, two vertices being adjacent iff they are disjoint (such graphs are called *Kneser-graphs*). Then

$$\Theta(K(\nu, n)) = \binom{\nu - 1}{n - 1}, \tag{23}$$

generalizing the Erdős – Ko – Rado theorem.

3. ASSOCIATION SCHEMES AND THE SHANNON CAPACITY

The theories of Delsarte and Lovász have a number of common characteristics. Both theories give an upper bound for the independence number of certain graphs, they apply eigenvalue techniques to matrices determined by these graphs, they yield relations between a graph and its complement, and they are applicable to allied structures as Johnson schemes and Kneser-graphs. In [14] and [16] (cf. [13]) interrelations between both theories have been shown; we here give, briefly, some results of [16].

Define for any graph G the number $\vartheta'(G)$ by

(24) $\vartheta'(G) = \max \{ \sum_{i,j} b_{ij} \mid B = (b_{ij})$ is a non-negative positive semi-definite $n \times n$-matrix with $\operatorname{Tr} B = 1$, and $b_{ij} = 0$ whenever $\{i, j\} \in E \}$.

So the difference with (20) is the restriction of the range to non-negative matrices B. It is easy to see that

(25) $$\alpha(G) \leqslant \vartheta'(G) \leqslant \vartheta(G);$$

the first inequality follows from taking, in [24], $b_{ij} = \frac{1}{\alpha(G)}$ if $i, j \in Y$, and $b_{ij} = 0$ otherwise, where Y is a set of independent vertices of size $\alpha(G)$. $\vartheta'(G)$ can be described equivalently as a minimum:

(26) $$\vartheta'(G) = \min \{\operatorname{lev} A \mid A = (a_{ij}) \text{ is a symmetric } n \times n\text{-matrix with } a_{ij} \geqslant 1 \text{ if } \{i, j\} \notin E\}$$

If G has, as edge set, the union of some classes of an association scheme then $\vartheta'(G)$ coincides with Delsarte's linear programming bound; that is, let $(X, \mathcal{R})$ be an association scheme, with $\mathcal{R} = (R_0, \dots, R_n)$, and let $0 \in M \subset \{0, \dots, n\}$. Let $G = (X, E)$ be the graph with $E = \bigcup_{k \notin M} R_k$. Then $\vartheta'(G)$ is equal to the linear programming bound (16) for M-cliques.

On the other hand, if G has such union of classes of an association scheme as edge set then

(27) $$\begin{aligned} \vartheta(G) = \max \{ \sum_{k=0}^{n} a_k \mid a_0 = 1;\ a_k = 0 \text{ for } k \notin M; \\ \sum_{k=0}^{n} a_k Q_k^u \geqslant 0 \text{ for } u = 0, \dots, n\} = \\ = \min \{ \sum_{u=0}^{n} b_u \mid b_0 = 1;\ b_0, \dots, b_n \geqslant 0; \\ \sum_{u=0}^{n} b_u P_k^u = 0 \text{ for } k \in M \setminus \{0\}\}. \end{aligned}$$

It can be proved from (27) that

(28) if the edge set of graph $G = (X, E)$ is the union of some classes of an association scheme $(X, \mathcal{R})$ then $\vartheta(G) \cdot \vartheta(\bar{G}) = |X|$.

4. EBERLEIN-POLYNOMIALS AND THE ERDŐS – KO – RADO THEOREM

Let ν, n and t be natural numbers $(\nu \geq n \geq t \geq 1)$ and let $K_{\nu,n,t}$ be the graph whose vertices are all n-subsets of a fixed ν-set, two of them being adjacent iff their intersection contains less than t elements. Erdős, Ko and Rado [7] proved that $\alpha(K_{\nu,n,t}) = \binom{\nu - t}{n - t}$ if ν is large enough with respect to n; that is, the maximum number of k-subsets of a ν-set such that any two of them intersect in at least t elements equals $\binom{\nu - t}{n - t}$ if ν is large enough. Lovász [12] proved that $\Theta(K_{\nu,n,1}) = \binom{\nu - 1}{n - 1}$ for $\nu \geq 2n$. We here show that $\Theta(K_{\nu,n,t}) = \binom{\nu - t}{n - t}$ for $\nu \to \infty$, by proving

Theorem. $\vartheta(K_{\nu,n,t}) = \binom{\nu - t}{n - t}$ *if* ν *is large with respect to* n.

Proof. Since, trivially, $\vartheta(K_{\nu,n,t}) \geq \binom{\nu - t}{n - t}$, it is sufficient to prove that $\vartheta(K_{\nu,n,t}) \leq \binom{\nu - t}{n - t}$ $(\nu \to \infty)$. As the edge set of $K_{\nu,n,t}$ is the union of some classes in a Johnson scheme, by (28) it is enough to prove that

$$(29) \qquad \vartheta(\overline{K_{\nu,n,t}}) \geq \frac{\binom{\nu}{n}}{\binom{\nu - t}{n - t}},$$

for $\nu \to \infty$. So apply (27) to the Johnson scheme, with $M = \{0, n - t + 1, \ldots, n\}$ (two t-subsets are adjacent in $\overline{K_{\nu,n,t}}$ if their Johnson distance is not in M). To prove (29) we must find $a_0, \ldots, a_n$ such that

$$(30) \qquad \sum_{k=0}^{n} a_k \geq \frac{\binom{\nu}{n}}{\binom{\nu - t}{n - t}},$$

$$(31) \qquad a_0 = 1, \ a_1 = a_2 = \ldots = a_{n-t} = 0;$$

$$\sum_{k=0}^{n} a_k \frac{\mu_u}{\nu_k} E_k(u) \geqslant 0 \quad \text{for} \quad u = 0, \dots, n, \tag{32}$$

where μ_u, ν_k and $E_k(u)$ are as given in (12) and (13). To this end take

$$a_k = \frac{1}{\binom{\nu - t}{n - t}} \binom{n}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{\nu - ((n-j) \wedge t)}{\nu - n}, \tag{33}$$

where $p \wedge q$ denotes the minimum of p and q. Note that if a $t-(\nu, n, 1)$ design exists $(a_0, \dots, a_n)$ is the inner distribution of this design; in that case (30), (31) and (32) are satisfied by (15). But we have to prove them in general (for $\nu \to \infty$).

We first prove (30).

$$\begin{aligned} \sum_{k=0}^{n} a_k &= \binom{\nu - t}{n - t}^{-1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{\nu - ((n-j) \wedge t)}{\nu - n} \times \\ &\quad \times \sum_{k=j}^{n} (-1)^k \binom{n-j}{k-j} = \\ &= \binom{\nu - t}{n - t}^{-1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{\nu - ((n-j) \wedge t)}{\nu - n} (-1)^n \delta_{0,n-j} = \\ &= \frac{\binom{\nu}{n}}{\binom{\nu - t}{n - t}}. \end{aligned} \tag{34}$$

Also (31) is easily seen to be fulfilled. Trivially $a_0 = 1$, and, moreover, for $k = 1, \dots, n-t$, one has

$$\begin{aligned} &\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{\nu - ((n-j) \wedge t)}{\nu - n} = \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{\nu - t}{\nu - n} = 0, \end{aligned} \tag{35}$$

hence $a_1 = \dots = a_{n-t} = 0$.

We finally prove (32). Set

$$b_u = \sum_{k=0}^{n} a_k \frac{\mu_u}{\nu_k} E_k(u), \tag{36}$$

for $u = 0, \dots, n$. Let n and t be fixed; then b_u only depends on ν, and sometimes we shall write $b_u(\nu)$ instead of b_u. We prove

$$(37) \qquad b_0 = \frac{\binom{\nu}{n}}{\binom{\nu - t}{n - t}},$$

$$(38) \qquad b_1 = \dots = b_t = 0,$$

$$(39) \qquad \lim_{\nu \to \infty} b_u(\nu) > 0 \quad \text{for} \quad u = t + 1, \dots, n,$$

which suffices to prove (32) for $\nu \to \infty$.

(37) is easy to see:

$$(40) \qquad b_0 = \sum_{k=0}^{n} a_k \frac{\mu_0}{\nu_k} E_k(0) = \sum_{k=0}^{n} a_k = \frac{\binom{\nu}{n}}{\binom{\nu - t}{n - t}}.$$

To prove (38) and (39) we use

$$(41) \qquad \sum_{u=0}^{n} b_u \binom{n - u}{n - m}\binom{\nu - m - u}{n - m} = \frac{\binom{\nu}{n}\binom{n}{m}\binom{\nu - (m \wedge t)}{\nu - n}}{\binom{\nu - t}{n - t}},$$

for $m = 0, \dots, n$. To see this first observe that

$$(42) \qquad \begin{aligned} \binom{\nu}{n}\delta_{iu} &= \sum_{k=0}^{n} E_k(i) \frac{\mu_u}{\nu_k} E_k(u) = \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j}\binom{n-i}{j}\binom{\nu - n + j - i}{j} \frac{\mu_u}{\nu_k} E_k(u) = \\ &= \sum_{j=0}^{n} \Big[\binom{n-i}{j}\binom{\nu - n + j - i}{j}\Big] \times \\ &\times \Big[\sum_{k=j}^{n} (-1)^{k-j} \binom{n-j}{k-j} \frac{\mu_u}{\nu_k} E_k(u)\Big] \end{aligned}$$

using (8), (9) and (13). Hence the two forms between [] represent inverse matrices (modulo the factor $\binom{\nu}{n}$); so also

$$\sum_{u=0}^{n}\Big[\sum_{k=j}^{n}(-1)^{k-j}\binom{n-j}{k-j}\frac{\mu_u}{\nu_k}E_k(u)\Big]\times \tag{43}$$
$$\times\Big[\binom{n-u}{h}\binom{\nu-n+h-u}{h}\Big]=\binom{\nu}{n}\delta_{jh}.$$

Therefore

$$\sum_{u=0}^{n} b_u\binom{n-u}{n-m}\binom{\nu-m-u}{n-m}=$$
$$=\sum_{u=0}^{n}\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{\nu-t}{n-t}^{-1}(-1)^{k-j}\binom{n}{k}\binom{k}{j}\times \tag{44}$$
$$\times\binom{\nu-((n-j)\wedge t)}{\nu-n}\frac{\mu_u}{\nu_k}E_k(u)\binom{n-u}{n-m}\binom{\nu-m-u}{n-m}=$$
$$=\sum_{j=0}^{n}\binom{\nu-t}{n-t}^{-1}\binom{n}{j}\binom{\nu-((n-j)\wedge t)}{\nu-n}\binom{\nu}{n}\delta_{j,n-m}=$$
$$=\binom{\nu-t}{n-t}^{-1}\binom{n}{m}\binom{\nu-(m\wedge t)}{\nu-n}\binom{\nu}{n}$$

(note that $\binom{n}{k}\binom{k}{j}=\binom{n}{j}\binom{n-j}{k-j}$ if $k\geqslant j$, and 0 otherwise), proving (41).

Now we prove (38) by induction. Suppose we have shown that $b_1=\ldots=b_w=0$ for some $0\leqslant w<t$. We prove $b_{w+1}=0$ by substituting $m=w+1$ in (41).

$$\frac{\binom{\nu}{n}\binom{n}{w+1}\binom{\nu-w-1}{\nu-n}}{\binom{\nu-t}{n-t}}=$$
$$=\sum_{u=0}^{w+1} b_u\binom{n-u}{n-w-1}\binom{\nu-w-1-u}{n-w-1}= \tag{45}$$
$$=b_0\binom{n}{w+1}\binom{\nu-w-1}{n-w-1}+b_{w+1}\binom{\nu-2w-2}{n-w-1}=$$
$$=\frac{\binom{\nu}{n}\binom{n}{w+1}\binom{\nu-w-1}{\nu-n}}{\binom{\nu-t}{n-t}}+b_{w+1}\binom{\nu-2w-2}{n-w-1},$$

which implies $b_{w+1} = 0$.

We finally show (39). It follows from (37), (38) and (41) that

$$\sum_{u=t}^{n} b_u \binom{n-u}{n-m}\binom{v-m-u}{n-m} = \binom{v}{n}\binom{n}{m}\left(1 - \frac{\binom{v-m}{n-m}}{\binom{v-t}{n-t}}\right) \tag{46}$$

for $m = t, \ldots, n$. Let μ'_u, v'_k and $E'_k(u)$ be the parameters of the Johnson scheme with v replaced by $v - 2t$, and n by $n - t$. Now

$$\begin{aligned} b_u\binom{v-2t}{n-t} &= \sum_{i=t}^{n} b_i\binom{v-2t}{n-t}\delta_{i-t,u-t} = \\ &= \sum_{i=t}^{n}\sum_{j=0}^{n-t}\sum_{k=j}^{n-t}\binom{n-i}{j}\binom{v-n+j-i}{j}\times \\ &\times(-1)^{k-j}\binom{n-t-j}{k-j}\frac{\mu'_u}{v'_k}E'_k(u-t)b_i = \\ &= \sum_{j=0}^{n-t}\sum_{k=j}^{n-t}(-1)^{k-j}\binom{n-t-j}{k-j}\frac{\mu'_u}{v'_k}E'_k(u-t)\times \\ &\times\binom{v}{n}\binom{n}{j}\left(1 - \frac{\binom{v-n+j}{j}}{\binom{v-t}{n-t}}\right), \end{aligned} \tag{47}$$

for $u = t, \ldots, n$, by applying (42). Now, for $u = t + 1, \ldots, n$, we have

$$\begin{aligned} \lim_{v\to\infty} b_u(v)\frac{\binom{v-2t}{n-t}}{\mu'_u\binom{v}{n}} &= \sum_{j=0}^{n-t}\sum_{k=j}^{n-t}(-1)^{k-j}\binom{n-t-j}{k-j}\binom{n}{j}\times \\ &\times\lim_{v\to\infty}\frac{E'_k(u-t)}{v'_k}\left(1 - \frac{\binom{v-n+j}{j}}{\binom{v-t}{n-t}}\right). \end{aligned} \tag{48}$$

It follows from (12) and (13) that

$$(49)\qquad \lim_{\nu\to\infty} \frac{E'_k(u-t)}{\nu'_k} = \frac{\binom{n-u}{k}}{\binom{n-t}{k}};$$

also

$$(50)\qquad \lim_{\nu\to\infty} \left(1 - \frac{\binom{\nu-n+j}{j}}{\binom{\nu-t}{n-t}}\right) = 1 \quad \text{if} \quad 0 \leqslant j < n-t,$$

$$= 0 \quad \text{if} \quad j = n-t.$$

Therefore, (48) is equal to

$$(51)\qquad \sum_{j=0}^{n-t} \sum_{k=j}^{n-t}{}' (-1)^{k-j} \frac{\binom{n-t-j}{k-j}\binom{n}{j}\binom{n-u}{k}}{\binom{n-t}{k}}$$

(note that for $j = k = n-t$ the limit (49) equals 0 since $u > t$). As

$$(52)\qquad \begin{aligned} &\sum_{j=0}^{k} (-1)^{k-j}\binom{n-t-j}{k-j}\binom{n}{j} = \\ &= \sum_{j=0}^{k} \binom{-n+t+k-1}{k-j}\binom{n}{j} = \binom{t+k-1}{k} \end{aligned}$$

we have

$$(53)\qquad \lim_{\nu\to\infty} b_u(\nu)\frac{\binom{\nu-2t}{n-t}}{\mu'_u\binom{\nu}{n}} = \sum_{k=0}^{n-t}{}' \frac{\binom{t+k-1}{k}\binom{n-u}{k}}{\binom{n-t}{k}} > 0,$$

for $u = t+1, \ldots, n$.∎

The question remains how large ν has to be with respect to n and t for having $\Theta(K_{\nu,n,t}) = \binom{\nu-t}{n-t}$; e.g., is it possible to deduce with the above method the good bounds of Frankl [8]?

One could try to extend the method to obtain the ϑ-values of graphs whose edge set is the union of some arbitrary classes of a Johnson scheme; e.g., if the vertices of the graph G are all n-subsets of a ν-set, two of them being adjacent iff the size of their intersection is in a certain set $\{t_1, \ldots$ $\ldots, t_h\}$, is it true that

$$(54) \qquad \vartheta(G) \leqslant \prod_{i=1}^{h} \frac{v - t_i}{n - t_i}$$

for $v \to \infty$? (Cf. Deza, Erdős and Frankl [6]).

Results analogous to the above theorem could be obtained in Hamming schemes to obtain the ϑ-value of graphs derived from those schemes; is there any relation between the asymptotic operativeness of upper bounds and certain asymptotic non-existence results for perfect codes (cf. Van Lint [11] and Best [1])?

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