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FRACTIONAL PACKING AND COVERING

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INTRODUCTION

Let $H = (V,E)$ be a *hypergraph* (i.e., V is a finite set (of *points* or *vertices*), and E is a family of subsets of V (called the *edges*)). Packing problems ask for the maximum number $\nu(H)$ of pairwise disjoint edges of H; trivially, $\nu(H)$ is never more than the minimum number $\tau(H)$ of points representing each edge, and one may ask: when do we have $\nu(H) = \tau(H)$? In a number of cases a useful tool to answer this question is the theory of *fractional packing and covering*.

Usually, in a packing an edge occurs a certain integral number (0 or 1) of times; we can extend this by allowing each edge to occur a *fractional* number of times. We obtain a *fractional packing* by assigning to each edge a nonnegative rational number such that, for each point, the sum of the numbers given to the edges containing that point, is at most one. So, if only integers are assigned, we have a (usual) packing. Therefore, $\nu(H) \le \nu^*(H)$, where $\nu^*(H)$ equals the maximum sum of the assigned numbers in any fractional packing. Similarly, one defines $\tau^*(H)$ to be the minimum sum of rational numbers assigned to the points such that the sum of the numbers assigned to the points in any edge is at least one. So $\tau^*(H) \le \tau(H)$, and it is not difficult to see that $\nu^*(H) \le \tau^*(H)$. In fact we have $\nu^*(H) = \tau^*(H)$ since

(1) $$\nu^*(H) = \max\{|y| \mid y \ge 0,\ yM \le 1\}$$

and

(2) $$\tau^*(H) = \min\{|x| \mid x \ge 0,\ Mx \ge 1\},$$

where M is the *incidence matrix* of H (i.e. M is a (0,1)-matrix with rows and columns indexed by E and V, respectively, the entry in the (E,v)-th position being a one iff $v \in E$), $|y|$ and $|x|$ denote the sums of the entries in the (appropriately sized) vectors x and y, respectively, and 1 is an all-one vector. Since, by the Duality theorem of linear programming, for any matrix A and vectors b and w

$$\max\{yb \mid y \geq 0,\ yA \leq w\} = \min\{wx \mid x \geq 0,\ Ax \geq b\} \tag{3}$$

(and this also holds if we restrict ourselves to rational A, b, w, x, and y), we conclude from (1) and (2) that $\nu^*(H) = \tau^*(H)$. There is a reasonably good procedure (the *simplex method*) to calculate (3), which, by (1) and (2), may be used to determine $\nu^*(H)$ and $\tau^*(H)$.

What can we say about $\nu(H)$ and $\tau(H)$ if we know $\nu^*(H)$? Clearly, $\nu(H)$ is equal to the right hand side of (1) if one restricts the range of y to integral (i.e., integer coordinate) vectors; $\tau(H)$ can be described similarly. Therefore, we want methods to determine the left and right hand sides of (3) when we restrict ourselves to integral y and x (obviously, we lose equality in (3) in general); the search for those methods is a main goal of the theory of *integer linear programming*.

The branch of combinatorics which solves combinatorial problems with the help of fractional packing and covering and linear programming sometimes is called *polyhedral combinatorics*, since polyhedral representations are used to solve the problems. Chvátal's claim that "combinatorics = number theory + linear programming" seems to be particularly valid for polyhedral combinatorics, searching for lattice points in polyhedra. For instance, the right hand side of (3) asks for the minimum value of wx where x is in the polyhedron

$$P = \{x \geq 0 \mid Ax \geq b\}. \tag{4}$$

If we know that all the vertices of P have integer coordinates we may deduce that, in (3), we can restrict ourselves to integral x, without loss of generality. In general it is useful to have a procedure to derive from (4) a matrix A' and a vector b' such that the set

$$P' = \{x \geq 0 \mid A'x \geq b'\} \tag{5}$$

is the convex hull of the integral vectors in P. For from (5) we may conclude that

$$\min\{wx \mid x \geq 0,\ x \text{ integral},\ Ax \geq b\} = \min\{wx \mid x \geq 0,\ A'x \geq b'\} = \max\{yb' \mid y \geq 0,\ yA' \leq w\}, \tag{6}$$

and then the simplex method is applicable. Indeed Chvátal has given a general procedure, which is, in a sense, related to Gomory's "*cutting plane method*" for solving integer linear programs.

However, in the present paper, to keep the size in hand, we confine ourselves mainly to finding classes of linear programming problems one or both sides of which are achieved by integral vectors. That is, specializing to hypergraphs, we focus our attention on classes of hypergraphs for which $\nu(H) = \nu^*(H)$ or $\tau^*(H) = \tau(H)$. Often these classes turn out to have nice structural properties. E.g., if we have $\nu = \nu^*$ for a certain hypergraph and certain derived hypergraphs, then also $\tau = \tau^*$, i.e. $\nu = \tau$. Or, if $\tau = \tau^*$ for certain hypergraphs, then $\tau = \tau^*$ also for certain other hypergraphs.

Often the content of the results is the assertion that certain polyhedra have integral vertices, or the result consists of the determination of the faces of the convex hull of a given set of vertices.

A further restriction is that our approach will be rather theoretical; we shall not discuss algorithms to find packings and coverings. It must be said, however, that algorithms and combinatorial optimization form an important motivation for many of the results mentioned in this paper.

The reader whose interest exceeds the bounds we have set ourselves here is referred to CHVÁTAL [18,19] for a procedure to find the faces of the convex hull of integral vectors in a polyhedron, to GOMORY [61,62,63] for a description of the "cutting plane algorithm", to ROSENBERG [136] for a comparison of Chvátal's procedure with Gomory's algorithm, to CHVÁTAL [20] for a nice informal discussion on polyhedral combinatorics, to LOVÁSZ [103] and STEIN [150] for investigations comparing τ and τ^*, and to LAWLER [93] for a survey of algorithmic methods in combinatorial optimization.

In the present paper we assume familiarity with basic definitions and properties of graphs, hypergraphs and polyhedra, and with the Duality theorem of linear programming (knowing (3) is sufficient).

Background references are BONDY & MURTY [16] and BERGE [7] for graph and hypergraph theory, DANTZIG [25] for an extensive survey of linear programming techniques, GARFINKEL & NEMHAUSER [59] and HU [82] for information about integer linear programming (see JOHNSON [84] for a review of some more books), and STOER & WITZGALL [151] for convexity in relation to optimization.

Survey papers related to the present one are BERGE [13], EDMONDS [35] and WOODALL [175].

Organization of the paper

Section 1 of this paper collects some general and special properties of polyhedra and lattice points, and their interaction, needed for the other sections. In Section 2 we investigate classes of hypergraphs H for which $\nu(H) = \nu^*(H)$ or $\tau^*(H) = \tau(H)$; it includes Fulkerson's theory of *blocking* and *anti-blocking polyhedra and hypergraphs*, and Lovász's *perfect graph theorem*.

Section 3 gives Hoffman & Kruskal's result on *totally unimodular matrices* and Berge's results on *balanced hypergraphs*. Finally, in Section 4 a recently developed method of Edmonds & Giles is described, solving some special classes of integer linear programming problems with *"submodular" functions* and *"cross-free" families*; furthermore Edmonds' characterization of *matching polyhedra* is discussed.

In each of the Sections 2, 3 and 4 we first present some general theorems as tools, which are then applied to a number of examples. Some of these examples emerge several times throughout the text, viz. "bipartite graphs" (Examples 2, 5, 9 and 16), "network flows" (Examples 1, 10, 17, 18 and 21), "partially ordered sets" (Examples 3 and 6), "graphs" (Examples 7 and 11, and § 4.3), "matroids" (Examples 8 and 20), "directed cuts" (Examples 12, 19 and 23), "arborescences" (Examples 13 and 22). Sometimes in describing an application, we anticipate results obtained in a subsequent section.

Some conventions

Throughout this paper we work within rational vector spaces rather than real or complex ones. Also any matrix is assumed to be rational-valued. This will not cause much loss of generality since, on the one hand, results will be needed often only in their rational form, and, on the other hand, most of the assertions can be straightforwardly extended to the real field.

When talking about a maximum or minimum the assertions in question are meant to hold only in case the maximum or minimum exists; e.g., if we say that a certain maximum is an integer, we mean that the maximum is an integer if it exists.

When using notations like $Mx \geq b$ and wx, where M is a matrix and b, w and x are vectors, we implicitly assume compatibility of sizes of M, b, w, and x (wx denotes the usual inner product). Moreover, 0 and 1 stand for appropriately sized all-zero and all-one vectors.

If the rows and columns of a matrix M are indexed by sets X and Y, respectively, then M is said to be an $X \times Y$-matrix. Furthermore, we identify functions with vectors; e.g., a function $\phi: V \to \mathbb{Q}$ may be considered as a vector in $\mathbb{Q}^V$, and conversely.

$\mathbb{Q}_+$ and $\mathbb{Z}_+$ denote the sets of nonnegative rationals and integers, respectively.

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1. POLYHEDRA AND INTEGRAL POINTS

Here we collect some general and special information about polyhedra and integral points, and especially about their interaction.

1.1. Convexity and integrality

Convexity and integrality represent the two sides of polyhedral combinatorics. Two parallel aspects of convexity and integrality, respectively, are given by the following two basic properties of a matrix A and a vector c:

(1) *there exists a nonnegative vector* y *such that* $yA = c$, *if and only if for each vector* x *one has* $cx \geq 0$ *whenever* $Ax \geq 0$

(Farkas' lemma; cf. Chapter 2, Proposition 10, or HALL [70], Theorem 8.2.1), and

(2) *there exists an integral vector* y *such that* $yA = c$, *if and only if for each vector* x *one has* $cx \in \mathbb{Z}$ *whenever* Ax *is integral*

(cf. Van der WAERDEN [169] Section 108).

(1) says that if C is the smallest convex cone containing the points $a_1,\ldots,a_m$ (represented by the rows of A), that is, if C is the set of nonnegative scalar combinations of $a_1,\ldots,a_m$, then C is the intersection of all closed half-spaces (i.e. sets of the form $\{x|\ bx \geq 0\}$ for any vector b) containing $a_1,\ldots,a_m$.

Similarly, (2) says that if S is the smallest lattice (additive subgroup) containing the points $a_1,\ldots,a_m$, that is, if C is the set of integral scalar combinations of $a_1,\ldots,a_m$, then C is the intersection of all sets of the form $\{x|\ bx$ is an integer$\}$ (for any b) containing $a_1,\ldots,a_m$. So $\mathbb{Q}_+$ and $\mathbb{Z}$ have parallel properties; it would be very helpful for many problems in polyhedral combinatorics if the set $\mathbb{Z}_+$ had an analogous property, but alas, this is not the case, not even for dimension one ($m = 1$). Fortunately there are some other useful results relating convexity with integrality.

1.2. Polyhedra

A *(convex) polyhedron* in $\mathbb{Q}^n$ is a subset P of $\mathbb{Q}^n$ determined by a finite set of linear inequalities, that is, P is a polyhedron iff

(1) $P = \{x \in \mathbb{Q}^n|\ Ax \le b\}$

for some matrix A and vector b. P is a *polytope* in $\mathbb{Q}^n$ if P is the convex hull of a finite number of points in $\mathbb{Q}^n$. A classical result is:

(2) *P is a polytope iff P is a bounded polyhedron.*

A point v in a polyhedron P is a *vertex* of P if $P\backslash\{v\}$ is convex. So a polytope is the convex hull of its vertices. A polyhedron has a number of *faces*; these can be described as nonempty subsets F of P such that

(3) $F = \{x \in P|\ A'x = b'\}$,

where A' and b' arise from A and b by deleting some rows of A and the corresponding components in b.

A central problem in this field consists of determining (the equations for) the faces of a polyhedron if its vertices are known, or conversely. The advantage of knowing the faces is that one can apply linear programming techniques to find "optimal" vertices: if we know that (1) is the convex hull of a finite set S of vectors then

(4) $\max\{wx|\ x \in S\} = \max\{wx|\ Ax \le b\} = \min\{yb|\ y \ge 0,\ yA = w\}$.

E.g., let S be the set of characteristic vectors of stable subsets in a graph. In general, it is a difficult problem to find the faces (to find A and b) of the convex hull of S (see CHVÁTAL [19], cf. [18], NEMHAUSER &

TROTTER [120] and PADBERG [125]), although we shall see that for some classes of graphs (perfect graphs and line-graphs) these faces can be found simply.

It is not difficult to see that a face F is a minimal face (with respect to inclusion) of (1) iff

(5) $$F = \{x \in \mathbb{Q}^n \mid A'x = b'\}$$

for some A' and b' (arising from A and b as before); so minimal faces are exactly those faces which are affine subspaces of $\mathbb{Q}^n$.

Note that if x is not in the polyhedron P in $\mathbb{Q}^n$ then there is a hyperplane separating x from P, i.e., there exists a $w \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$ such that $wx > r$ and $wv \le r$ for all $v \in P$. So two polyhedra P and R are equal iff for all $w \in \mathbb{Q}^n$ we have:

(6) $$\max\{wx \mid x \in P\} = \max\{wx \mid x \in R\}.$$

1.3. Blocking and anti-blocking polyhedra

Often we shall be concerned with polyhedra P of one of the types

(1) $$P = \{x \in \mathbb{Q}^n_+ \mid Cx \le 1\}, \quad \text{or} \quad P = \{x \in \mathbb{Q}^n_+ \mid Cx \ge 1\}$$

where C is a nonnegative matrix. FULKERSON [48,50,51] developed a theory for polyhedra of these types, called the *theory of blocking and anti-blocking polyhedra*.

For a polyhedron P of the first type, let

(2) $$A(P) = \{y \in \mathbb{Q}^n_+ \mid yx \le 1 \text{ for } x \in P\}$$

be the *anti-blocking polyhedron* of P; and for a polyhedron P of the second type, let

(3) $$B(P) = \{y \in \mathbb{Q}^n_+ \mid yx \ge 1 \text{ for } x \in P\}$$

be the *blocking polyhedron* of P. Clearly, A(P) and B(P), respectively, are of the same type as P.

A pair (P,R) is called an *anti-blocking pair of polyhedra* if P is a polyhedron of the first type and R = A(P). The pair (P,R) is called a *blocking pair of polyhedra* if P is a polyhedron of the second type and $R = B(P)$. We list various equivalent characterizations of (anti-)blocking pairs of polyhedra.

THEOREM 1. (FULKERSON [50,51], LEHMAN [94]) *Let* $P = \{x \in \mathbb{Q}^n_+ \mid Cx \le 1\}$ *and* $R = \{z \in \mathbb{Q}^n_+ \mid Dz \le 1\}$*, where* C *and* D *are nonnegative matrices with row vectors* $c_1,\dots,c_m$ *and* $d_1,\dots,d_k$*, respectively. Then the following assertions are equivalent:*

(i) (P,R) *is an anti-blocking pair of polyhedra;*

(ii) R *consists of all vectors* x *such that* $x \le c$ *for some convex combination* c *of* $c_1,\dots,c_m$*;*

(iii) *for all* $w \in \mathbb{Q}^n_+$: $\max\{wc_1,\dots,wc_m\} = \min\{|y| \mid y \ge 0,\ yD \ge w\}$*;*

(iv) $xz \le 1$ *for* $x \in P$ *and* $z \in R$*, and for all* $\ell, w \in \mathbb{Q}^n_+$:
$\max\{wx \mid x \in P\} \cdot \max\{\ell z \mid z \in R\} \ge \ell w$ *("length-width-inequality");*

(v) (R,P) *is an anti-blocking pair of polyhedra.*

PROOF. (i) $\leftrightarrow$ (ii). Since

(4)
$$\begin{aligned} A(P) &= \{z \in \mathbb{Q}^n_+ \mid xz \le 1 \text{ for } x \in P\} = \\ &= \{z \in \mathbb{Q}^n_+ \mid \max\{zx \mid x \in P\} \le 1\} = \\ &= \{z \in \mathbb{Q}^n_+ \mid \max\{zx \mid x \ge 0,\ Cx \le 1\} \le 1\} = \\ &= \{z \in \mathbb{Q}^n_+ \mid \min\{|y| \mid y \ge 0,\ yC \ge z\} \le 1\} = \\ &= \{z \in \mathbb{Q}^n_+ \mid z \le yC \text{ for some } y \ge 0 \text{ with } |y| \le 1\}, \end{aligned}$$

we have that A(P) consists of all vectors x such that $x \le c$ for some convex combination c of $c_1,\dots,c_m$. Hence $R = A(P)$ iff (ii) holds.

(ii) $\leftrightarrow$ (iii). This follows directly from the Duality theorem of linear programming:

(5)
$$\min\{|y| \mid y \ge 0,\ yD \ge w\} = \max\{wz \mid z \ge 0,\ Dz \le 1\} = \max\{wz \mid z \in R\}.$$

(i) $\leftrightarrow$ (iv). Clearly, the assertion "$R \subset A(P)$" is equivalent to the first half of (iv). We prove that $A(P) \subset R$ iff the second half of (iv) holds.

It is easy to see that $A(P) \subset R$ iff

(6)
$$\forall \ell \in \mathbb{Q}^n_+: \max\{\ell z \mid z \in A(P)\} \le \max\{\ell z \mid z \in R\}.$$

By scalar multiplication of ℓ we see that (6) is equivalent to

(7) $\forall \ell \in \mathbb{Q}^n_+$: $\max\{\ell z \mid z \in R\} \le 1$ implies $\max\{\ell z \mid z \in A(P)\} \le 1$.

(8) is a reformulation of (7):

(8) $\forall \ell \in \mathbb{Q}^n_+$: $(\forall z \in R: \ell z \le 1)$ implies $\forall w \in A(P): \ell w \le 1$.

It follows from the definition of the anti-blocking polyhedron A(P) that (8) is equivalent to:

(9) $\forall \ell \in \mathbb{Q}^n_+$: $(\forall z \in R: \ell z \le 1)$ implies $\forall w \in \mathbb{Q}^n_+$ $((\forall x \in P: wx \le 1)$ implies $\ell w \le 1)$,

and hence to:

(10) $\forall \ell, w \in \mathbb{Q}^n_+$: $\max\{wx \mid x \in P\} \le 1$ and $\max\{\ell z \mid z \in R\} \le 1$ together imply $\ell w \le 1$.

Again by using scalar multiplications of ℓ and w, we see that (10) holds if and only if:

(11) $\forall \ell, w \in \mathbb{Q}^n_+$: $\max\{wx \mid x \in P\} \cdot \max\{\ell z \mid z \in R\} \ge \ell w$,

which is the second half of (iv).

(iv) $\leftrightarrow$ (v). By symmetry of (iv) this equivalence can be proved in a manner analogous to the previous one. □

REMARK. Since each rational vector is a nonnegative scalar multiple of an integral vector and since the (in-)equalities in question are stable under nonnegative multiplication, in the assertions (iii) and (iv) we may replace the conditions $w \in \mathbb{Q}^n_+$ and $\ell \in \mathbb{Q}^n_+$, by $w \in \mathbb{Z}^n_+$ and $\ell \in \mathbb{Z}^n_+$, respectively.

By changing terminology (replacing, anti-blocking, $\le$, min, max, by blocking, $\ge$, max, min and so on) one similarly proves the blocking analogue of Theorem 1:

THEOREM 1. (FULKERSON [48,50], LEHMAN [94]) *Let* $P = \{x \in \mathbb{Q}^n_+ \mid Cx \ge 1\}$ *and let* $R = \{z \in \mathbb{Q}^n_+ \mid Dz \ge 1\}$, *where* C *and* D *are nonnegative matrices with row vectors* $c_1, \ldots, c_m$ *and* $d_1, \ldots, d_k$, *respectively. Then the following assertions* *are equivalent:*

(i) (P,R) *is a blocking pair of polyhedra;*

(ii) R *consists of all vectors* x *such that* $x \geq c$ *for some convex combination* c *of* $c_1,\ldots,c_m$*;*

(iii) *for all* $w \in \mathbb{Q}^n_+$: $\min\{wc_1,\ldots,wc_m\} = \max\{|y| \mid y \geq 0,\ yD \leq w\}$*;*

(iv) $xz \geq 1$ *for* $x \in P$ *and* $z \in R$, *and for all* $\ell, w \in \mathbb{Q}^n_+$:
$\min\{wx \mid x \in P\} \cdot \min\{\ell z \mid z \in R\} \leq \ell w$ *("length-width-inequality");*

(v) (R,P) *is a blocking pair of polyhedra.*

PROOF. Analogous to the previous proof. □

The theory of blocking and anti-blocking polyhedra is a useful tool for fractional packing and covering problems.

1.4. Integrality of vertices

It will be useful to have a characterization of polytopes the vertices of which all are integral; more general, a characterization is sought of polyhedra all faces of which contain an integral vector. That is a characterization of polyhedra P such that for all $w \in \mathbb{Q}^n$

(1) $$\max\{wx \mid x \in P\}$$

is achieved by an integral x. The following theorem characterizes such polyhedra (in case all minimal faces of the polyhedron are vertices the theorem can be proved in a simpler way).

THEOREM 3. (EDMONDS & GILES [37]) *Let* P *be a polyhedron in* $\mathbb{Q}^n$. *Each face of* P *contains an integral vector, if and only if* $\max\{wx \mid x \in P\}$ *is an integer for each* $w \in \mathbb{Z}^n$.

PROOF. The "only if" part being straightforward, we prove "if". So suppose that for all $w \in \mathbb{Z}^n$ $\max\{wx \mid x \in P\}$ is an integer and let $P = \{x \in \mathbb{Q}^n \mid Ax \leq b\}$, for some matrix A and vector b. Let $F = \{x \in \mathbb{Q}^n \mid A'x = b'\}$ be a minimal face of P (cf. § 1.2); we may suppose that the rows of A' are linearly independent. We have to prove that $A'x = b'$ for some $x \in \mathbb{Z}^n$. By (2) of § 1.1 it suffices to show that for each vector y: yA' is integral implies yb' is an integer. So let y be a vector such that yA' is integral. F is a minimal face, hence there is an open convex cone $U \subset \mathbb{Q}^n$ such that, for all $w \in U$,

$\max\{wx \mid x \in P\}$ is achieved by all vectors x in F. Since U is an open convex cone there are integral vectors w_1 and w_2 in U such that $yA' = w_1 - w_2$. Since, for all $x \in F$, w_1x and w_2x are integers (independent of the choice of $x \in F$), we have, for $x \in F$:

(2) $$yb' = yA'x = w_1x - w_2x$$

which is again an integer. As F is nonempty we have proved that $yb' \in \mathbb{Z}$. □

Let M be an n×m-matrix and let b be an integral vector of length n. Consider the series of inequalities, for $w \in \mathbb{Z}^m$:

(3) $$\begin{aligned} &\max\{wx \mid x \in \mathbb{Z}^m,\ Mx \le b\} \le \max\{wx \mid x \in \mathbb{Q}^m,\ Mx \le b\} = \\ &= \min\{yb \mid y \in \mathbb{Q}^n_+,\ yM = w\} \le \min\{yb \mid y \in \tfrac{1}{2}\mathbb{Z}^n_+,\ yM = w\} \le \\ &\le \min\{yb \mid y \in \mathbb{Z}^n_+,\ yM = w\}. \end{aligned}$$

Trivially, if the first and the last expressions are equal then also the last two minima are equal. The next theorem asserts that the converse also holds: if, *for each* $w \in \mathbb{Z}^m$, the last two minima are equal, then all five optima are the same (for each $w \in \mathbb{Z}^m$). The theorem is a combination of results of EDMONDS & GILES [37] and LOVÁSZ [105,106].

THEOREM 4. *For each* $w \in \mathbb{Z}^n$ *both sides of the linear programming duality equation*

(4) $$\max\{wx \mid x \in \mathbb{Q}^m,\ Mx \le b\} = \min\{yb \mid y \in \mathbb{Q}^n_+,\ yM = w\}$$

are attained by integral vectors x *and* y, *if and only if for each* $w \in \mathbb{Z}^n$

(5) $$\min\{yb \mid y \in \tfrac{1}{2}\mathbb{Z}^n_+,\ yM = w\}$$

is attained by an integral y.

PROOF. By (3) if suffices to prove the "if" part. So suppose (5) is achieved by an integral vector y, for each $w \in \mathbb{Z}^n$. Then for each natural number k we have:

(6) $$\min\{yb \mid y \in 2^{-(k+1)}\mathbb{Z}^n_+,\ yM = w\} = \min\{yb \mid y \in 2^{-k}\mathbb{Z}^n_+,\ yM = w\},$$

since this is equivalent to

(7) $$2^{-k}.\min\{yb \mid y \in \tfrac{1}{2}\mathbb{Z}^n_+,\ yM = 2^k.w\} = 2^{-k}.\min\{yb \mid y \in \mathbb{Z}^n_+,\ yM = 2^k.w\},$$

which holds by assumption. Therefore, by induction, for each natural number k

(8) $$\min\{yb \mid y \in 2^{-k}\mathbb{Z}^n_+,\ yM = w\} = \min\{yb \mid y \in \mathbb{Z}^n_+,\ yM = w\}.$$

Hence, since

(9) $$\min\{yb \mid y \in \mathbb{Q}^n_+,\ yM = w\} = \inf_k\ (\min\{yb \mid y \in 2^{-k}\mathbb{Z}^n_+,\ yM = w\}),$$

we have that

(10) $$\min\{yb \mid y \in \mathbb{Q}^n_+,\ yM = w\} = \min\{yb \mid y \in \mathbb{Z}^n_+,\ yM = w\}.$$

By the Duality theorem of linear programming

(11) $$\max\{wx \mid x \in \mathbb{Q}^m,\ Mx \le b\} = \min\{yb \mid y \in \mathbb{Q}^n_+,\ yM = w\}.$$

Since b is integral, it follows from (10) and (11) that $\max\{wx \mid x \in \mathbb{Q}^m, Mx \le b\}$ is an integer, for each $w \in \mathbb{Z}^n$. Therefore, by Theorem 3, each face of the polyhedron $\{x \in \mathbb{Q}^n \mid Mx \le b\}$ contains integral vectors. Therefore

(12) $$\max\{wx \mid x \in \mathbb{Z}^n,\ Mx \le b\} = \max\{wx \mid x \in \mathbb{Q}^n,\ Mx \le b\}$$

for each $w \in \mathbb{Z}^n$ (and hence also for each $w \in \mathbb{Q}^n$). (10), (11) and (12) together imply the required property of (4). □

An immediate corollary is:

<u>COROLLARY 5</u>. *Let* M *be a nonnegative matrix and let* b *be an integral vector. For each* $w \in \mathbb{Z}^n_+$ *both sides of the linear programming duality equation*

(13) $$\max\{wx \mid x \ge 0,\ Mx \le b\} = \min\{yb \mid y \ge 0,\ yM \ge w\}$$

are attained by integral vectors x *and* y, *if and only if for each* $w \in \mathbb{Z}^n_+$

(14) $$\min\{yb \mid y \in \tfrac{1}{2}\mathbb{Z}^n_+,\ yM \ge w\}$$

is attained by an integral vector y.

EDMONDS & GILES [37] call a system of linear inequalities $Mx \le b$ *totally dual integral* if for all integral vectors w the minimization problem

(15) $\min\{yb \mid y \ge 0,\ yM = w\}$

has an integral solution y. It follows from Theorem 3 that if $Mx \le b$ is totally dual integral and b is integer-valued then each face of the polyhedron $\{x \mid Mx \le b\}$ contains integral vectors.

2. HYPERGRAPHS

2.1. Notation

A classical theorem of MENGER [113] says the following. Suppose we have a directed graph G, with two fixed vertices r and s. Call the set of arrows in a directed path from r to s an *r-s-path*. Then the maximum number of pairwise disjoint r-s-paths is equal to the minimum number of arrows meeting each r-s-path.

To formulate this result in a wider context define, just as in the introduction, for each hypergraph $H = (V, \mathcal{E})$ the numbers

(1) $\nu(H)$ = the maximum number of pairwise disjoint edges of H,

and

(2) $\tau(H)$ = the minimum size of a subset V' of V intersecting each edge.

It is clear that $\nu(H) \le \tau(H)$. If V is the arrow set of the digraph G and $\mathcal{E}$ is the collection of all r-s-paths in G then the content of Menger's theorem is that $\nu(H) = \tau(H)$.

More generally, define, for hypergraphs $H = (V, \mathcal{E})$ and natural numbers k:

(3) $\nu_k(H) = \max\{\sum_{E\in\mathcal{E}} g(E) \mid g\colon \mathcal{E} \to \mathbb{Z}_+ \text{ such that } \sum_{E\ni v} g(E) \le k \text{ for all } v \in V\}$

and

(4) $\tau_k(H) = \min\{\sum_{v\in V} f(v) \mid f\colon V \to \mathbb{Z}_+^n \text{ such that } \sum_{v\in E} f(v) \ge k \text{ for all } E \in \mathcal{E}\}.$

One easily sees that $\nu(H) = \nu_1(H)$, $\tau(H) = \tau_1(H)$ and $\nu_k(H) \le \tau_k(H)$. Moreover, let

(5) $$\nu^{*}(H) = \sup_{k} \frac{\nu_k(H)}{k} = \lim_{k\to\infty} \frac{\nu_k(H)}{k},$$

and

(6) $$\tau^{*}(H) = \inf_{k} \frac{\tau_k(H)}{k} = \lim_{k\to\infty} \frac{\tau_k(H)}{k};$$

the right hand side equalities follow from the facts that $\nu_{k+\ell}(H) \geq \nu_k(H)+\nu_\ell(H)$ and $\tau_{k+\ell}(H) \leq \tau_k(H)+\tau_\ell(H)$, respectively (using "Fekete's lemma").

We may put (5) and (6) in a linear programming form. Let M be the incidence matrix of H. Then

(7) $$\nu^{*}(H) = \max\{|y| \mid y \in \mathbb{Q}_+^E,\ yM \leq 1\}$$

and

(8) $$\tau^{*}(H) = \min\{|x| \mid x \in \mathbb{Q}_+^V,\ Mx \geq 1\}.$$

The Duality theorem of linear programming gives us that $\nu^{*}(H) = \tau^{*}(H)$. Since the matrix M and the all-one vectors are rational-valued, the simplex-method for solving linear programming problems delivers rational-valued vectors y and x in (7) and (8); this implies that we may replace in (5) and (6) the "sup" and "inf" by "max" and "min", respectively.

Summarizing we have for natural numbers k and ℓ:

(9) $$\nu(H) \leq \frac{\nu_k(H)}{k} \leq \frac{\nu_{k\ell}(H)}{k\ell} \leq \nu^{*}(H) = \tau^{*}(H) \leq \frac{\tau_{k\ell}(H)}{k\ell} \leq \frac{\tau_k(H)}{k} \leq \tau(H).$$

In particular, if $\nu(H) = \tau(H)$ then all inequalities become equalities. It can be considered as one of the aims of this paper to determine those k for which $\nu_k(H) = k.\nu^{*}(H)$, or $k.\tau^{*}(H) = \tau_k(H)$. Often it amounts to investigating to what extent the equality of certain terms in (9) implies the equality of other terms.

It is easy to see that $\nu_k(H) = k.\nu^{*}(H)$ if and only if the maximum in (7) is attained by a vector $y \in 1/k.\mathbb{Z}_+$, i.e., by a vector y having integral multiples of 1/k as coordinates.

The question of determining $\nu(H)$ may be viewed as a packing problem; we now introduce its covering counterpart. A basic example (in a sense the counterpart of Menger's theorem) is DILWORTH's theorem [26]: let $(V,\leq)$ be a finite partially ordered set; then the minimum number of chains needed to cover V is equal to the maximum number of elements in an antichain (an *(anti-) chain* is a set of pairwise (in-)comparable elements).

In hypergraph language: define for each hypergraph $H = (V,E)$ the numbers

(10) $\rho(H)$ = the minimum number of edges needed to cover V,

and

(11) $\alpha(H)$ = the maximum number of points no two of which are contained in an edge.

Now we have $\rho(H) \geq \alpha(H)$. If V is the set of elements of a partially ordered set and E its collection of chains, then Dilworth's theorem tells us that $\rho(H) = \alpha(H)$.

Again, define more generally for hypergraphs $H = (V,E)$ and natural numbers k:

(12) $$\rho_k(H) = \min\{\sum_{E\in E} g(E) \mid g\colon E \to \mathbb{Z}_+ \text{ such that } \sum_{E\ni v} g(E) \geq k \text{ for all } v \in V\}$$

and

(13) $$\alpha_k(H) = \max\{\sum_{v\in V} f(v) \mid f\colon V \to \mathbb{Z}_+ \text{ such that } \sum_{v\in E} f(v) \leq k \text{ for all } E \in E\}.$$

Now we have: $\rho(H) = \rho_1(H)$, $\alpha(H) = \alpha_1(H)$ and $\rho_k(H) \geq \alpha_k(H)$. Moreover, let

(14) $$\rho^*(H) = \inf_k \frac{\rho_k(H)}{k} = \lim_{k\to\infty} \frac{\rho_k(H)}{k} = \min_k \frac{\rho_k(H)}{k},$$

and

(15) $$\alpha^*(H) = \sup_k \frac{\alpha_k(H)}{k} = \lim_{k\to\infty} \frac{\alpha_k(H)}{k} = \max_k \frac{\alpha_k(H)}{k};$$

just as before these equalities follow from Fekete's lemma and the rationality of linear programming solutions. The Duality theorem yields $\rho^*(H) = \alpha^*(H)$. Summarizing we have, for natural numbers k and ℓ:

(16) $$\rho(H) \geq \frac{\rho_k(H)}{k} \geq \frac{\rho_{k\ell}(H)}{k\ell} \geq \rho^*(H) = \alpha^*(H) \geq \frac{\alpha_{k\ell}(H)}{k\ell} \geq \frac{\alpha_k(H)}{k} \geq \alpha(H).$$

We shall also investigate when these inequalities become equalities.

2.2. Conormal and Fulkersonian hypergraphs

Now we shall deal with problems concerning the functions ν, τ, ρ, and α. Comparing the pair α,ρ with the pair τ,ν, it turns out that they sometimes share analogous properties, but at times their properties diverge.

In this subsection we exhibit some of their common features. Subsection 2.3 is devoted to the perfect graph theorem, being a base for many results on α and ρ. Subsections 2.4 and 2.5 show some of the divergent properties of α,ρ and τ,ν, respectively.

We first need some further definitions. Let $H = (V,\mathcal{E})$ be a hypergraph. *Multiplying a vertex* $v \in B$ by some number $k \geq 0$ means that we replace v by k new vertices $v_1,\ldots,v_k$, and each edge E containing v by k new edges $(E\backslash\{v\})\cup\{v_1\},\ldots,(E\backslash\{v\})\cup\{v_k\}$. E.g., if V is the set of arrows of a directed graph, with two fixed vertices r and s, and $\mathcal{E}$ is the collection of r-s-paths, then multiplying v by k corresponds with replacing, in the digraph, the arrow v by k parallel arrows.

Multiplying a vertex v by 0 is the same as removing the vertex v and all edges containing v.

More generally, for a function $w:V \to \mathbb{Z}_+$, the hypergraph H^w arises from H by multiplying, successively, every vertex v by $w(v)$. So the class of hypergraphs arising from digraphs as described above is closed under the transition $H \to H^w$. A class with this property will be called "*closed under multiplication of vertices*".

The *hereditary closure* $\hat{H}$ of H is the hypergraph having the same vertex set as H, with edges all sets contained in any edge of H. H is *hereditary* if $H = \hat{H}$. Similarly, $\check{H}$ again has the same vertex set as H, now with edges all subsets containing some edge of H.

The *anti-blocker* $A(H)$ and *blocker* $B(H)$ of H are hypergraphs with vertex set V, while the edge set of $A(H)$ is the collection

$$\text{(1)} \qquad \{V' \subset V \mid |V' \cap E| \leq 1 \text{ for all } E \in \mathcal{E}\};$$

the edge set of $B(H)$ is

$$\text{(2)} \qquad \{V' \subset V \mid |V' \cap E| \geq 1 \text{ for all } E \in \mathcal{E}\}.$$

So $\alpha(H)$ is equal to the maximum size of edges in $A(H)$, and $\tau(H)$ is equal to the minimum size of edges in $B(H)$.

Clearly, $A(H) = A(\hat{H})$ and $B(H) = B(\check{H})$. It is easy to see that $B(B(H)) = \check{H}$ (cf. EDMONDS & FULKERSON [36], and SEYMOUR [143]). An analogous property does not hold for the anti-blocker; in fact

$$\text{(3)} \qquad A(A(H)) = \hat{H} \text{ if and only if } H \text{ is } \textit{conformal},$$

that is, by definition, iff any subset V' of V is contained in an edge of H whenever each pair of vertices in V' is contained in an edge. In particular, for each hypergraph H the hypergraph A(H) is conformal.

If M is the incidence matrix of H a straightforward analysis of H^w, ν and τ yields:

$$(4)\quad \begin{aligned} \tau(H^w) &= \min\{wx \mid x \in \mathbb{Z}_+^V,\ Mx \geq 1\} \\ \tau^*(H^w) &= \min\{wx \mid x \in \mathbb{Q}_+^V,\ Mx \geq 1\} \\ \nu^*(H^w) &= \max\{|y| \mid y \in \mathbb{Q}_+^E,\ yM \leq w\} \\ \nu(H^w) &= \max\{|y| \mid y \in \mathbb{Z}_+^E,\ yM \leq w\}. \end{aligned}$$

Moreover, if H is hereditary we have:

$$(5)\quad \begin{aligned} \alpha(H^w) &= \max\{wx \mid x \in \mathbb{Z}_+^V,\ Mx \leq 1\} \\ \alpha^*(H^w) &= \max\{wx \mid x \in \mathbb{Q}_+^V,\ Mx \leq 1\} \\ \rho^*(H^w) &= \min\{|y| \mid y \in \mathbb{Q}_+^E,\ yM \geq w\} \\ \rho(H^w) &= \min\{|y| \mid y \in \mathbb{Z}_+^E,\ yM \geq w\}. \end{aligned}$$

REMARK. In (5) we have to require that H is hereditary since otherwise we must adapt, for the α,ρ-case the definition of "multiplying a vertex by 0". In the τ,ν-case removing a point v together with the edges incident with it in case $w(v) = 0$ gives no problems, but in the α,ρ-case this does not work unless we assume that H is hereditary. This causes no loss of generality since in α,ρ-problems passing from H to $\hat{H}$ mostly does not change those problems.

Now we have two analogous theorems, based on the theory of blocking and anti-blocking polyhedra (subsection 1.3).

THEOREM 6. (FULKERSON [50,51], LEHMAN [94]) *Let* H *and* K *be hypergraphs such that* $K = A(H)$ *and* $H = A(K)$. *Then the following assertions are equivalent:*

(i) $\alpha^*(H^w)$ *is an integer for each function* $w: V \to \mathbb{Z}_+$;

(ii) $\alpha^*(H^w) = \alpha(H^w)$ *for each function* $w: V \to \mathbb{Z}_+$;

(iii) $\alpha(H^w)\alpha(K^\ell) \geq \sum_{v \in V} \ell(v)w(v)$ *for all functions* $\ell, w: V \to \mathbb{Z}_+$;

(iv) $\alpha^*(K^\ell) = \alpha(K^\ell)$ *for each function* $\ell: V \to \mathbb{Z}_+$;

(v) $\alpha^*(K^\ell)$ *is an integer for each function* $\ell: V \to \mathbb{Z}_+$.

REMARK. Let M and N be the incidence matrices of H and K, respectively. Let

(6) $P = \{x \in \mathbb{Q}_+^V | \ Mx \le 1\}$

and

(7) $R = \{z \in \mathbb{Q}_+^V | \ Nx \le 1\}$.

So, by (5), $\alpha^*(H^w) = \max\{wx | x \in P\}$ and $\alpha(H^w) = \max\{wx | x \in \mathbb{Z}_+^V, x \in P\}$ (since $H = A(K)$, H is hereditary). This means that (ii) is equivalent to saying that P has integral vertices. Similarly, (iv) is equivalent to saying that R has integral vertices.

All five assertions (i) - (v) *are equivalent to:* (P,R) *is an anti-blocking pair of polyhedra.*

PROOF. Evidently, (ii) → (i) and (iv) → (v).
(i) → (ii). Assertion (i) says that, for each $w: V \to \mathbb{Z}_+$, the number $\max\{wx | x \in P\}$ is an integer. It follows that for each $w: V \to \mathbb{Z}$ this number is an integer. Consequently, by Theorem 3, each vertex of P is integral, that is, (ii) holds.
The proof of (v) → (iv) is similar.

So the equivalence of (i) and (ii), and that of (iv) and (v), is based on Theorem 3; Theorem 1 is a basis for the equivalence of (ii), (iii) and (iv). We show that each of (ii), (iii), (iv) is equivalent to the pair (P,R) being an anti-blocking pair of polyhedra.

As mentioned, (ii) is equivalent to P having integral vertices, that is, to P consisting of all vectors $v \le c$ for some convex combination c of characteristic vectors of A(H). But these characteristic vectors are the row vectors of N, hence, by Theorem 1, (ii) is equivalent to (P,R) being an anti-blocking pair of polyhedra.

Similarly, (iv) is equivalent to (P,R) being an anti-blocking pair of polyhedra. Finally we show that assertion (iii) is equivalent to assertion (iv) of Theorem 1. To this end let $R' = A(P)$ and $P' = A(R)$. So R' consists of all vectors $v \le c$ for some convex combination c of row vectors of M; P' consists of all vectors $v \le d$ for some convex combination d of row vectors of N.

Hence $\alpha^*(H^w) = \max\{wx | x \in P'\}$ and $\alpha^*(K^\ell) = \max\{\ell z | z \in R'\}$, and for all $x \in P'$ and $z \in R'$ one has $xz \le 1$. Therefore (iii) implies, by (iv) of Theorem 1, that (P',R') is an anti-blocking pair, hence also (P,R) is an anti-blocking pair.

Conversely, if (P,R) is an anti-blocking pair also (P',R') is an anti-blocking pair. But then (iv) of Theorem 1, applied to the pair (P',R'), implies (iii). □

By using Theorem 3 together with Theorem 2 we can derive the blocking analogue:

<u>THEOREM 7</u>. (FULKERSON [48,50], LEHMAN [94]) *Let* H *and* K *be hypergraphs such that* K = B(H) *and* H = B(K). *Then the following assertions are equivalent:*

(i) $\tau^*(H^w)$ *is an integer for each function* $w\colon V \to \mathbb{Z}_+$;

(ii) $\tau^*(H^w) = \tau(H^w)$ *for each function* $w\colon V \to \mathbb{Z}_+$;

(iii) $\tau(H^w)\tau(K^\ell) \le \sum_{v\in V} \ell(v)w(v)$ *for all functions* $\ell, w\colon V \to \mathbb{Z}_+$;

(iv) $\tau^*(K^\ell) = \tau(K^\ell)$ *for each function* $\ell\colon V \to \mathbb{Z}_+$;

(v) $\tau^*(K^\ell)$ *is an integer for each function* $\ell\colon V \to \mathbb{Z}_+$.

<u>PROOF</u>. Adapt the previous proof. □

By giving one example we indicate how these theorems can be used; in the other subsections more examples can be found.

<u>EXAMPLE 1</u>: *Network flows* (cf. FULKERSON & WEINBERGER [55]). Suppose we have a directed graph, with two fixed vertices r and s. Let V be the set of arrows of the digraph, and let $\mathcal{E}$ be the collection of subsets of V containing an r-s-path. Let $\mathcal{F}$ be the collection of subsets of V intersecting each r-s-path; such sets are called r-s-*disconnecting sets*. Let $H = (V,\mathcal{E})$ and $K = (V,\mathcal{F})$; hence B(H) = K and B(K) = H.

Proving $\tau(K) = \nu(K)$ is easy: the length of a shortest r-s-path is equal to the maximum number of pairwise disjoint r-s-disconnecting sets. Since multiplication of vertices of K corresponds to replacing arrows by paths, one even has: $\tau(K^\ell) = \nu(K^\ell)$, for all $\ell\colon V \to \mathbb{Z}_+$. In particular: $\tau(K^\ell) = \tau^*(K^\ell)$ for all $\ell\colon V \to \mathbb{Z}_+$. Hence by Theorem 7, $\tau(H^w) = \tau^*(H^w) = \nu^*(H^w)$ for each $w\colon V \to \mathbb{Z}_+$.

So if we consider a function $w\colon V \to \mathbb{Z}_+$ as a "capacity function" defined on the arrows of the digraph, then $\tau(H^w)$ is equal to the minimum capacity of an r-s-disconnecting set: $\nu^*(H^w)$ is equal to the maximum amount of "flow" which can go "through" the arrows of the digraph, from r to s, such that through no arrow is there a flow bigger than the capacity of the arrow. $\tau(H^w) = \nu^*(H^w)$ therefore, is the content of FORD & FULKERSON's *max-flow min-cut theorem* [43].

It is even true that, for $w\colon V \to \mathbb{Z}_+$, $\tau(H^w) = \nu(H^w)$ (Ford & Fulkerson's *integer-flow theorem*), but this cannot be derived straightforwardly from Theorem 7; it will be discussed in subsection 2.5. For an extensive survey on "Flows in Networks" we refer to FORD & FULKERSON's fundamental book with this title [44]. For a covering analogue see LINIAL [96].

We shall call a hypergraph H' *conormal* if H' is conformal such that one, and hence each, of the conditions mentioned in Theorem 6 holds for the pair $H = \hat{H}'$ and $K = A(H)$.

We call a hypergraph H' *Fulkersonian* if one, and hence each, of the conditions mentioned in Theorem 7 holds for the pair $H = \check{H}'$ and $K = B(H)$. So

(8) H *is Fulkersonian iff* B(H) *is Fulkersonian,*

and, *if* H *is conformal,*

(9) H *is conormal iff* A(H) *is conormal.*

(Fulkersonian hypergraphs are called by SEYMOUR [145,147] hypergraphs with the $\mathbb{Q}_+$*-Max-flow Min-cut property*. Conormal hypergraphs are those hypergraphs whose duals are *normal* - see LOVÁSZ [98,100].)

The relationship between α,ρ and τ,ν has further counterparts: anti-blocking versus blocking; A(H) versus B(H); conormal versus Fulkersonian. As said earlier, the theory of α,ρ is not completely analogous to that of τ,ν. The necessity of adding the conditions of hereditarity and conformality each time shows one point of anomaly. However, this implies a simpler representation for conormal hypergraphs, namely by perfect graphs (see § 2.3).

It will turn out that another divergence is that in Theorem 6 (the α,ρ-case) we may replace in the assertions (i)-(v) the conditions $w\colon V \to \mathbb{Z}_+$ and $\ell\colon V \to \mathbb{Z}_+$ by $w\colon V \to \{0,1\}$ and $\ell\colon V \to \{0,1\}$, respectively. Furthermore, we may extend (ii) to: $\alpha(H^w) = \rho(H^w)$ for all $w\colon V \to \mathbb{Z}_+$. These extensions and sharpenings will be discussed in subsection 2.4.

Analogous sharpenings and extensions are *not* valid for Theorem 7. Replacing $\mathbb{Z}_+$ there by $\{0,1\}$ yields assertions which are not equivalent to the original ones. Also the assertion "$\tau(H^w) = \nu(H^w)$ for all $w\colon V \to \mathbb{Z}_+$" is provably stronger than assertion (ii) of Theorem 7. For more details see subsection 2.5.

2.3. Perfect graphs

Let $\gamma(G)$ and $\omega(G)$ denote the chromatic number and clique number (maximum size of a clique) of the graph G. Clearly, $\omega(G) \le \gamma(G)$. The property "$\omega = \gamma$" does not say much about the internal structure of a graph: by adding a disjoint large clique each graph can be extended to a graph with this property. The property

(1) $\omega(G') = \gamma(G')$ for each induced subgraph G' of G

says more; graphs G satisfying (1) are called *perfect*.

Examples of perfect graphs are: (i) bipartite graphs (trivially); (ii) *transitively orientable graphs* (i.e., graphs with vertices the elements of a partially ordered set, two of them being adjacent iff they are comparable; the perfectness of these graphs is easy to see). The content of KÖNIG's theorem [86] and DILWORTH's theorem [26], respectively, is that complements of bipartite and of transitively orientable graphs are perfect. This caused BERGE [3,4] to conjecture that the complementary graph $\bar{G}$ of a perfect graph G is again perfect. This "perfect graph conjecture" was proved in 1972 by LOVÁSZ [98] (unknowingly extending one of Fulkerson's ideas), after partial results of BERGE [7], BERGE & LAS VERGNAS [14], SACHS [139], and FULKERSON [49,50,51].

THEOREM 8. (LOVÁSZ's perfect graph theorem [98]) *A graph* G *is perfect if and only if* $\bar{G}$ *is perfect*.

PROOF. I. We first show that if $G = (V,E)$ is perfect, then the graph G_v is perfect, where G_v arises from G by replacing the vertex v by two new vertices v' and v", each of them being adjacent to those vertices which were adjacent in G to v; moreover v' and v" are adjacent. The adjacency within $V\setminus\{v\}$ remains unchanged.

Choose an arbitrary vertex v. To prove that G_v is perfect it is, by induction, sufficient to show that $\omega(G_v) = \gamma(G_v)$. If $\omega(G_v) = \omega(G)+1$, then $\omega(G_v) = \gamma(G_v)$, since $\gamma(G_v) \le \gamma(G)+1 = \omega(G)+1$. Therefore suppose $\omega(G_v) = \omega(G)$. Now colour G with $\omega(G)$ colours, and suppose the vertex v is in the colour class W. Consider the subgraph G' of G_v induced by $(V\setminus W)\cup\{v'\}$; this graph is isomorphic to the subgraph of G induced by $(V\setminus W)\cup\{v\}$, so G' is perfect. Also we have $\omega(G') = \omega(G)-1$, since if $(V\setminus W)\cup\{v'\}$ contains a clique of size $\omega(G)$ it must contain v' (there is no clique of size $\omega(G) = \gamma(G)$ contained in $V\setminus W$), and hence $\omega(G_v) = \omega(G)+1$.

Since G' is perfect, $\omega(G') = \gamma(G')$ and so G' can be coloured with $\omega(G') = \omega(G_v)-1$ colours. Adding the colour class $(W\setminus\{v\})\cup\{v''\}$ yields a colouring with $\omega(G_v)$ colours.

II. Now suppose G is a smallest (under taking induced subgraphs) perfect graph such that $\bar{G}$ is not perfect. Hence we know that $\omega(\bar{G}) < \gamma(\bar{G})$, and also that each stable subset of $\bar{G}$ is disjoint from some clique of $\bar{G}$ of size $\omega(\bar{G})$ (otherwise we could split off such a stable subset as a colour class to obtain a smaller counterexample). That is, each clique of G is disjoint from some stable subset of G of size $\alpha(G)$.

Let $C_1,\ldots,C_m$ be all cliques of G. Let $V_1,\ldots,V_m$ be $\alpha(G)$-sized stable subsets of V such that C_i is disjoint from V_i, for $i = 1,\ldots,m$. Now make a graph G'', having vertex set the disjoint sum of $V_1,\ldots,V_m$, such that two "new" vertices $v_i \in V_i$ and $v_j \in V_j$ $(i\neq j)$ are adjacent iff the "old" vertices v_i and v_j are equal or adjacent (each set V_i is stable in G''). It is easy to see that G'' arises from G by splitting points, as described in part I of this proof. So G'' is perfect.

But $\alpha(G'') = \alpha(G)$, and $\omega(G'') < m$, since each clique is disjoint from one of the sets V_i. Since the number of vertices of G'' is equal to $m.\alpha(G)$, G'' cannot be covered by $\omega(G'')$ stable subsets of G'', i.e. $\omega(G'') < \gamma(G'')$, contradicting the perfectness of G''. □

The following examples are applications of the perfect graph theorem (see also BERGE [5,11], SHANNON [149], TUCKER [154]).

EXAMPLE 2: *Bipartite graphs*. As remarked earlier, any bipartite graph is trivially perfect, hence the complements of bipartite graphs are perfect. This is the content of a theorem of KÖNIG [87] and EGERVÁRY [42]: *the maximum cardinality of a stable subset of a bipartite graph is equal to the minimum number of edges needed to cover all points* (the theorem is easily adapted if the graph has isolated vertices).

A theorem of GALLAI [56,57] says that, for any graph G without isolated vertices one has:

(2) $\alpha(G) + \tau(G) = \nu(G) + \rho(G) =$ *the number of points of* G.

So the König-Egerváry theorem, together with Gallai's theorem, gives KÖNIG's theorem [87]: *the maximum number of pairwise disjoint edges in a bipartite* *graph is equal to the minimum number of points representing all edges.* This is equivalent to saying that the complement $\overline{L(G)}$ of the line-graph L(G) of a bipartite graph G is perfect. By the perfect graph theorem also the line-graph L(G) itself is perfect, which is the content of another theorem of KÖNIG [86]: *the minimum number of colours needed to colour the edges of a bipartite graph such that no two edges of the same colour meet, is equal to the maximum degree of the graph.*

EXAMPLE 3: *Partially ordered sets.* A transitively orientable graph is trivially perfect, hence its complementary graph is perfect, which is the content of DILWORTH's theorem [26]: *the minimum number of chains needed to cover a partially ordered set is equal to the maximum size of an anti-chain.*

EXAMPLE 4: *Triangulated graphs.* A graph G is called *triangulated* if each circuit having at least four edges contains a chord. Dirac (cf. FULKERSON [51]) showed that each triangulated graph contains a vertex v all of whose neighbours together form a clique, i.e., v is in only one maximal clique. From this one easily derives that $\alpha(G) = \gamma(\bar{G})$ for triangulated graphs G. Since each induced subgraph of a triangulated graph is triangulated again, it follows that complements of triangulated graphs are perfect (HAJNAL & SURÁNYI [69]). Hence, by the perfect graph theorem, triangulated graphs are perfect.

If G is perfect then $\omega(G).\alpha(G)$ is not less than the number of vertices of G, since colouring the vertices with $\omega(G) = \gamma(G)$ colours, each colour class contains at most $\alpha(G)$ vertices. Each induced subgraph of G clearly has this property. In fact this characterizes perfect graphs, as LOVÁSZ [99] has proved the following sharpening of the perfect graph theorem (suggested by A. Hajnal).

THEOREM 9. (LOVÁSZ [99]) *A graph* G *is perfect iff* $\omega(G')\omega(\overline{G'})$ *is not less than the number of vertices of* G', *for each induced subgraph* G' *of* G.

The following sharpening of Theorem 9 (and of the perfect graph theorem) is a conjecture of Berge and Gilmore, which is still unsolved.

STRONG PERFECT GRAPH CONJECTURE (BERGE [6]): A graph G is perfect iff no induced subgraph of G is isomorphic to the odd circuit C_{2n+1} or to its complement $\overline{C_{2n+1}}$, for $n \geq 2$.

So it is conjectured that each minimal nonperfect graph is isomorphic to an odd circuit or to the complement of an odd circuit.

Several partial results on this conjecture have been found: CHVÁTAL [21] showed that the strong perfect graph conjecture is equivalent to the conjecture that each minimal nonperfect graph G has a spanning subgraph isomorphic to $C^{\alpha-1}_{\alpha\omega-1}$, where $\alpha = \alpha(G)$ and $\omega = \omega(G)$ (a *spanning subgraph* of G arises from G by deleting some of the edges; C^k_n is the graph with vertices $1,\ldots,n$, two vertices i and j being adjacent iff $0 < |i-j| \le k \pmod n$)); PARTHASARATHY & RAVINDRA [130] showed the truth of the strong perfect graph conjecture for graphs having no $K_{1,3}$ as an induced subgraph (e.g. line-graphs; see also TROTTER [153] and De WERRA [173]) (this implies that, to show the conjecture, it is enough to show that any minimal nonperfect graph has no $K_{1,3}$ as induced subgraph) and for graphs having no K_4 minus one edge as an induced subgraph [131]; they investigated also perfectness of product graphs (see [135]); TUCKER proved the strong perfect graph conjecture for planar graphs [155], "circular arc" graphs [156], and 3-chromatic graphs [157]; GALLAI [58], SACHS [139] and MEYNIEL [114] showed that if every odd circuit in G of length at least five contains at least two non-crossing (Gallai)/ crossing (Sachs)/arbitrary (Meyniel) chords, then G is perfect; OLARU [122] and PADBERG [125,126,128] have derived several properties of minimal nonperfect graphs (e.g., PADBERG [125] showed that every minimal nonperfect graph G with n points contains exactly n cliques of size $\omega(G)$; their characteristic vectors form a nonsingular matrix).

2.4. Conormal hypergraphs

The theory of perfect graphs can be described and extended smoothly within the context of hypergraphs.

Let $G = (V,E)$ be a graph; let the hypergraph $H_G = (V,\mathcal{E})$ have edges all stable subsets of V. So H is conformal iff $\hat{H} = H_G$ for some (uniquely determined) graph G. Then, as can be seen straightforwardly, the property "$\omega(G) = \gamma(G)$" coincides with "$\alpha(H_G) = \rho(H_G)$".

If G' is the subgraph of G induced by $V' \subset V$, then $H_{G'}$ equals H^w_G, where w is the characteristic vector of V' (writing H^w_G for $(H_G)^w$). It follows that G is perfect if and only if $\alpha(H^w_G) = \rho(H^w_G)$ for each $w\colon V \to \{0,1\}$. Part I of the proof of the perfect graph theorem implies that G is perfect iff $\alpha(H^w_G) = \rho(H^w_G)$ for each function $w\colon V \to \mathbb{Z}_+$. In particular, if G is perfect then H_G is conormal. The next theorem implies even that:

(1) G *is perfect if and only if* H_G *is conormal,*

H *is conormal if and only if* $\hat{H} = H_G$ *for some perfect graph* G.

Hence the theories of perfect graphs and conormal hypergraphs pursue parallel courses. Formulations in terms of hypergraphs sometimes reveal underlying structures and create better understanding.

For each graph G one has: $H_{\bar{G}} = A(H_G)$. The perfect graph theorem now can be formulated and extended within the theory of hypergraphs as follows, yielding an extension of Theorem 6.

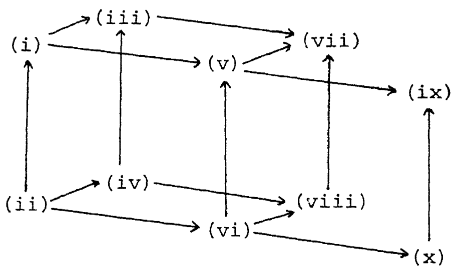
THEOREM 10. (FULKERSON [50,51], LEHMAN [95], LOVÁSZ [98,99,100], BERGE [10]) *Let* $H = (V,E)$ *be a hereditary, conformal hypergraph. Each of the following assertions is equivalent to* H *being conormal:*

(i)	$\alpha(H^w) = \rho(H^w)$ *for each* $w: V \to \{0,1\}$;	(ii)	id...$w:V \to \mathbb{Z}_+$;
(iii)	$\alpha(H^w) = \alpha^*(H^w)$ *for each* $w: V \to \{0,1\}$;	(iv)	id...$w:V \to \mathbb{Z}_+$;
(v)	$\rho^*(H^w) = \rho(H^w)$ *for each* $w: V \to \{0,1\}$;	(vi)	id...$w:V \to \mathbb{Z}_+$;
(vii)	$\alpha^*(H^w) \in \mathbb{Z}$ *for each* $w: V \to \{0,1\}$;	(viii)	id...$w:V \to \mathbb{Z}_+$;
(ix)	$\rho_2(H^w) = 2.\rho(H^w)$ *for each* $w: V \to \{0,1\}$;	(x)	id...$w:V \to \mathbb{Z}_+$;
(xi)	$\alpha(H^w)r(H^w) \geq \sum_V w(v)$ *for each* $w: V \to \{0,1\}$;	(xii)	id...$w:V \to \mathbb{Z}_+$;
(xiii)	$\alpha(H^w)\alpha(A(H)^\ell) \geq \sum_V w(v)\ell(v)$ *for each* $\ell,w:V \to \{0,1\}$;	(xiv)	id..$\ell,w:V \to \mathbb{Z}_+$;

(i')-(xii'), *arising from* (i)-(xii) *by replacing* H *by* $A(H)$.

PROOF. We shall not give a complete proof of this theorem, but discuss some parts of it and refer to the original papers for the details of the other parts.

It is clear, by using (16) of subsection 2.1, that



where arrows stand for implications.
The equivalence of the conormality of H to each of the assertions (iv),

(viii), (xiv), (iv') and (viii') is true by definition (cf. Theorem 6). The implication (iv) → (ii) was proved by FULKERSON [51]. This implies that (ii) and (ii') are equivalent, being the content of FULKERSON's "pluperfect graph theorem" [49,50,51] which says: if each graph arising from a graph G by a series of splittings of points (as in the first part of the proof of the perfect graph theorem) is perfect, then the same holds for the complementary graph $\bar{G}$. So, knowing the pluperfect graph theorem, to prove the perfect graph theorem it is enough to show that the class of perfect graphs is closed under splitting of points, and this was shown by LOVÁSZ [98] (part I of the proof of Theorem 8). Theorem 5 of [98] also shows the implication (vii) → (viii), and hence the equivalence of (i)-(viii).

(x) → (vi) is straightforward by observing that $\rho_{k\ell}(H^w) = \rho_{\ell}(H^{kw})$. If $2\rho(H^w) = \rho_2(H^w)$ for all $w: V \to \mathbb{Z}_+$, then

(2) $$\rho_{2^{i+1}}(H^w) = \rho_2(H^{2^i w}) = 2\rho(H^{2^i w}) = 2\rho_{2^i}(H^w),$$

hence, by induction on i, we have for all i

(3) $$\rho_{2^i}(H^w) = 2^i\rho(H^w),$$

i.e., for all i:

(4) $$\frac{\rho_{2^i}(H^w)}{2^i} = \rho(H^w).$$

Since $\rho^*(H^w) = \lim_{k\to\infty} (\rho_k(H^w))/k$ (cf. (14) in subsection 2.1) it follows that $\rho^*(H^w) = \rho(H^w)$.

The implication (ix) → (x), and hence the equivalence of (i)-(x), follows from BERGE [10] (cf. LOVÁSZ [100]).

Clearly (xii) → (xi) and (xiv) → (xiii). Furthermore (i) → (xi) and (ii) → (xii), since for each hypergraph H we have that $\rho(H).r(H)$ is at least the number of points in H.

It is easy to see that, in (xiii), we lose no generality if we assume that $\ell = w$. Since, for $w: V \to \{0,1\}$, $r(H^w) = \alpha(A(H)^w)$ the equivalence (xi) ↔ (xiii) is clear.

Also, for $w: V \to \mathbb{Z}_+$, $r(H^w) = \alpha(A(H)^{\ell})$, where ℓ arises from w by replacing each positive entry by 1. So (xiv) → (xii) is true. Finally, the implication (xi) → (i) follows from Theorem 7 (LOVÁSZ [99], cf. [100],

PADBERG [128], SAKAROVITCH [140]).

Hence the assertions (i)-(xiv) and (i')-(xii') all are equivalent. □

Note that each of the assertions (i)-(xii) implies that H is conformal, even if this were not required in advance (but hereditarity is still required). For suppose H is not conformal; let $V' \subset V$ be such that: (i) $V' \notin E$; (ii) each pair of elements of V' together forms an edge of H; and (iii) $|V'| = k$ is minimal (under the conditions (i) and (ii)). Let w be the characteristic vector of V'. Then: $\alpha(H^w) = 1$, $\alpha^*(H^w) = \frac{k}{k-1} = \rho^*(H^w)$, $r(H^w) = k-1$, $\sum_{v \in V} w(v) = k$, $\rho_2(H^w) = 3$, and $\rho(H^w) = 2$. This contradicts each of the assertions (i)-(xii).

A hypergraph is *normal* if the dual hypergraph is conormal. It follows from Theorem 10 that $H = (V,E)$ is normal if and only if $\nu(H') = \tau(H')$ for all hypergraphs $H' = (V,E')$ with $E' \subset E$.

The perfect graph theorem is contained in Theorem 10. It also follows that, to prove the strong perfect graph conjecture, it is sufficient to show that if a graph $G = (V,E)$ has no circuit C_{2n+1} or its complement $(n \geq 2)$ as induced subgraph, then the maximum value of $\sum_{v \in V} f(v)$ is an integer, where f is a nonnegative function defined on the vertices such that the sum of the numbers assigned to the vertices in any clique does not exceed 1.

A straightforward sharpening of the results mentioned in Section 1 gives that for each hypergraph H and natural number k:

(5) $\alpha_k(H^w) = k\alpha^*(H^w)$ *for all* $w: V \to \mathbb{Z}_+$, *if and only if* $k\alpha^*(H^w)$ *is an integer, for all* $w: V \to \mathbb{Z}_+$.

Hence also

(6) $\rho_k(H^w) = \alpha_k(H^w)$ *for all* $w: V \to \mathbb{Z}_+$, *if and only if* $\rho_k(H^w) = k\rho^*(H^w)$ *for all* $w: V \to \mathbb{Z}_+$, *and also, if and only if* $2\rho_k(H^w) = \rho_{2k}(H^w)$ *for all* $w: V \to \mathbb{Z}_+$.

What happens when we replace $\mathbb{Z}_+$ by $\{0,1\}$ in (5) and (6)? For k = 1, 2 or 3 they remain valid (k = 1: Theorem 10 (LOVÁSZ [98]); k = 2: LOVÁSZ [102]; k = 3: LOVÁSZ [106]), but for k = 60 we may not replace in (5) or (6) $\mathbb{Z}_+$ by $\{0,1\}$ (SCHRIJVER & SEYMOUR [142]).

Finally we discuss some examples.

EXAMPLE 5: *Bipartite graphs.* Let $G = (V,E)$ be a bipartite graph. Then G, $\bar{G}$,

$L(G)$ and $\overline{L(G)}$ are perfect (Example 2). It follows from Theorem 10 that:

(i) for each function $w: V \to \mathbb{Z}_+$, the maximum value of $w(v')+w(v'')$, where $\{v',v''\} \in E$, is equal to the minimum number of stable subsets of V (possibly taking a subset more than once) such that any vertex v is in at least $w(v)$ of these subsets;

(ii) for each function $w: E \to \mathbb{Z}_+$, the maximum value of $w(e_1)+\ldots w(e_k)$, where $e_1,\ldots,e_k$ are pairwise disjoint edges, is equal to the minimum value of $\sum_{v\in V} f(v)$, where $f: V \to \mathbb{Z}_+$ such that $f(v')+f(v'') \geq w(\{v',v''\})$ for each $\{v',v''\} \in E$;

(iii) each function $w: E \to \mathbb{Q}_+$ such that $\sum_{e \ni v} w(e) \leq 1$ for each $v \in V$, is a convex combination of characteristic vectors of matchings in G (BIRKHOFF [15] and Von NEUMANN [121]).

For a survey of several linear programming applications to bipartite graphs see FORD & FULKERSON [44], HOFFMAN [71] and HOFFMAN & KUHN [77].

EXAMPLE 6: *Partially ordered sets*. Theorem 10 also characterizes the convex hull of (characteristic vectors of) chains/antichains in a partially ordered set: this convex hull consists exactly of those nonnegative functions whose sum is at most 1 on each antichain/chain.

This characterization (and also Dilworth's theorem) has been extended by GREENE & KLEITMAN [64,65], cf. HOFFMAN & SCHWARTZ [79].

EXAMPLE 7: *Graphs*. Let $G = (V,E)$ be a graph without isolated vertices, and let $\mathcal{E}$ be the set $E\cup\{\{v\} \mid v \in V\}\cup\{\emptyset\}$. Set $H = (V,\mathcal{E})$, i.e., $H = \hat{G}$. It is easy to see that $\rho_4(H) = 2\rho_2(H)$. Since the class of hypergraphs H obtained this way from graphs is closed under multiplication of vertices, we derive from (6) that $\rho_2(H) = \alpha_2(H)$, i.e., $\rho_2(G) = \alpha_2(G)$ (cf. LOVÁSZ [102]).

EXAMPLE 8: *Matroids*. Let $H = (V,\mathcal{I})$ be a *matroid*, i.e. let $\mathcal{I}$ be a nonempty collection of subsets of V such that:

(i) if $V'' \subset V' \in \mathcal{I}$ then $V'' \in \mathcal{I}$;

(ii) if $V',V'' \in \mathcal{I}$ and $|V'| < |V''|$ then $V'\cup\{v\} \in \mathcal{I}$ for some $v \in V''\setminus V'$.

We furthermore assume that each singleton is in $\mathcal{I}$. The sets in $\mathcal{I}$ are called the *independent sets* of the matroid. H determines a *rank-function* $r: \mathcal{P}(V) \to \mathbb{Z}_+$, given by

(7) $$r(V') = \max\{|V''| \mid V'' \subset V' \text{ and } V'' \text{ is independent}\},$$

for $V' \subset V$. So $V' \in \mathcal{I}$ iff $r(V') = |V'|$.

Examples of matroids are given by:

(i) V is the set of edges of an undirected graph,
$\mathcal{I}$ consists of all sets of edges containing no circuit;

(ii) V is the set of edges of a connected, undirected graph,
$\mathcal{I}$ consists of all sets of edges the removal of which does not disconnect the graph;

(iii) V is a set of vectors in a vector space,
$\mathcal{I}$ consists of all linearly independent subsets of V;

(iv) V is a collection of subsets of a finite set,
$\mathcal{I}$ consists of all subcollections of V having a system of distinct representatives (cf. MIRSKY [116]).

For more background information about matroids see WELSH [172].

EDMONDS [32] (cf. [35]) showed, by means of the so-called *greedy algorithm*, that, for $w: V \to \mathbb{Z}_+$, the maximum value of $\sum_{v \in V'} w(v)$, where V' is independent, is equal to the minimum value of

$$(8) \qquad r(V_1) + \ldots + r(V_k)$$

where $V_1,\ldots,V_k$ are subsets of V (for some k) such that each element v of V occurs in at least w(v) sets of $V_1,\ldots,V_k$. In the language of matrices, let M be the $\mathcal{P}(V) \times V$-matrix such that the row with index $V' \in \mathcal{P}(V)$ is the characteristic vector of V'. Then Edmonds' result can be restated as: for each $w: V \to \mathbb{Z}_+$

$$(9) \qquad \max\{wx \mid x \in \mathbb{Z}_+^V,\ Mx \le r\} = \min\{yr \mid y \in \mathbb{Z}_+^{\mathcal{P}(V)},\ yM \ge w\}.$$

Let M' arise from M by dividing any row with index V' by r(V') (and deleting the row with index $\emptyset$). Then (9) implies that the polyhedron

$$(10) \qquad P = \{x \ge 0 \mid M'x \le 1\}$$

is the convex hull of characteristic vectors of independent sets of H. So the anti-blocking polyhedron of P is

$$(11) \qquad R = \{z \ge 0 \mid Nz \le 1\}$$

where N is the incidence matrix of H. By Theorem 1 R consists of all vectors $v \le c$ for some convex combination c of row vectors of M'. So the left hand side of the linear programming duality equality

(12) $$\max\{|z| \mid z \ge 0, Nz \le 1\} = \min\{|y| \mid y \ge 0, yN \ge 1\}$$

is equal to

(13) $$\max_{\emptyset \ne V' \subset V} \frac{|V'|}{r(V')} = \alpha^*(H) = \rho^*(H).$$

In fact, EDMONDS [28,33] and NASH-WILLIAMS [119] proved that $\rho(H) = \lceil \rho^*(H) \rceil$, i.e., *the minimum number of independent sets needed to cover* V *is equal to*

(14) $$\max_{\emptyset \ne V' \subset V} \frac{\lceil |V'| \rceil}{r(V')} .$$

This can be used to determine the minimum number of forests needed to cover the edges of a graph (NASH-WILLIAMS [118]; for a directed analogue see FRANK [47]). This theory can be dualized to get, e.g., the maximum number of disjoint spanning forests - see EDMONDS [29], NASH-WILLIAMS [117], TUTTE [162], WELSH [172].

2.5. Fulkersonian hypergraphs

The assertions for τ,ν analogous to those in Theorem 10, are not all equivalent to each other, that is, we may not sharpen Theorem 7 by replacing $\mathbb{Z}_+$ by $\{0,1\}$, nor we may extend Theorem 7 by setting $\tau = \nu$ for $\tau = \tau^*$. However, there are still some equivalences.

THEOREM 11. (LOVÁSZ [100]) *Let* $H = (V,E)$ *be a hypergraph. Then the following are equivalent:*

(i) $\tau^*(H^w)$ *is an integer for each* $w: V \to \{0,1\}$, *and*
(ii) $\tau(H^w) = \tau^*(H^w)$ *for each* $w: V \to \{0,1\}$.

PROOF. Since obviously (ii) → (i), we prove (i) → (ii). Suppose (i) is true and (ii) is false. Let $w: V \to \{0,1\}$ be such that $\tau^*(H^w) < \tau(H^w)$, and assume $|w|$ is as small as possible. Without loss of generality we may assume that $H = H^w$.

So for all $u: V \to \{0,1\}$ we have $\tau(H^u) = \tau^*(H^u)$ whenever $u(v) = 0$ for some $v \in V$. Let $z: V \to \mathbb{Q}_+$ be such that $\sum_{v \in E} z(v) \ge 1$ for all $E \in E$, and

$\tau^*(H) = |z|$. Let v' be a vertex such that $z(v') > 0$. Let $u(v) = 1$ if $v \neq v'$, and $u(v') = 0$. Then

$$(1) \qquad \tau^*(H) = |z| > |z| - z(v') = uz \geq \tau^*(H^u) \geq \tau^*(H) - 1.$$

Hence, since by (i) $\tau^*(H^u)$ and $\tau^*(H)$ are integers, $\tau^*(H) = 1 + \tau^*(H^u)$. As $\tau(H^u) = \tau^*(H^u)$ and $\tau(H) \leq 1 + \tau(H^u)$ it follows that $\tau(H) = \tau^*(H)$. □

Direct consequences of Theorem 11 are:

COROLLARY 12. *Let* $H = (V,E)$ *be a hypergraph. Then the following two assertions are equivalent:*

(i) $\nu(H^w) = \nu^*(H^w)$ *for all* $w: V \to \{0,1\}$;
(ii) $\nu(H^w) = \tau(H^w)$ *for all* $w: V \to \{0,1\}$.

COROLLARY 13. (cf. LOVÁSZ [105]) *Let* $H = (V,E)$ *be a hypergraph. Then the following three assertions are equivalent:*

(i) $\nu(H^w) = \nu^*(H^w)$ *for all* $w: V \to \mathbb{Z}_+$;
(i) $\nu(H^w) = \tau(H^w)$ *for all* $w: V \to \mathbb{Z}_+$;
(iii) $\nu_2(H^w) = 2.\nu(H^w)$ *for all* $w: V \to \mathbb{Z}_+$.

Corollary 13 follows from Corollary 12 by applying Corollary 12 for each H^w apart. Assertion (iii) can be seen in the same way as the implication (x) → (vi) of Theorem 10.

A hypergraph H satisfying (i) and (ii) of Corollary 12 is called *seminormal*; if H satisfies (i), (ii) and (iii) of Corollary 12, H is called *Mengerian*. It is not difficult to see that each normal hypergraph (cf. subsection 2.4) is seminormal.

The following theorem gives a characterization of hypergraphs H for which the blocker B(H) is Mengerian. A *k-cover* of $H = (V,E)$ is a function $\ell: V \to \mathbb{Z}_+$ such that $\sum_{v \in E} \ell(v) \geq k$ for all $E \in E$.

THEOREM 14. *Let* $H = (V,E)$ *be a hypergraph. Then* B(H) *is Mengerian if and only if, for each natural number* k, *any* k*-cover is the sum of* k 1*-covers of* H.

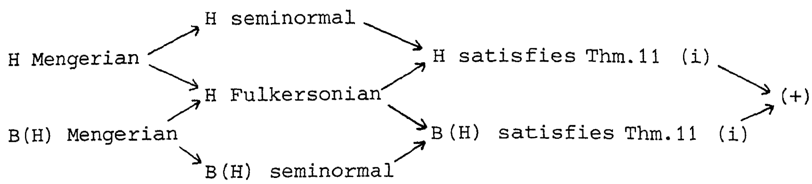
PROOF. By definition, B(H) is Mengerian iff $\nu(B(H)^\ell) = \tau(B(H)^\ell)$, for each $\ell: V \to \mathbb{Z}_+$. Now $\tau(B(H)^\ell)$ equals the minimum value of $\sum_{v \in E} \ell(v)$, for $E \in E$. Moreover, $\nu(B(H)^\ell)$ equals the maximum number k of 1-covers $\ell_1,\ldots,\ell_k$ such that $\ell_1(v)+ \ldots +\ell_k(v) \le \ell(v)$ for each $v \in V$. So, for each natural number k we have: for each $\ell: V \to \mathbb{Z}_+$: $\tau(B(H)^{\ell}) \ge k$ implies $\nu(B(H)^{\ell}) \ge k$, if and only if each k-cover is the sum of k 1-covers. □

Note that the right hand side of the equivalence of Theorem 14 directly implies (by definition of τ_k (Section 2.1)) that $\tau_k(H) = k\,\tau(H)$ for all k, that is, $\tau(H) = \tau^*(H)$.

The relations between the several classes of hypergraphs can be visualized in a diagram, where arrows stand for implications, and (+) denotes

(+) $$\tau(H^w)\tau(B(H)^{\ell}) \le \ell w, \text{ for all } \ell, w: V \to \{0,1\},$$

for $H = (V,E)$.

(2)


There are no more arrows (or equivalences) in this diagram (except for arrows following from the transitive closure of implications). To show this, it is enough to give an example of a non-seminormal hypergraph with Mengerian blocker, and an example of a seminormal hypergraph whose blocker does not satisfy (i) of Theorem 11.

The hypergraph Q_6, having vertices all edges of K_4 (the complete undirected graph on four points), with edges all triangles in K_4 (considered as triples of edges) is not seminormal, but $B(Q_6)$ is Mengerian (LOVÁSZ [100], SEYMOUR [145]). SEYMOUR [145] conjectures that a Fulkersonian hypergraph $H = (V,E)$ is Mengerian if it does not contain a minor whose minimal edges (under inclusion) form a hypergraph isomorphic to Q_6 (a hypergraph H' is a *minor* of H if it arises from H by a series of removals of points (i.e. multiplications by k = 0), and contractions of points (i.e., removal of the points from the vertex set and from the edges)). It is easy to see that any minor of a Mengerian hypergaph is Mengerian again. Validity of this conjecture implies the truth of Seymour's second conjecture that a hypergraph H is Mengerian if its blocker is Mengerian and H itself does not have Q_6 as a minor ("Both conjectures are based on a lack of counterexamples rather than a superfluity of supporting evidence.")[1]) The hypergraph with four points and with edges all three-element subsets containing a fixed point, is semi-normal, but its blocker does not satisfy assertion (i) of Theorem 11.

Again, Theorem 11 and its corollaries can be extended to:

(3) $k.\tau^*(H^w)$ *is an integer for each* $w: V \to \mathbb{Z}_+$, *if and only if* $k.\tau^*(H^w) = \tau_k(H^w)$ *for each* $w: V \to \mathbb{Z}_+$,

and

(4) $k.\nu^*(H^w) = \nu_k(H^w)$ *for each* $w: V \to \mathbb{Z}_+$, *if and only if* $\tau_k(H^w) = \nu_k(H^w)$ *for each* $w: V \to \mathbb{Z}_+$ *and also, if and only if* $\nu_{2k}(H^w) = 2\nu_k(H^w)$ *for each* $w: V \to \mathbb{Z}_+$,

for any hypergraph $H = (V,E)$ (LOVÁSZ [102,105], SCHRIJVER & SEYMOUR [142]).

There is a variety of classes of hypergraphs to which we can apply the results obtained in this subsection (for more examples see MAURRAS [110], WOODALL [175]).

EXAMPLE 9: *Bipartite graphs*. Let $H = (V,E)$ be a bipartite graph. It is very easy to show that $\nu_2(H) = 2\nu(H)$. Since the class of bipartite graphs is closed under multiplication of vertices we even know that $\nu_2(H^w) = 2\nu(H^w)$ for all $w: V \to \mathbb{Z}_+$. Hence, by Corollary 13, $\tau(H) = \nu(H)$, which is the content of KŌNIG's theorem [87].

Let K be the hypergraph obtained from the bipartite graph H by taking as vertices all edges of H, and as edges of K all *stars*, i.e., all sets $\{e \in E | v \in e\}$ for $v \in V$. Now K is Mengerian (see Example 16), and B(K) is Mengerian, which follows from a result of GUPTA [67,68]: *the maximum number of pairwise disjoint sets of edges in bipartite graph, each set covering all points, is equal to the minimum valency of the bipartite graph* (this result was also found by D. Kōnig (unpublished)). Note that the class of hypergraphs B(K) arising this way from a bipartite graph is closed under multiplication of vertices.

EXAMPLE 10: *Network flows*. Let $H = (V,E)$ be a hypergraph with vertices all arrows in a digraph, and edges all r-s-paths (where r and s are two fixed vertices of the digraph). By Corollary 13, to prove FORD & FULKERSON's max-flow min-cut theorem [43] (in the integer form) it suffices to prove that $\nu_2(H) = 2\nu(H)$ for each hypergraph H arising this way from digraphs. Corollary 13 then gives that $\tau(H^w) = \nu(H^w)$ for all $w: V \to \mathbb{Z}_+$, which is the content of the max-flow min-cut theorem.

EXAMPLE 11: *Graphs.* Let $G = (V,E)$ be a graph. After proving that $\nu_4(G) = 2\nu_2(G)$ (which is not difficult) and observing that the class of graphs is closed under multiplication of vertices, we deduce from (4) that $\tau_2(G) = \nu_2(G)$ (TUTTE [160], cf. BERGE [12]).

GALLAI [56,57] showed that $\alpha(G)+\tau(G) = \rho(G)+\nu(G) = |V|$ (assuming that $V = \cup E$). LOVÁSZ [102] observed that one proves similarly:

$$(5) \qquad \alpha_2(G) + \tau_2(G) = \rho_2(G) + \nu_2(G) = 2|V|.$$

Hence "$\tau_2(G) = \nu_2(G)$" can be derived from Example 7.

BERGE [2] derived from a result of TUTTE [158,161] that

$$(6) \qquad \nu(G) = \min_{V' \subset V} \frac{|V|+|V'|-o(V\backslash V')}{2}$$

where $o(V\backslash V')$ denotes the number of components having an odd number of vertices in the subgraph of G induecd by $V\backslash V'$. This result is known as the *Tutte-Berge theorem* - see subsection 4.3.

EXAMPLE 12: *Directed cuts.* Let $D = (V,A)$ be a digraph. A *directed cut* is a set of arrows of the form $(V\backslash V',V')$ whenever $\emptyset \neq V' \neq V$ and $(V',V\backslash V') = \emptyset$. Here (V',V'') denotes the set of arrows with tail in V' and head in V''. Consider the hypergraph H with vertices all arrows of D, and edges all directed cuts.

Call a set of arrows the contraction of which makes D strongly connected, a *diconnecting set*. That is, a set A' of arrows is diconnecting iff adding, for each arrow in A', an arrow in the reversed direction makes D strongly connected. Let K be the hypergraph with vertices all arrows, and with edges all diconnecting subsets of A. It is easy to see that $K = B(H)$.

In 1976 LUCCHESI & YOUNGER [108] proved that $\tau(H) = \nu(H)$ (this was conjectured by Robertson & Younger), i.e., *the minimum size of a diconnecting set is equal to the maximum number of pairwise disjoint directed cuts* (for a proof see Example 19). Since the class of hypergraphs H obtained this way from directed graphs is closed under multiplication of vertices, we even have that $\tau(H^w) = \nu(H^w)$ for each $w: A \to \mathbb{Z}_+$, i.e., H is Mengerian. This implies that H and $K = B(H)$ are Fulkersonian. Hence $\tau(K) = \tau^*(K)$. It is conjectured by EDMONDS & GILES [37] that, in fact, $\tau(K) = \nu(K)$, i.e. the minimum size of a directed cut is equal to the maximum number of pairwise disjoint diconnecting sets.[1)] Since the class of hypergraphs K obtained this way from digraphs is closed under multiplication of vertices by $k \neq 0$, a simple adaptation of the proof method for Corollary 13 shows that it is enough to prove that, in general, $\nu_2(K) = 2\nu(K)$.

Edmonds & Giles' conjecture has been proved by FRANK [46] (cf. Example 23) in case the digraph D has a vertex from which each other vertex is reachable by a directed path (this result also follows from Edmonds' arborescence theorem (Example 13)).

EXAMPLE 13: *Arborescences.* Let $D = (V,A)$ be a digraph, with fixed vertex r, called the root. An *r-arborescence* is a collection A' of arrows such that each vertex in V is reachable from r by a directed path consisting of arrows from A'. It is easy to see that a minimal (under inclusion) r-arborescence is a directed tree.

Let H be the hypergraph with vertex set A and edges all r-arborescences. EDMONDS [31,34] (cf. LOVÁSZ [105], TARJAN [152], and Example 22) proved that $\tau(H) = \nu(H)$, that is, *the maximum number of edge-disjoint r-arborescences is equal to the minimum "indegree" of any nonempty subset of* $V\backslash\{r\}$ *(Edmonds' arborescence* or *branching theorem)*. Here we used that the blocker $K = B(H)$ of H has edges all sets containing a set of edges of the form $(V\backslash V', V')$ for some $\emptyset \neq V' \subset V\backslash\{r\}$ (again, (V',V'') denotes the set of arrows from V' to V").

By Menger's theorem, Edmonds' result is equivalent to: *if there are* k *edge-disjoint paths from* r *to any other vertex, then there are* k *edge-disjoint* r*-arborescences*. A. Frank (personal communication) posed, as a conjecture, a vertex-disjoint version of this theorem:

CONJECTURE. If from r to any other vertex there are at least k vertex-disjoint paths, then there are k r-arborescences such that, for each vertex $s \neq r$, the (unique) paths from r to s within the respective r-arborescences are pairwise vertex-disjoint (clearly, except for their endpoints).

FRANK [45] also relates Edmonds' theorem to Tutte's theorem on the maximum number of disjoint spanning trees in a graph (cf. Example 8).

Since the class of hypergraphs H obtained this way from digraphs is closed under multiplication of vertices it is even true that $\tau(H^w) = \nu(H^w)$ for all $w: A \rightarrow \mathbb{Z}_+$. So H is Mengerian and Fulkersonian, hence also $K = B(H)$ is Fulkersonian. FULKERSON [52,53] (cf. LOVÁSZ [106]) showed that K is also Mengerian, i.e., *the minimum weight of an r-arborescence is equal to the* *maximum number of sets of the form* $(V\setminus V',V')$ $(V' \subset V\setminus\{r\})$ *such that no arrow occurs in more of these sets than its weight* (for any integral weight function defined on the edges) (see Example 22).

EXAMPLE 14: *Binary hypergraphs*. A hypergraph $H = (V,\mathcal{E})$ is called *binary* if $E_1 \Delta E_2 \Delta E_3 \in \mathcal{E}$ whenever $E_1, E_2, E_3 \in \mathcal{E}$ (Δ means symmetric difference); so the characteristic vectors of the edges may be regarded as vectors in a coset of a chain-group modulo 2 (for characterizations of binary hypergraphs, see LEHMAN [94] and SEYMOUR [114]).

It is easy to see that the class of hypergraphs $\check{H}$ arising from binary hypergraphs H is closed under multiplication of vertices. If H is binary, then $B(H) = \check{K}$ where K has edges all subsets of V intersecting each edge of H in an odd number of points. So K again is binary, and $B(K) = \check{H}$.

LOVÁSZ [102] proved that each binary hypergraph H has $\tau_2(H) = 2\tau(H)$. SEYMOUR [145] proved that a binary hypergraph is Mengerian if and only if H has no minor isomorphic to Q_6.

The class of binary Fulkersonian hypergraphs has, as yet, not been characterized this way, despite its nice structural properties (the class is closed under taking blockers). SEYMOUR [146] conjectures that a binary hypergraph is Fulkersonian if and only if it does not contain a minor whose minimal edges are "isomorphic" to: either the lines of the Fano-plane, or the edge-sets of odd circuits of K_5, or the minimal edge-sets in K_5 intersecting each odd circuit.

(SEYMOUR [145] in fact proved: let $H = (V,\mathcal{I})$ be a matroid, and let $\mathcal{C}$ be its set of *circuits* (i.e., minimal dependent sets); then *for each* $v \in V$ *the hypergraph* $(V\setminus\{v\},\{C\setminus\{v\} \mid v \in C \in \mathcal{C}\})$ *is Mengerian if and only if* H *is a binary matroid not containing the dual of the Fano-matroid as a minor* (binary and minor, for the moment, in the matroid sense). This generalizes Menger's theorem for undirected graphs. In this light it is interesting to see that MINTY [115] proved, for collections $\mathcal{C}$ and $\mathcal{D}$ of subsets of a set V: $\mathcal{C}$ *and* $\mathcal{D}$ *are the collections of circuits and cocircuits of a matroid, respectively, if and only if for each* v *in* V *the hypergraphs* $(V\setminus\{v\},\{C\setminus\{v\} \mid v \in C \in \mathcal{C}\})$ *and* $(V\setminus\{v\},\{D\setminus\{v\} \mid v \in D \in \mathcal{D}\})$ *have, as edges, the minimal edges of the blocker of each other*. So the class of matroids for which the hypergraphs $(V\setminus\{v\},\{C\setminus\{v\} \mid v \in C \in \mathcal{C}\})$ are Fulkersonian $(v \in V)$ is closed under taking duals.)

We give four examples of binary hypergraphs, each of them being derived from a graph $G = (V,E)$.

(i) Let r and s be two vertices of G. Let $\mathcal{E}$ consist of those subsets E' of E such that the graph (V,E') has an even valency at each point except at r and s. The hypergraph $H = (E,\mathcal{E})$ is binary, and the minimal edges are the r-s-paths. By Menger's theorem H is Mengerian, and also B(H) is Mengerian (trivially).

(ii) Let T be an even subset of V and call a subset E' of E a *T-join* if T coincides with the set of vertices having an odd valency in the graph (V,E'). Let $\mathcal{E}$ be the collection of T-joins. Then the hypergraph $H = (E,\mathcal{E})$ is binary.

A subsets E' of E is called a *T-cut* if E' is equal to $\delta(V')$ for some $V' \subset V$ with $|V' \cap T|$ odd ($\delta(V')$ is the set of edges intersecting V' in exactly one point). Let $\mathcal{F}$ consist of all T-cuts. The hypergraph $K = (E,\mathcal{F})$ again is binary. Furthermore $\check{H} = B(K)$ and $\check{K} = B(H)$. SEYMOUR [148] proved that, if G is bipartite, then $\nu_2(K) = 2\nu(K)$; this implies a result of LOVÁSZ [102] that, if G is arbitrary, $\nu_4(K) = 2\nu_2(K)$ (this implication can be seen by replacing each edge of G by two edges in series, thus obtaining a bipartite graph). Since the class of hypergraphs K obtained this way from graphs is closed under multiplication of vertices (this is *not* so if we restrict ourselves to bipartite graphs) (4) implies that $\nu_2(K) = \tau_2(K)$. As K is binary we know that $\tau_2(K) = 2\tau(K)$, hence $\tau(K) = \frac{1}{2}\nu_2(K)$ ((a) moreover if G is bipartite then $\tau(K) = \nu(K)$; (b) if $G = K_4$ and $T = V$ then $\tau(K) \neq \nu(K)$; (c) if we have $T = V$, then $\tau(K)$ is equal to the minimum size of a V-join; in that case $\tau(K) = \frac{1}{2}|V|$ if and only if G contains a perfect matching (cf. subsection 4.3) - LOVÁSZ [102] showed that Tutte's 1-factor theorem can be derived in this way).

In particular, $\tau(K) = \tau^*(K)$, hence by Theorem 7 $\tau(H) = \tau^*(H)$ (EDMONDS & JOHNSON [39], extending the "Chinese postman problem"), i.e., since the class of hypergraphs H obtained this way is closed under multiplication of vertices, H and K are Fulkersonian (but, in general it is not the case that $\frac{1}{2}\nu_2(H) = \tau(H)$).

(iii) Let r,s,r',s' be four distinct vertices of G. Let $\mathcal{E}$ be the collection of all subsets E' of E such that, in the graph (V,E'), either r and s, or r' and s' are the only two vertices of odd valency. So the minimal elements of $\mathcal{E}$ are the r-s-paths and the r'-s'-paths. Clearly, the hypergraph $H = (E,\mathcal{E})$ is binary.

Let $\mathcal{F}$ be the collection of all subsets $E' = \delta(V')$ of E such that $|V' \cap \{r,s\}| = |V' \cap \{r',s'\}| = 1$. Again $K = (E,\mathcal{F})$ is a binary hypergraph. Furthermore $\check{H} = B(K)$ and $\check{K} = B(H)$.

LOVÁSZ [104] proved that, if G is Eulerian, then $\nu_2(H) = 2\nu(H)$; this implies that, for arbitrary G, $\nu_4(H) = 2\nu_2(H)$ (make G Eulerian by replacing each edge by two parallel edges). Since the class of hypergraphs H obtained this way is closed under multiplication of vertices we know, by (4), that $\tau_2(H) = \nu_2(H)$. Moreover, since H is binary $\tau_2(H) = 2\tau(H)$, hence $\tau(H) = \frac{1}{2}\nu_2(H)$, which is the content of HU's *two-commodity-flow theorem* [81]. So, if G is Eulerian, then $\tau(H) = \nu(H)$, which is a result of ROTHSCHILD & WHINSTON [137]: *the maximum number of edge-disjoint paths connecting* r *with* s, *or* r' *with* s' *in the Eulerian graph* G *is equal to the minimum size of a collection of edges whose removal disconnects* r *from* s, *and* r' *from* s'.

Similarly, SEYMOUR [147] proved that, if G is bipartite, then $\nu_2(K) = 2\nu(K)$; hence, by an analogous reasoning, we know that $\tau(K) = \frac{1}{2}\nu_2(K)$ ($= \nu(K)$ if G is bipartite).

The classes of hypergraphs $\check{H}$ and $\check{K}$ arising this way are closed under multiplication of vertices, so it follows that H and K are Fulkersonian.

(iv) Suppose V partitions into R,S,R' and S'. Let H be the hypergraph with vertex set E, and edges all subsets E' of E such that, in the graph (V,E'), either there is an odd number of points with odd valency in each of R and S and an even number of points with odd valency in each of R' and S', or conversely.

So the minimal edges of H are the paths connecting either R with S or R' with S'. It is easy to see that H is binary.

KLEITMAN, MARTIN-LÖF, ROTHSCHILD & WHINSTON [85] proved that $\tau(H) = \nu(H)$. This can be derived from $\nu_2(H) = 2\nu(H)$: the class of hypergraphs H arising this way is closed under multiplication of vertices, hence, by Corollary 13, $\tau(H) = \nu(H)$.

<u>EXAMPLE 15</u>: *S-paths*. Let G = (V,E) be a graph and let S be a subset of V. Call a set of edges an *S-path* if it forms a path between two different points of S. Let H be the hypergraph with vertex set E and edges all S-paths. LOVÁSZ [104] proved that $\tau_2(H) = \nu_2(H)$; since the class of hypergraphs obtained this way is closed under multiplication of vertices it is sufficient to prove that $\nu_4(H) = 2\nu_2(H)$.

MADER [109] showed that

$$(7) \qquad \nu(H) = \min \frac{\Delta(V_1)+\ldots+\Delta(V_k)-\varepsilon(V\setminus(V_1\cup\ldots\cup V_k))}{2}$$

where the minimum is taken over all collections of pairwise disjoint sets $V_1,\ldots,V_k$ such that $S \subset V_1 \cup \ldots \cup V_k$ and each V_i intersects S in exactly one point (so $k = |S|$); $\Delta(V')$ is the number of edges intersecting V' in exactly one point, and $\varepsilon(V')$ denotes the number of components C of the subgraph induced by V' for which $\Delta(C)$ is odd.

Mader thus proved, inter alia, Gallai's conjecture that $\nu(H) \geq \frac{1}{2}\tau(H)$ (cf. LOVÁSZ [104]). Mader's result can be derived also from the matroid parity theorem for representable matroids of LOVÁSZ [107].

3. TOTAL UNIMODULARITY

3.1. Totally unimodular matrices

In the preceding section one of the main problems was to decide whether certain polyhedra have integral vertices, or, more generally, whether each of their faces contains integral vectors. Therefore, it would be nice to have a characterization of pairs of matrices M and vectors b such that each face of the polyhedron

(1) $$P = \{x \mid Mx \leq b\}$$

contains integral vectors. This problem has, as yet, not been solved in general; but a nice result in this direction was found by HOFFMAN & KRUSKAL [76]. A matrix M is called *totally unimodular* if each square submatrix of M has determinant +1, 0 or -1; it follows that M is a $\{+1,0,-1\}$-matrix.

THEOREM 15. (HOFFMAN & KRUSKAL [76]) *If* M *is a totally unimodular matrix and* b *is integer-valued then each face of the polyhedron* $P = \{x \mid Mx \leq b\}$ *contains integral vectors.*

PROOF. Let M be a totally unimodular matrix and let b be an integral vector. Let $F = \{x \mid M'x = b'\}$ be a minimal face of P (cf. Section 1.2), where the matrix M' consists of some rows of M and b' consists of the corresponding entries of b. We may assume that the rows of M' are linearly independent. Let $M' = M_1'M_2'$, where M_1' is nonsingular. Since $\det M_1' = \pm 1$ we find that the vector

(2) $$x = \begin{pmatrix} (M_1')^{-1} \\ 0 \end{pmatrix} . b'$$

is integer-valued. Since $M'x = b'$, the face F contains an integral vector. □

Let M be a totally unimodular matrix. Since the matrix

$$(3) \qquad \begin{pmatrix} I \\ -I \\ M \\ -M \end{pmatrix}$$

is totally unimodular as well, it follows that for all integral a,b,c and d, each face of the polyhedron $\{x \mid c \le x \le d,\ a \le Mx \le b\}$ contains integral vectors. In fact, Hoffman & Kruskal showed that this characterizes totally unimodular matrices.

THEOREM 16. (HOFFMAN & KRUSKAL [76], VEINOTT & DANTZIG [165]) *A matrix* M *is totally unimodular iff for each integral vector* b *each face of the polyhedron* $\{x \mid x \ge 0,\ Mx \le b\}$ *contains integral vectors.*

One implication follows directly from Theorem 15; the reverse implication is more difficult to prove - see e.g. GARFINKEL & NEMHAUSER [59].

In particular, it follows from Theorem 15 that if M is totally unimodular and b and w are integral vectors, then both sides of the linear programming duality equation

$$(4) \qquad \max\{wx \mid x \ge 0,\ Mx \le b\} = \min\{yb \mid y \ge 0,\ yM \ge w\}$$

can be solved with integral x and y.

Other characterizations of a matrix M to be totally unimodular are:

(i) each collection of rows of M can be split into two classes such that the sum of the rows in one class, minus the sum of rows in the other class, is a $0,\pm1$-vector (GHOUILA-HOURI [60]);

(i) M is a $(0,\pm1)$-matrix with no nonsingular submatrix containing an even number of nonzero entries in each row and in each column (CAMION [17]);

(iii) M is a $(0,\pm1)$-matrix with no square submatrix having determinant ±2 (Gomory, cf. CAMION [17]).

For more results concerning totally unimodular matrices, cf. COMMONER [22], HOFFMAN [73], PADBERG [129].

Hoffman & Kruskal's result can be applied to the following examples.

EXAMPLE 16: *Bipartite graphs. The incidence matrix of a graph is totally unimodular iff the graph is bipartite.* Let M be the incidence matrix of the bipartite graph $G = (V,E)$. By taking in (4) $w \equiv 1$ and $b \equiv 1$ one gets

$$(5) \qquad \max\{|x| \mid x \in \mathbb{Z}_+^V, \ Mx \le 1\} = \min\{|y| \mid y \in \mathbb{Z}_+^E, \ yM \ge 1\}$$

which is the content of the theorem of KŐNIG [87] and EGERVÁRY [42]: *the maximum number of pairwise nonadjacent points is equal to the minimum number of edges covering all points,* i.e., $\alpha(G) = \rho(G)$.

Similarly, one has that

$$(6) \qquad \min\{|x| \mid x \in \mathbb{Z}_+^V, \ Mx \ge 1\} = \max\{|y| \mid y \in \mathbb{Z}_+^E, \ yM \le 1\}$$

or: *the maximum number of pairwise disjoint edges is equal to the minimum number of points representing each edge* (KŐNIG's theorem [87]), i.e. $\tau(G) = \nu(G)$.

Clearly, by letting w and b arbitrary, we can obtain more general results, e.g., for all $w: E \to \mathbb{Z}_+$

$$(7) \qquad \min\{yw \mid y \in \mathbb{Z}_+^E, \ yM \ge 1\} = \max\{|x| \mid x \in \mathbb{Z}_+^V, \ Mx \le w\}$$

which implies that the hypergraph K of Example 9 is Mengerian.

EXAMPLE 17: *Network flows.* The incidence matrix of a digraph $D = (V,A)$ is the $A \times V$-matrix M with:

$$(8) \qquad \begin{aligned} M_{a,v} &= 1, \text{ if } v \text{ is head of arrow } a, \\ M_{a,v} &= -1, \text{ if } v \text{ is tail of arrow } a, \\ M_{a,v} &= 0, \text{ otherwise.} \end{aligned}$$

The incidence matrix of a digraph is totally unimodular (this was first conjectured by POINCARÉ [132]).

Let r and s be two vertices of a digraph $D = (V,A)$, and let D' be derived from D by adding a new arrow a' with tail s and head r. Let M' be the incidence matrix of D'. Consider the linear programming duality equation

$$(9) \qquad \max\{yf \mid 0 \le y \le d, \ yM' \le 0\} = \min\{dz \mid z \ge 0, \ x \ge 0, \ z+M'x \ge f\}$$

where f is a vector with a one in the position of the new arrow a', and zeros in the other positions, and d is any integral vector.

We may view d as a capacity function defined on the arrows of D', and y as a flow function. The condition "$yM' \leq 0$" can be interpreted as saying that no vertex of D receives a larger amount of flow than departs from it. Since the total amount of incoming flow is equal to the total amount of outgoing flow, $yM' \leq 0$ implies $yM' = 0$. The value of yf equals the flow in D' through the new arrow a'. So the maximum value of yf is equal to the maximum flow through the arrows of D from r to s, subject to the capacity function d (restricted to D), if we take d(a') large enough. By the total unimodularity of M this flow y can be taken to be integral.

The right hand side of (9) is equal to the minimum value of dz where $z: A \to \mathbb{Z}_+$ and $x: V \to \mathbb{Z}_+$ such that

$$(10) \qquad z(a) + x(w) - x(v) \geq 0$$

for each arrow a = (v,w) of D, and $z(a')+x(r)-x(s) \geq 1$, by the definition of f. If d(a') is large enough, a pair z,x achieving the minimum has z(a') = 0, so $x(r) \geq 1 + x(s)$. It follows straightforwardly that the minimum value of dz is equal to the minimum capacity of an r-s-disconnecting set.

So from the total unimodularity of M one can derive FORD & FULKERSON's max-flow min-cut theorem [43]: *the maximum amount of flow from* r *to* s *subject to the capacity function* d *is equal to the minimum capacity of an* r-s-*disconnecting set*. If all capacities are integers then the optimal flow can be taken to be integral ("integer flow theorem"). If each capacity is 1 then Menger's theorem follows.

If we impose not only an upper bound d, but also a lower bound function c for the flow through arrows, where $0 \leq c \leq d$, (9) gives: the maximum flow in D from r to s subject to the upper bound d and the lower bound c, is equal to the minimum value of

$$(11) \qquad \sum_{\substack{(v,w)\in E \\ v\in V',w\in V''}} d((v,w)) - \sum_{\substack{(w,v)\in E \\ w\in V'',v\in V'}} c((w,v))$$

where V',V" partitions V such that $r \in V'$ and $s \in V''$ (cf. HOFFMAN [71]). If we impose only lower bounds and no upper bounds one can derive, inter alia, Dilworth's theorem (Example 3) (cf. also HOFFMAN [72] and HOFFMAN & SCHWARTZ [79]).

Let $D = (V,A)$ be a directed graph, and let A' be a set of arrows together forming a spanning tree for D. Let M be the $A' \times A$-matrix given by

(12)
$$\begin{aligned} M_{a,e} &= 0, \text{ if the unique v-w-path in } A' \text{ does not pass a;} \\ M_{a,e} &= 1, \text{ if the unique v-w-path in } A' \text{ pass a forwardly;} \\ M_{a,e} &= -1, \text{ if the unique v-w-path in } A' \text{ pass a backwardly;} \end{aligned}$$

for $a \in A'$ and $e = (v,w) \in A$. Then M is totally unimodular; this can be derived from the above by using elementary linear algebra arguments (TUTTE [163], cf. BONDY & MURTY [16]).

3.2. Unimodular, balanced and normal hypergraphs

A hypergraph $H = (V,\mathcal{E})$ is called *unimodular* if its incidence matrix is totally unimodular. H is *balanced* if for all $E_1,\ldots,E_k$, $x_1 \in E_1 \cap E_2, \ldots, x_{k-1} \in E_{k-1} \cap E_k$, $x_k \in E_k \cap E_1$, where k is odd, there exists an E_i $(1 \leq i \leq k)$ containing at least three elements from $x_1,\ldots,x_k$. Formulated otherwise, H is balanced iff its incidence matrix does not contain an odd-sized square submatrix with exactly two ones in each row and each column. It follows from Gomory's and Camion's characterizations of totally unimodular matrices (subsection 3.1) that each unimodular hypergraph is balanced.

Unimodular and balanced hypergraphs form, in a sense, a mixture of hypergraphs "nice" for α,ρ-problems and those "nice" for τ,ν-problems.

Berge and Las Vergnas characterized balanced hypergraphs. A hypergraph $H' = (V',\mathcal{E}')$ is called a *partial subhypergraph* of $H = (V,\mathcal{E})$ if $V' \subset V$ and $\mathcal{E}' \subset \{E \cap V' \mid E \in \mathcal{E}\}$.

THEOREM 17. (BERGE [8,9], BERGE & LAS VERGNAS [14]) *Let $H = (V,\mathcal{E})$ be a hypergraph. The following assertions are equivalent:*

(i) H *is balanced;*

(ii) $\tau(H') = \nu(H')$, *for each partial subhypergraph* H' *of* H;

(iii) $\alpha(H') = \rho(H')$, *for each partial subhypergraph* H' *of* H;

(iv) $\gamma(H') = r(H')$, *for each partial subhypergraph* H' *of* H;

(v) $q(H') = \delta(H')$, *for each partial subhypergraph* H' *of* H;

(vi) $\kappa(H') = r'(H')$, *for each partial subhypergraph* H' *of* H;

(vii) $\varepsilon(H') = \delta'(H')$, *for each partial subhypergraph* H' *of* H.

Here: $\gamma(H')$ = the minimum number of colours needed to colour the vertices of H' such that no edge contains the same colour twice;

$r(H')$ and $r'(H')$ denote the maximum and minimum size, respectively, of edges of H';

$\delta(H')$ and $\delta'(H')$ denote the maximum and minimum valency, respectively, of H';

$q(H')$ = minimum number of collections of pairwise disjoint edges, such that each edge is in at least one of these collections;

$\kappa(H')$ = maximum number of pairwise disjoint subsets of the vertex set of H', each of them intersecting each edge;

$\varepsilon(H')$ = maximum number of pairwise disjoint edge collections, each covering the vertex set of H'.

PROOF. To prove that each of (ii)-(vii) implies (i) is easy: if H is not balanced H contains, as a partial subhypergraph, an odd circuit graph, for which none of (ii)-(vii) is valid.

For a proof of (i) → (ii) we refer to BERGE & LAS VERGNAS [14] or BERGE [7]. Since the dual of a balanced hypergraph is trivially balanced again, a proof of (i) → (ii) is also a proof of (i) → (iii).

In fact, (iii) is equivalent to: each partial subhypergraph is conormal. So, by Theorem 10, for each partial subhypergraph H' the anti-blocker $A(H')$ is conormal, i.e.,

$$(1) \qquad \gamma(H') = \rho(A(H')) = \alpha(A(H')) = r(H').$$

So (iii) implies (iv). Since (iv) implies that each partial subhypergraph of H is conformal, also (iv) → (iii). Since (v) arises from (iv) by replacing H by its dual hypergraph, it follows that (i)-(v) are equivalent. For the equivalence of (vi) and (vii) to (i)-(v) we refer to BERGE [7]. □

A graph is balanced iff it is bipartite, so Theorem 17 can be considered as extending several theorems of KÖNIG [86,87], GUPTA [67,68] (cf. Examples 2, 5 and 16).

It follows from Theorem 17 that any balanced hypergraph is normal and conormal. The relations between some classes of hypergraphs are represented by the following diagram, where an arrow denotes implication. There are no more arrows other than those arising from making the transitive closure (cf. BERGE [7]).

(2)

```
                                   ↗ H conormal
                     ↗ H balanced ─→ H normal ───→ H seminormal ───→ H satisfies
H unimodular ─────────────────────→ H Mengerian ↗→ H Fulkersonian ↗  Thm.11(i)
```

We close this section with a rather technical theorem surveying the characterizations and interrelations given so far, in the language of matrices (cf. PADBERG [127], FULKERSON, HOFFMAN & OPPENHEIM [54]). If in vector b the entry ∞ occurs then the rows in the inequality $Mx \le b$ corresponding to ∞ do not impose any condition on x. Similarly if we minimize yb then we take any entry of y to be 0 if the corresponding entry in b is ∞.

THEOREM 18. *Let* M *be an* $m\times n$-(0,1)-*matrix.*

(a) *The following are equivalent:*

(i) M *is the incidence matrix of a unimodular hypergraph;*

(ii) $\forall b \in \mathbb{Z}_+^m, \forall w \in \mathbb{Z}_+^n$ $\min\{yb \mid y \ge 0, yM \ge w\}$ *is achieved by an integral* y;

(iii) $\forall b \in \mathbb{Z}_+^m, \forall w \in \mathbb{Z}_+^n$ $\max\{wx \mid x \ge 0, Mx \le b\}$ *is achieved by an integral* x;

(iv) $\forall b \in \mathbb{Z}_+^m, \forall w \in \mathbb{Z}_+^n$ $\max\{yb \mid y \ge 0, yM \le w\}$ *is achieved by an integral* y;

(v) $\forall b \in \mathbb{Z}_+^m, \forall w \in \mathbb{Z}_+^n$ $\min\{wx \mid x \ge 0, Mx \ge b\}$ *is achieved by an integral* x.

(b) *The following are equivalent:*

(i) M *is the incidence matrix of a balanced hypergraph;*

(ii) $\forall b\in\{1,\infty\}^m, \forall w\in\{0,1\}^n$ $\min\{yb \mid y \ge 0, yM \ge w\}$ *is achieved by an integral* y;

(iii) $\forall b\in\{1,\infty\}^m, \forall w\in\mathbb{Z}_+^n$ $\min\{yb \mid y \ge 0, yM \ge w\}$ *is achieved by an integral* y;

(iv) $\forall b\in\{1,\infty\}^m, \forall w\in\{0,1\}^n$ $\max\{wx \mid x \ge 0, Mx \le b\}$ *is achieved by an integral* x;

(v) $\forall b\in\{1,\infty\}^m, \forall w\in\mathbb{Z}_+^n$ $\max\{wx \mid x \ge 0, Mx \le b\}$ *is achieved by an integral* x;

(vi) $\forall b\in\{0,1\}^m, \forall w\in\{1,\infty\}^n$ $\max\{yb \mid y \ge 0, yM \le w\}$ *is achieved by an integral* y;

(vii) $\forall b\in\mathbb{Z}_+^m, \forall w\in\{1,\infty\}^n$ $\max\{yb \mid y \ge 0, yM \le w\}$ *is achieved by an integral* y;

(viii) $\forall b\in\{0,1\}^m, \forall w\in\{1,\infty\}^n$ $\min\{wx \mid x \ge 0, Mx \ge b\}$ *is achieved by an integral* x;

(ix) $\forall b\in\mathbb{Z}_+^m, \forall w\in\{1,\infty\}^n$ $\min\{wx \mid x \ge 0, Mx \ge b\}$ *is achieved by an integral* x.

(c) *The following are equivalent:*

(i) M *is the incidence matrix of a conormal hypergraph;*

(ii) *if* $b\equiv 1$, $\forall w \in \{0,1\}^n$ $\min\{yb \mid y \ge 0, yM \ge w\}$ *is achieved by an integral* y;

(iii) *if* $b\equiv 1$, $\forall w \in \mathbb{Z}_+^n$ $\min\{yb \mid y \ge 0, yM \ge w\}$ *is achieved by an integral* y;

(iv) *if* $b\equiv 1$, $\forall w \in \{0,1\}^n$ $\max\{wx \mid x \ge 0, Mx \le b\}$ *is achieved by an integral* x;

(v) *if* $b\equiv 1$, $\forall w \in \mathbb{Z}_+^n$ $\max\{wx \mid x \ge 0, Mx \le b\}$ *is achieved by an integral* x;

(d) *The following are equivalent:*

(i) M *is the incidence matrix of a normal hypergraph;*

(ii) $\forall b \in \{0,1\}^m$, *if* $w\equiv 1$, $\max\{yb \mid y \ge 0, yM \le w\}$ *is achieved by an integral* y;

(iii) $\forall b \in \mathbb{Z}_+^m$, *if* $w\equiv 1$, $\max\{yb \mid y \ge 0, yM \le w\}$ *is achieved by an integral* y;

(iv) $\forall b \in \{0,1\}^m$, *if* $w\equiv 1$, $\min\{wx \mid x \geq 0, Mx \geq b\}$ *is achieved by an integral* x;

(v) $\forall b \in \mathbb{Z}_+^m$, *if* $w\equiv 1$, $\min\{wx \mid x \geq 0, Mx \geq b\}$ *is achieved by an integral* x.

(e) *The following are equivalent:*

(i) M *is the incidence matrix of a Fulkersonian hypergraph;*

(ii) *if* $b\equiv 1$, $\forall w \in \mathbb{Z}_+^n$ $\min\{wx \mid x \geq 0, Mx \geq b\}$ *is achieved by an integral* x.

(f) *The following are equivalent:*

(i) M *is the incidence matrix of a Mengerian hypergraph;*

(ii) *if* $b\equiv 1$, $\forall w \in \mathbb{Z}_+^n$ $\max\{yb \mid y \geq 0, yM \leq w\}$ *is achieved by an integral* y.

(g) *The following are equivalent:*

(i) M *is the incidence matrix of a seminormal hypergraph;*

(ii) *if* $b\equiv 1$, $\forall w \in \{0,1\}^n$ $\max\{yb \mid y \geq 0, yM \leq w\}$ *is achieved by an integral* y.

4. SUBMODULAR FUNCTIONS AND NESTED FAMILIES

In this section we exhibit a method of proof designed by EDMONDS & GILES [37], based on ideas of EDMONDS [32], LOVÁSZ [105] and N. Robertson. We shall not give a general description of this method but present three instances of its employment. The first one, due to Edmonds & Giles, is based on defining a submodular function on a "crossing" family, and is applicable to network flows, matroids and directed cuts. The second one, due to FRANK [46], defines a supermodular function on a "kernel system", yielding results again for flows and directed cuts, and for arborescences. The third instance applies Edmonds & Giles' method to matchings in graphs (SCHRIJVER & SEYMOUR [141]).

4.1. Submodular functions on graphs

The results in this subsection are based on EDMONDS & GILES [37]. Let $D = (V,A)$ be a digraph. Call a collection $\mathcal{F} \subset \mathcal{P}(V)$ *crossing* if

(1) $T,U \in \mathcal{F}$, $T \cap U \neq \emptyset$, $T \cup U \neq V$ implies $T \cap U \in \mathcal{F}$ and $T \cup U \in \mathcal{F}$.

A function $f\colon \mathcal{F} \to \mathbb{Q}$ is *submodular* if

(2) $f(T) + f(U) \geq f(T\cap U) + f(T\cup U)$

whenever T, U, $T \cap U$, $T \cup U \in \mathcal{F}$.

Suppose we have a crossing family $\mathcal{F} \subset \mathcal{P}(V)$ and a submodular function f on $\mathcal{F}$. Furthermore suppose there are functions $d,b,c\colon A \to \mathbb{Q}$. Consider the following problem.

(3) What is the maximum value of cx, where x is a "flow" function defined on the arrows such that:

(i) $d \le x \le b$;

(ii) for each $T \in \mathcal{F}$ the loss of flow is at most $f(T)$, i.e., the total amount of flow going out of T, minus the total amount of flow coming into T is at most $f(T)$?

When does an integer-valued flow exists?

We remark that we do not require that in each vertex the amount of incoming flow equals the amount of outgoing flow. By taking $\mathcal{F} = \{\{v\} | v \in V\}$ and $f = 0$ problem (3) becomes a problem about this "classic" form of flow. So this is one of the problems derivable from (3) but there are more; we discuss them at the end of this subsection.

We can put problem (3) in the language of linear programming. To this end let M be the $\mathcal{F} \times A$-matrix with

(4)
$$M_{T,a} = 1, \text{ if the tail of a is in T and its head is not in T,}$$
$$M_{T,a} = -1, \text{ if the head of a is in T and its tail is not in T,}$$
$$M_{T,a} = 0, \text{ otherwise,}$$

for $T \in \mathcal{F}$ and $a \in A$. Now condition (ii) of (3) is equivalent to: $Mx \le f$. So (3) asks for

(5) $$\max\{cx | d \le x \le b,\ Mx \le f\}$$

which is, by the Duality theorem of linear programming, equal to

(6) $$\min\{zb-wd+yf | z,w \in \mathbb{Q}_+^A,\ y \in \mathbb{Q}_+^{\mathcal{F}},\ z - w + yM = c\}.$$

Now we can formulate Edmonds & Giles' result:

<u>THEOREM 19</u>. (EDMONDS & GILES [37]) *If* b, d, c *and* f *are integral then both* (5) *and* (6) *have integral solutions* x, z, w *and* y.

<u>REMARK</u>. It follows that if only b, d and f are integral then (5) has an integral solution x; if only c is integral, then (6) can be solved by integral z,w,y.

DESCRIPTION OF THE METHOD OF PROOF

A collection $\mathcal{F}'$ of subsets of V is called *cross-free* if for all $T,U \in \mathcal{F}'$:

(7) $$T \subset U, \text{ or } U \subset T, \text{ or } T \cap U = \emptyset, \text{ or } T \cup U = V.$$

By induction on $|\mathcal{F}'|$ one can prove: a collection $\mathcal{F}'$ is cross-free if and only if there exists a directed tree, with vertex set V' and arrow set A', and a function $\phi: V \to V'$, such that for each set T in $\mathcal{F}'$ there is an arrow a in the tree with the property: T consists exactly of all $v \in V$ such that the arrow a points to $\phi(v)$ (i.e., such that, if we should remove a from the tree, $\phi(v)$ is in the same component as the head of a). In fact one can make a one-to-one correspondence between $\mathcal{F}'$ and the arrows of the tree.

Call a vector $y \in \mathbb{Q}_+^{\mathcal{F}}$ *cross-free* if the collection $\{T \in \mathcal{F} | y_T > 0\}$ is cross-free.

<u>Step 1</u>. *The minimum* (6) *is achieved by some* z,w,y *where* y *is cross-free.*

<u>PROOF</u>. Let z,w,y achieve the minimum, so that

(8) $$\sum_{T \in \mathcal{F}} y_T . |T| . |V \setminus T| \text{ is as small as possible.}$$

We prove that y is cross-free. For suppose that $y_T \geq y_U > 0$, for $T,U \in \mathcal{F}$, such that $T \not\subset U \not\subset T$, $T \cap U \neq \emptyset$ and $T \cup U \neq V$. Since $\mathcal{F}$ is crossing, $T \cap U \in \mathcal{F}$ and $T \cup U \in \mathcal{F}$. Now let $y': \mathcal{F} \to \mathbb{Q}_+$ be given by

(9) $$y'_U = 0, \qquad y'_T = y_T - y_U,$$
$$y'_{T\cap U} = y_{T\cap U} + y_U, \qquad y'_{T\cup U} = y_{T\cup U} + y_U,$$

and y' coincides with y in the remaining coordinates. Straightforward checking shows that $y'f \leq yf$, $y'M = yM$ (so z,w,y' achieve the minimum (6)), and

(10) $$\sum_{T\in\mathcal{F}} y'_T . |T| . |V \setminus T| < \sum_{T \in \mathcal{F}} y_T . |T| . |V \setminus T|$$

contradicting (8). □

<u>Step 2</u>. *If* c *is integral the minimum* (6) *is attained by integral* z,w,y.

<u>PROOF</u>. Let z,w,y achieve (6) such that y is cross-free. Let M' and f' arise

from M and f by deleting rows of M and entries of f, respectively, corresponding with the 0-coordinates of y. So the rows of M' correspond to the cross-free family $F' = \{T \in F | y_T > 0\}$. Thus (6) is equal to

$$(11) \qquad \min\{zb-wd+y'f' | z,w \in \mathbb{Q}_+^E,\ y' \in \mathbb{Q}_+^{F'},\ z-w+y'M' = c\}.$$

Straightforward checking, using the definition of M, the tree representation of cross-free families and Example 17 (last paragraph), shows that M' is totally unimodular. Hence (11) can be attained by integral z,w,y'. By lengthening y' with zero-coordinates, thus getting y, we obtain an integral solution z,w,y for (6). □

Step 3. *If* c,d,b *and* f *are integral, both* (5) *and* (6) *are attained by integral* x,z,w,y.

PROOF. Since we have proved that for each integral c the minimum (6) has an integral solution, by Theorem 3 (or 4) also for each c the maximum (5) has an integral solution x. □

Theorem 19 can be restated as: *for integral* b,d *and* f *the system of linear inequalities*

$$(12) \qquad b \le x \le d,\ Mx \le f$$

is totally dual integral (cf. subsection 1.4).

The theorem of Edmonds and Giles has been extended to so-called lattice polyhedra by HOFFMAN & SCHWARTZ [80], HOFFMAN [74,75] (cf. KORNBLUM [88, 89,90]). See also JOHNSON [83].

We now give some applications of Theorem 19.

EXAMPLE 18: *Network flows.* If we take $F = \{\{v\} | v \in V\}$ and $f \equiv 0$, the equalities (5) and (6) pass to those treated in Example 17.

EXAMPLE 19: *Directed cuts.* Let D = (V,A) be a digraph. Let F be the collection of subsets V' of V such that $\emptyset \neq V' \neq V$ and no arrow leaves V'. So the sets (V\V',V'), for $V' \in F$, are exactly the directed cuts of D (Example 12). It is easy to check that F is a crossing family. Also the function $f \equiv -1$ (defined on F) is trivially submodular. Taking $b \equiv 0$, $d \equiv -\infty$ (or very small), $c \equiv 1$ Theorem 19 passes into the theorem of LUCCHESI & YOUNGER [108]: *the* *maximum number of disjoint directed cuts is equal to the minimum size of a set of arrows intersecting each directed cut* (this was proved for bipartite directed graphs by McWHIRTHER & YOUNGER [112]). For (5) = (6) changes to

(13) $$\max\{|x| \mid x \le 0,\ Mx \le -1\} = \min\{-|y| \mid y \ge 0,\ yM \le 1\}$$

i.e.,

(14) $$\min\{|x| \mid x \ge 0,\ Mx \ge 1\} = \max\{|y| \mid y \ge 0,\ yM \le 1\},$$

both sides still having integral solutions x and y. The left hand side of (14) is equal to the minimum cardinality of a set intersecting each directed cut (a diconnecting set), and the right hand side equals the maximum number of disjoint directed cuts.

EXAMPLE 20: *Matroids*. Let $(V,\mathcal{I}_1)$ and $(V,\mathcal{I}_2)$ be matroids, with rank-functions r_1 and r_2, respectively. The theorem of Edmonds & Giles can be used to prove EDMONDS' intersection theorem [32] (cf. TUTTE [164]) giving the maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$. This can be done as follows.

Let V_1 and V_2 be disjoint copies of V, and make a digraph D with vertex set $V_1 \cup V_2$ by drawing an arrow from any point in V_1 to its corresponding point in V_2. Let $\mathcal{F}$ be the collection

(15) $$\mathcal{F} = \{V_1' \mid V_1' \subset V_1\} \cup \{V_1 \cup V_2' \mid V_2' \subset V_2\},$$

which is crossing. Let $f\colon \mathcal{F} \to \mathbb{Z}_+$ be given by

(16) $$\begin{aligned} f(V_1') &= r_1(V_1'), && \text{for } V_1' \subset V_1, \\ f(V_1 \cup V_2') &= r_2(V_2 \setminus V_2'), && \text{for } V_2' \subset V_2 \end{aligned}$$

(losing no generality we assume that $r_1(V_1) = r_2(V_2)$). Then f is submodular (this can be derived from the well-known submodularity of r_1 and r_2). Now let $c \equiv 1$, $d \equiv 0$, and $b \equiv 1$. Then (5) becomes

(17) $$\max\{|x| \mid 0 \le x \le 1,\ Mx \le f\}$$

and, since an integral solution x exists, this is the maximum cardinality of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$. Expression (6) equals

(18) $\min\{|z| + yf \mid z,y \geq 0,\ z+yM \geq 1\}$.

This is (again since (6) has integral solutions) the minimum value of

(19) $|V_0| + r_1(V_1^1) + \ldots + r_1(V_1^k) + r_2(V_2^1) + \ldots + r_2(V_2^\ell)$

such that $V = V_0 \cup V_1^1 \cup \ldots \cup V_1^k \cup V_2^1 \cup \ldots \cup V_2^\ell$. But always $r_1(V_0) \leq |V_0|$, $r_1(V_1^1) + \ldots + r_1(V_1^k) \geq r_1(V_1^1 \cup \ldots \cup V_1^k)$ and $r_2(V_2^1) + \ldots + r_2(V_2^\ell) \geq r_1(V_2^1 \cup \ldots \cup V_2^\ell)$, hence the minimum value of (19) is equal to the minimum value of $r_1(V') + r_2(V'')$, where V', V'' partitions V. So *Edmonds' matroid intersection theorem* can be derived: *the maximum cardinality of a common independent set is equal to*

(20) $\min_{V' \subset V} (r_1(V') + r_2(V \backslash V'))$.

Of course, by taking c arbitrary, the Edmonds-Giles theorem gives the maximum weight of a common independent set as well (cf. EDMONDS [32,33], LAWLER [92]). A corollary is that the intersection of the convex hulls P_1 and P_2 of all characteristic vectors of independent sets in I_1 and I_2, respectively, only has integral vertices. Also results on "polymatroids" are derivable - see EDMONDS & GILES [37]. (For other extensions of Edmonds' matroid intersection theorem see CUNNINGHAM [23] and McDIARMID [111] (proving a conjecture of FULKERSON [50], cf. WEINBERGER [170,171]).)

4.2. Kernel systems on directed graphs

A second framework for proving min-max theorems, having many features in common with the proof method described above but with a number of different applications, has been drawn up by FRANK [46].

Let $D = (V,A)$ be a directed graph, with a fixed vertex r, called the *root*. For subsets U of V, the *indegree* $\rho(U)$ and *outdegree* $\delta(U)$ of U is the number of arrows entering U and leaving U, respectively. A collection F of subsets of $V \backslash \{r\}$ is called a *kernel system* with respect to D if

(1) (i) $\rho(U) > 0$ for all $U \in F$, and
(ii) if $T,U \in F$ and $T \cap U \neq \emptyset$, then $T \cap U \in F$ and $T \cup U \in F$.

A function $f\colon F \to \mathbb{Q}_+$ is *supermodular* if

(2) $f(T) + f(U) \leq f(T \cap U) + f(T \cup U)$

whenever $T, U \in \mathcal{F}$ and $T \cap U \neq \emptyset$.

Suppose we have a kernel system $\mathcal{F}$ and a supermodular function f on $\mathcal{F}$. Furthermore suppose there is a function $c\colon A \to \mathbb{Q}_+$. Consider the problem:

(3) What is the minimum value of cx for a "flow" $x\colon A \to \mathbb{Q}_+$ such that, for each $T \in \mathcal{F}$, the total amount of flow coming into T is at least f(T)?
When does an integral optimal flow exist?

Again, we delay the discussion of particular instances of this problem until the end of this subsection.

First we put the problem in the language of linear programming. Let M be the $\mathcal{F} \times A$-matrix with

(4)
$M_{T,a} = 1$, if the head of a is in T and its tail is not in T.
$M_{T,a} = 0$, otherwise,

for $T \in \mathcal{F}$ and $a \in A$. The condition mentioned in (3) is equivalent to: $Mx \geq f$. So (3) asks for

(5) $\min\{cx \mid x \geq 0,\ Mx \geq f\}$

which is, by the Duality theorem of linear programming, equal to

(6) $\max\{yf \mid y \in \mathbb{Q}_+^{\mathcal{F}},\ yM \leq c\}$.

If y is integral and $yM \leq c$, y can be interpreted as a subcollection $\mathcal{F}'$ of $\mathcal{F}$, possibly taking sets repeatedly, such that no arrow a enters more than c(a) of sets in $\mathcal{F}'$.

Now Frank's theorem is:

THEOREM 20. (FRANK [46]) *If* c *and* f *are integral then both* (5) *and* (6) *are achieved by integral* x *and* y.

DESCRIPTION OF THE METHOD OF PROOF

Call a collection F' of subsets of $V\setminus\{r\}$ *laminar* if, for all $T,U \in F'$, $T \subset U$, or $U \subset T$, or $T \cap U = \emptyset$. Laminar collections again have a nice, tree-like structure; their Venn-diagram is "planar". Laminar collections can be split up into *levels*. The first level consists of all maximal (with respect to inclusion) sets in F'; the (i+1)-th level consists of all maximal sets in F' properly contained in some set of the i-th level. Each level consists of pairwise disjoint sets.

Each laminar collection, being cross-free (subsection 4.1), has a *tree-representation* by a directed tree; this tree can be taken to be rooted, i.e., the tree contains a vertex from which directed paths are going to any other vertex of the tree.

A vector $y \in \mathbb{Q}_+^F$ is called *laminar* if the collection $F' = \{T \in F | y_T > 0\}$ is laminar.

Step 1. *The maximum* (6) *is achieved by some laminar* y.

PROOF. Let y achieve the maximum (6) such that

(7) $$\sum_{T\in F} y_T \cdot |T| \cdot |V\setminus T| \text{ is as small as possible.}$$

Suppose y is not laminar, and let $T,U \in F$ be such that $y_T \geq y_U > 0$, $T \cap U \neq \emptyset$, and $T \not\subset U \not\subset T$. Now let

(8) $$\begin{aligned} y'_U &= 0, & y'_T &= y_T - y_U, \\ y'_{T\cap U} &= y_{T\cap U} + y_U, & y'_{T\cup U} &= y_{T\cup U} + y_U, \end{aligned}$$

and let y' coincide with y in the remaining coordinates. Straightforward checking shows that $y'f \geq yf$, $y'M = yM$ (so y' achieves the maximum (6)) and

(9) $$\sum_{T\in F} y'_T \cdot |T| \cdot |V\setminus T| < \sum_{T\in F} y_T \cdot |T| \cdot |V\setminus T|$$

contradicting our assumption (7). □

Step 2. *If* c *is integral the maximum* (6) *is achieved by an integral* y.

PROOF. Let y achieve the maximum (6) such that y is laminar. Let $F' = \{T \in F | y_T > 0\}$ and let M' and f' arise from M and f by deleting rows and entries corresponding with positions whose index is not in F'. So (6) is equal to

(10) $\max\{y'f' \mid y' \in \mathbb{Q}_+^{F'}, y'M' \le c\}.$

Straightforward checking, using the definition of M, the (rooted) tree-representation of F' and the last paragraph of Example 17, shows that M' is totally unimodular; hence (10) is achieved by some integral y'. By lengthening y' with zero-coordinates we obtain an integral solution y for (6). □

Step 3. *If* c *and* f *are integral then both* (5) *and* (6) *are achieved by integral* x *and* y.

PROOF. Since for each integral c the maximum (6) has an integral solution, by Theorem 3 (or 4), also the minimum (5) has an integral solution x, if f is integral. □

So Frank's theorem says: *if* f *is integer-valued then the system of linear inequalities*

(11) $x \ge 0,\ Mx \ge f$

is totally dual integral (cf. subsection 1.4).

Before giving applications of Frank's theorem we mention a second theorem of Frank. Let be given a digraph $D = (V,A)$, with fixed root r, and a kernel system $F \subset P(V\backslash\{r\})$. Call a subset $A' \subset A$ *k-entering* if for each $T \in F$ there are at least k arrows in A' entering T.

THEOREM 21. (FRANK [46]) *A subset* A' *of* A *is* k*-entering iff* A' *is the disjoint union of* k 1*-enterings*.

For a proof we refer to [46]. We can translate this theorem in the language of hypergraphs by defining the hypergraph $H = (A,E)$, where E consists of all sets $(V\backslash T,T)$, for $T \in E$ (as usual, $(V\backslash V',V')$ denotes the set of arrows entering V'). By taking $c \equiv 1$ and $f \equiv 1$ in Theorem 20 one sees that $\tau(H) = \nu(H)$, or, more generally, that $\tau(H^w) = \nu(H^w)$ for all $w\colon A \to \mathbb{Z}_+$ (by taking $c = w$). So H is Mengerian. Let K be the blocker of H; so the edges of K are the 1-entering sets of arrows. From Theorem 14 it follows that Theorem 21 is equivalent to: K is Mengerian. In particular, $\tau(K) = \nu(K)$.

We now apply Theorems 20 and 21 to some examples.

EXAMPLE 21: *Network flows.* Let $D = (V,A)$ be a digraph, with fixed vertices r and s, such that an r-s-path exists. Let F be the collection of all subsets of $V\setminus\{r\}$ containing s. So F is a kernel system, with root r. It is easy to see that Theorem 21 applied to this kernel system gives us Menger's theorem.

EXAMPLE 22: *Arborescences.* Let $D = (V,A)$ be a digraph, with root r, having at least one r-arborescence. Now let $F = P(V\setminus\{r\})\setminus\{\phi\}$. Then Theorem 21 applied to this kernel system is equivalent to *Edmonds' arborescence* or *branching theorem* [34] (cf. LOVÁSZ [105]): *the maximum number of pairwise edge-disjoint* r*-arborescences is equal to the minimum indegree of sets in* F. For let H and K be as described after Theorem 21, then K has, as edges, all r-arborescences; hence $\tau(K) = \nu(K)$, which is the content of Edmonds' theorem (see VIDYASANKAR [166] for a covering analogue).

By taking $f \equiv 1$ Theorem 20 passes into: given a "weight" function c, defined on the arrows, *the minimum weight of an* r*-arborescence is equal to the maximum number* ℓ *of nonempty sets* $V_1,\ldots,V_\ell \subset V\setminus\{r\}$, *such that each arrow* a *enters at most* c(a) *of these sets*, that is, H is Mengerian (this is a result of FULKERSON [52], cf. LOVÁSZ [106]).

EXAMPLE 23: *Directed cuts.* Let $D = (V,A)$ be a directed graph, with root r, having an r-arborescence. Let F be the collection of all nonempty subsets of $V\setminus\{r\}$ having zero outdegree. So the edges of the hypergraph H, as described after Theorem 21 are all directed cuts. Theorem 21 implies a conjecture of EDMONDS & GILES [37] (cf. Example 12) that *the minimum size of a directed cut is equal to the maximum number of pairwise arrow-disjoint diconnecting sets* (this follows also from Edmonds' branching theorem).

4.3. Matchings in graphs

Finally we apply Edmonds-Giles-like techniques to prove total dual integrality for some linear inequalities derived from matchings in graphs. This was proved for the first time by CUNNINGHAM & MARSH [24] (cf. HOFFMAN & OPPENHEIM [78]); the present proof method is taken from SCHRIJVER & SEYMOUR [141]. We omit many technical details which are straightforward to check. Let $G = (V,E)$ be an undirected graph. A famous theorem of TUTTE [158] (cf. LOVÁSZ [101], see EDMONDS [27] and WITZGALL & ZAHN [174] for algorithms) asserts the following.

(1) *G has a 1-factor if and only if for each subset V' of V the number of odd components of <V\V'> does not exceed |V'|.*

[Here <V\V'> is the subgraph of G induced by V\V', and an *odd component* is a component having an odd number of vertices. A *1-factor* is a collection of pairwise disjoint edges covering all points.]
This theorem has turned out to be fundamental for subsequent investigations in matching theory. [A *matching* is a collection of pairwise disjoint edges.] For example, by adding new vertices one can deduce the following theorem of BERGE [2] (cf. ANDERSON [1]).

(2) *The maximum cardinality of a matching in* G (i.e., $\nu(G)$) *equals*

$$\min_{V' \subset V} \frac{|V|+|V'|-\mathcal{O}(V\backslash V')}{2}.$$

[In this formula $\mathcal{O}(V\backslash V')$ denotes the number of odd components of <V\V'>.]
This result is known as the *Tutte-Berge theorem*.

Much research has been done on matching theory by J. Edmonds and his co-workers (cf. EDMONDS [27,30], EDMONDS, JOHNSON & LOCKHART [40], EDMONDS & PULLEYBLANK [41], PULLEYBLANK & EDMONDS [134], PULLEYBLANK [133]). EDMONDS [30] studied maximum *weighted* matchings, and he gave a good algorithm for finding one (given a weigthing of the edges). An interesting theoretical byproduct is his *matching polyhedron theorem:*

(3) *A vector* $g \in \mathbb{Q}_+^E$ *is expressible as a convex combination of (characteristic vectors of) matchings if and only if*
(i) $\sum_{e \ni v} g(e) \le 1$, *for each vertex* v, *and*
(ii) $\sum_{e \subset V'} g(e) \le \lfloor \frac{1}{2}|V'| \rfloor$ *for each subset* V' *of* V.

Clearly, the inequalities (i) and (ii) are satisfied by any convex combination of matchings, since each matching itself satisfies them - the content of the theorem is the converse. Edmonds' theorem gives the faces of the convex hull of the matchings; it may be considered as an extension of the characterization of Birkhoff and Von Neumann (Example 5).

We can restate (3) in matrix terminology. Let M be the V×E-*incidence-matrix* of G, i.e., $M_{v,e} = 1$ if $v \in e$, and $M_{v,e} = 0$ if $v \notin e$, for $v \in V$, $e \in E$.

Define the $\mathcal{P}(V) \times E$-matrix N by $N_{V',e} = 1$ if $e \subset V'$, and $N_{V'e} = 0$, if $e \not\subset V'$, for $e \in E'$, $V' \subset V$. So the rows of N are the collections of edges of induced subgraphs of G. The function $f\colon \mathcal{P}(V) \to \mathbb{Q}_+$ is defined by $f(V') = f_{V'} = \lfloor \frac{1}{2}|V'| \rfloor$, for $V' \subset V$. Now (3) says that the convex hull P of the collection of matchings equals

$$(4) \qquad P = \{x \geq 0 \mid Mx \leq 1,\ Nx \leq f\}.$$

Since the matchings are the extreme points of P we have that the maximum weight of a matching equals

$$(5) \qquad \max\{wx \mid x \in \mathbb{Z}_+^E,\ Mx \leq 1,\ Nx \leq f\} = \max\{wx \mid x \in \mathbb{Q}_+^E,\ Mx \leq 1,\ Nx \leq f\}$$

for any "weight" function $w\colon E \to \mathbb{Q}$.

The left hand side of (5) is the maximum weight of a matching; the Duality theorem of linear programming is applicable to the right hand side, yielding

$$(6) \qquad \max\{wx \mid x \geq 0,\ Mx \leq 1,\ Nx \leq f\} = \min\{|y| + tf \mid y \geq 0,\ t \geq 0,\ yM+tN \geq w\}.$$

For the case $w \equiv 1$ we have, by the Tutte-Berge theorem (2), a stronger result since (2) may be formulated as

$$(7) \qquad \max\{|x| \mid x \in \mathbb{Z}_+^E,\ Mx \leq 1,\ Nx \leq f\} = \min\{|y| + tf \mid y \in \mathbb{Z}_+^V,\ t \in \mathbb{Z}_+^{\mathcal{P}(V)},\ yM+tN \geq 1\},$$

that is, also the minimum in (6) is achieved by an integral solution y,t. We shall show here that this is true for *each* integer-valued weight function w, i.e.

<u>THEOREM 22</u>. (CUNNINGHAM & MARSH [24], cf. SCHRIJVER & SEYMOUR [142]) *Both sides of the linear programming duality equality* (6) *are achieved by integral* x,y,t *if* w *is integral.*

As already mentioned, (1), (2) and (3) follow from this. Theorem 22 is equivalent to: *the system of linear inequalities*

$$(8) \qquad x \geq 0,\ Mx \leq 1,\ Nx \leq f$$

is totally dual integral (cf. subsection 1.4).

DESCRIPTION OF THE METHOD OF PROOF

Again we use the terminology of laminar subcollections $\mathcal{F}$ of $\mathcal{P}(V)$ and laminar vectors in $\mathbb{Q}_+^{\mathcal{P}(V)}$ (cf. subsection 4.2).

Step 1. *For each* $w \in \mathbb{Z}^E$

(9) $$\min\{|y| + tf \mid y \in \mathbb{Z}_+^V,\ t \in \mathbb{Z}_+^{\mathcal{P}(V)},\ yM+tN \geq w\}$$

is achieved by some y,t, *where* t *is laminar.*

PROOF. Let $w \in \mathbb{Z}^E$, and choose $y \in \mathbb{Z}_+^V$, $t \in \mathbb{Z}_+^{\mathcal{P}(V)}$ such that y and t attain the minimum in (9) and such that

(10) $$\sum_{U\subset V} t_U \cdot |U| \cdot (|V\backslash U| + 1) \text{ is as small as possible.}$$

We prove that t is laminar. Suppose t is not laminar, and let $t_T \geq t_U > 0$, with $T \nsubseteq U \nsubseteq T$ and $T \cap U \neq \emptyset$.

First suppose $|T \cap U|$ is odd. Define

(11) $$t'_U = 0, \qquad t'_T = t_T - t_U,$$
$$t'_{T\cap U} = t_{T\cap U} + t_U, \quad t'_{T\cup U} = t_{T\cup U} + t_U,$$

and let t' be equal to t in the remaining coordinates, i.e.,

(12) $$t' = t + t_U\{T\cap U, T\cup U\} - t_U\{T,U\},$$

identifying subsets of $\mathcal{P}(V)$ with their characteristic vectors in $\mathbb{Q}^{\mathcal{P}(V)}$. It can be checked straightforwardly that $|y|+t'f \leq |y|+tf$ and $yM+t'N \geq yM+tN$, so y,t' achieves the minimum (9), and

(13) $$\sum_{U\subset V} t'_U \cdot |U| \cdot (|V\backslash U|+1) < \sum_{U\subset V} t_U \cdot |U| \cdot (|V\backslash U|+1),$$

contradicting (10).

Secondly assume that $|T \cap U|$ is even. Let

(14) $$y' = y + t_U \cdot (T \cap U),$$
$$t' = t + t_U\{T\backslash U, U\backslash T\} - t_U\{T,U\},$$

again identifying characteristic vectors and subsets. Now we have that $|y'|+t'f \le |y|+ tf$, $y'M+t'N \ge yM+tN$, so y',t' achieves the minimum (6), and, furthermore, (13) holds for this t', again contradicting (10). □

Step 2. *For each* $w \in \mathbb{Z}^E$

(15) $$\min\{|y|+tf \mid y \in \tfrac{1}{2}\mathbb{Z}_+^V,\ t \in \tfrac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)},\ yM+tN \ge w\}$$

is attained by integral y *and* t.

PROOF. Since M and N are nonnegative we need to consider only $w \in \mathbb{Z}_+^E$. Suppose (15) is not attained by an integral solution y,t, and let $w \in \mathbb{Z}_+^E$ be a fixed counterexample to this, such that $|w|$ is as small as possible. Then each $y \in \tfrac{1}{2}\mathbb{Z}_+^V$, $t \in \tfrac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)}$ attaining the minimum (15) is such that $y \in \{0,\tfrac{1}{2}\}^V$ and $t \in \{0,\tfrac{1}{2}\}^{\mathcal{P}(V)}$, except, possibly, the (inessential) t-values on singletons and the empty set. If this were not the case, there would exist, as can be seen easily, a counterexample w' with $|w'| < |w|$.

Since (15) is equal to

(16) $$\tfrac{1}{2}\min\{|y|+tf \mid y \in \mathbb{Z}_+^V,\ t \in \mathbb{Z}_+^{\mathcal{P}(V)},\ yM+tN \ge 2w\}$$

it follows from step 1 that (15) is attained by some half-integer-valued y,t, where t is laminar. We may assume that t equals zero on singletons and the empty set. We may also assume that y and t are chosen such that $|y|$ is as large as possible, under the condition that t is laminar.

Now we define the laminar collection

(17) $$\mathcal{F} = \{U \subset V \mid t_U = \tfrac{1}{2}\},$$

and let

(18) $$S = \{v \in V \mid y_v = \tfrac{1}{2}\}.$$

First suppose $\mathcal{F} = \emptyset$, i.e., $t \equiv 0$. Define $y' \equiv 0, t' \equiv \{S\}$. It can be checked easily that

(19) $$|y'| + t'f \le |y| + tf,$$ $$y'M + t'N \ge \lfloor yM + tN \rfloor \ge w,$$

(vector $\lfloor u \rfloor$ arises from vector u by taking coordinate-wise lower integer parts) so y',t' reaches the minimum in (15); this contradicts our assumption that for this w there are no integral y,t attaining (15).

If $\mathcal{F} \neq \emptyset$, there are sets on an odd level of the laminar collection $\mathcal{F}$; let U be a minimal set (under inclusion) in $\mathcal{F}$ on an odd level, i.e., U is a minimal set such that $|\{T \in \mathcal{F} | U \subset T\}|$ is odd. Let $T_1,\ldots,T_k$ be the sets in $\mathcal{F}$ properly contained in U (possibly $k = 0$). So $T_1,\ldots,T_k$ are pairwise disjoint. It is easy to see that either

$$(20)\qquad \lfloor \tfrac{1}{2}|U| \rfloor + \lfloor \tfrac{1}{2}|T_1| \rfloor + \ldots + \lfloor \tfrac{1}{2}|T_k| \rfloor \geq |U \cap S| + 2(\lfloor \tfrac{1}{2}|T_1 \setminus S| \rfloor + \ldots + \lfloor \tfrac{1}{2}|T_k \setminus S| \rfloor)$$

or

$$(21)\qquad \lfloor \tfrac{1}{2}|U| \rfloor + \lfloor \tfrac{1}{2}|T_1| \rfloor + \ldots + \lfloor \tfrac{1}{2}|T_k| \rfloor \geq |U \setminus S| + 2(\lfloor \tfrac{1}{2}|T_1 \cap S| \rfloor + \ldots + \lfloor \tfrac{1}{2}|T_k \cap S| \rfloor).$$

If (20) is true, let

$$(22)\qquad \begin{aligned} y' &= y + \tfrac{1}{2}(U \cap S), \\ t' &= t - \tfrac{1}{2}\{U,T_1,\ldots,T_k\} + \{T_1 \setminus S,\ldots,T_k \setminus S\}. \end{aligned}$$

Since, as can be checked straightforwardly,

$$(23)\qquad \begin{aligned} |y'| + t'f &\leq |y| + tf, \\ y'M + t'N &\geq \lfloor yM + tN \rfloor \geq w, \end{aligned}$$

y',t' reaches the minimum (15). Hence y',t' are $\{0,\tfrac{1}{2}\}$-valued which implies that the right hand side of (20) equals zero. Since the left hand side of (20) is not zero this yields a strict inequality in the first line of (23), contradicting the minimality of $|y| + tf$.

Similarly we can deal with the case that (21) holds. Now let

$$(24)\qquad \begin{aligned} y' &= y + \tfrac{1}{2}(U \setminus S), \\ t' &= t - \tfrac{1}{2}\{U,T_1,\ldots,T_k\} + \{T_1 \cap S,\ldots,T_k \cap S\}. \end{aligned}$$

Again, for this y',t', (23) holds. Since t' is laminar we have that $|y'| \leq |y|$; moreover t' is $\{0,\tfrac{1}{2}\}$-valued. Hence the right hand side of (21) equals zero. This leads to a contradiction in the same way as before. □

<u>Step 3</u>. *Both sides of the linear programming duality equality* (6) *are attained by integral* x,y,t, *if* w *is integral.*

PROOF. This follows directly from step 2 and Theorem 4. □

As already mentioned a corollary of Theorem 22 is that any vector $x \in \mathbb{Q}_+^E$ is a convex combination of matchings if $Mx \le 1$ and $Nx \le f$. Let N' be the matrix arising from N by dividing any arrow with index U by $\lfloor \frac{1}{2}|U| \rfloor = f(U)$ (deleting the row if this number is zero). So the convex hull of matchings in G is equal to the polyhedron

(25) $$P = \{x \ge 0 \mid Mx \le 1,\ N'x \le 1\}.$$

The anti-blocking polyhedron R of P can be described as

(26) $$R = \{z \ge 0 \mid Lz \le 1\}$$

where L is a matrix whose rows are the characteristic vectors of matchings. By the theory of anti-blocking polyhedra R consists of all vectors $z \le c$ for some convex combination c of row vectors of M and N'. So

(27) $$\max\{|z| \mid z \ge 0,\ Lz \le 1\} = \max\{\Delta(G),\ \max_{U\subset V} \frac{\text{number of edges in } \langle U\rangle}{\lfloor \frac{1}{2}|U| \rfloor}\}$$

where $\Delta(G)$ is the maximum valency of G. By the Duality theorem of linear programming (27) equals

(28) $$\min\{|y| \mid y \ge 0,\ yL \ge 1\}.$$

If this minimum has an integral solution y then (28) can be interpreted as the minimum number $\chi(G)$ of colours needed to colour the edges of G such that no two edges of the same colour intersect each other. However, the Petersen-graph shows that (28) does not always have an integral solution y. The value of (28) can be interpreted as the *"fractional edge-colouring number"* $\chi^*(G)$ of G; so (27) and (28) together yield a min-max relation for $\chi^*(G)$. Note that, if G is simple, then $\chi(G) = \Delta(G)$ or $\chi(G) = \Delta(G)+1$, following a theorem of VIZING [168] and GUPTA [66]. (See SEYMOUR [146] for results relating matchings and edge-colouring to T-joins (Example 14 (ii)) and the Chinese postman problem.)

GALLAI's theorem [56,57] (cf. Example 11) says that $\nu(G)+\rho(G) = |V|$, for any graph G. Together with the Tutte-Berge theorem (2) this implies that

(29) $$\rho(G) = \max_{U\subset V} \frac{\mathcal{O}(U)+|U|}{2}.$$

Also a covering analogue of Edmonds' matching polyhedron theorem (3) can be proved: *for a vector* $g \in \mathbb{Q}_+^E$ *we have that* $g \geq c$ *for some convex combination* c *of* (characteristic vectors of) *edge sets covering all points, if and only if*

(30) $$\sum_{e \cap U \neq \emptyset} g(e) \geq \lceil \tfrac{1}{2}|U| \rceil, \text{ for each subset } U \text{ of } V.$$

More generally, it can be proved (in a way similar to the above proof of Theorem 22) that the system of linear inequalities (30) is totally dual integral.

This method of proof may also be extended to get results about *f-factors*, i.e. subgraphs such that the vertices v have prescribed valencies f(v) (cf. TUTTE [159,161], ORE [125,126], LOVÁSZ [97] and LAS VERGNAS [91]), and to get results about subgraphs whose valencies obey prescribed upper and lower bounds (cf. SCHRIJVER & SEYMOUR [141]).

The *"matroid parity problem"*, posed by LAWLER (cf. [93]), generalizes both the matching problem and the matroid intersection theorem: given a graph G = (V,E) and a matroid M = (V,$\mathcal{I}$), what is the maximum number of pairwise disjoint edges whose union is an independent set in the matroid? LOVÁSZ [107] recently gave an answer in case M is linear (i.e., $\mathcal{I}$ consists of the linear independent subsets of a vector space).

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[1]) These conjectures of Seymour and of Edmonds and Giles have been disproved by the example given in: A. SCHRIJVER, *A counterexample to a conjecture of Edmonds and Giles,* Discrete Math. 32 (1980) 213-214.