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UNIFORM HYPERGRAPHS

A.E. BROUWER & A. SCHRIJVER

INTRODUCTION

Let X be a fixed n-set (an *n-set* is a set having n elements). Consider the set $P_k(X)$ consisting of all k-subsets of X. There are various problems of a "packing & covering"-nature presented by the set $P_k(X)$. In this chapter we shall deal with some of them, mainly grouped around the following four questions:

1. What is the maximum number of pairwise disjoint sets in $P_k(X)$?
2. What is the maximum number of pairwise intersecting sets in $P_k(X)$?
3. What is the minimum number of classes into which $P_k(X)$ can be split up such that any two sets in any class are disjoint?
4. What is the minimum number of classes into which $P_k(X)$ can be split up such that any two sets in any class intersect?

We shall first give, briefly, the answers to these questions; they are treated more extensively in the Sections 1-4. To streamline the answers we assume, for the moment, that n is at least 2k (for smaller n the questions are not difficult).

The answer to the first problem is trivially $\lfloor\frac{n}{k}\rfloor$ ($\lfloor x\rfloor$ and $\lceil x\rceil$ denote the lower and upper integer part of a real number x, respectively).

The answer to the second question is easily seen to be at least $\binom{n-1}{k-1}$: take all k-subsets containing a fixed element of X. The content of the Erdös-Ko-Rado theorem (1961) is that one cannot have more: $\binom{n-1}{k-1}$ is indeed the answer to question 2.

The answer to the third question must be at least

$$(1)\qquad \lceil \binom{n}{k}/\lfloor\frac{n}{k}\rfloor\rceil$$

since each of the classes partitioning the $\binom{n}{k}$ elements of $P_k(X)$ contains at most $\lfloor n/k \rfloor$ elements. In 1973 Baranyai proved that indeed $\mathcal{P}_k(X)$ can be split into this many classes each consisting of pairwise disjoint sets. This is particularly interesting in case n is a multiple of k: then this splitting yields $\binom{n-1}{k-1}$ partitions of X, containing each k-subset exactly once.

In a similar manner we have that the answer to question 4 must be at least

$$(2) \qquad \left\lceil \binom{n}{k} / \binom{n-1}{k-1} \right\rceil = \left\lceil \frac{n}{k} \right\rceil .$$

An upper bound for the answer is given by the following construction (where we may suppose, without loss of generality, that $X = \{1,\dots,n\}$): let K_i be the collection of k-subsets of X whose smallest element is i $(i = 1,\dots,n)$; then

$$(3) \qquad K_1, K_2, \dots, K_{n-2k+1}, \; K_{n-2k+2} \cup \dots \cup K_n$$

are n-2k+2 classes of pairwise intersecting k-subsets of X, with union $\mathcal{P}_k(X)$. So the answer to problem 4 is at most n-2k+2. Kneser conjectured in 1955 that n-2k+2 indeed is the answer; in 1977 Lovász was able to prove this conjecture, using homotopy theory and topology of the sphere.

We may set the problems described above in the language of graphs. The graph K(n,k), usually called a *Kneser-graph*, has, by definition, the set $\mathcal{P}_k(X)$ as vertex set, two vertices being adjacent iff they are disjoint (as k-subsets). Now let, for any graph G, $\alpha(G)$, $\omega(G)$ and $\gamma(G)$ be its stability number, clique number and colouring number, respectively. In Chapter 1 we saw that

$$(4) \qquad \omega(G) = \alpha(\bar{G}),\ \omega(G) \le \gamma(G) \quad \text{and} \quad \frac{v}{\alpha(G)} \le \gamma(G),$$

where v is the number of vertices of G. The solutions to the problems 1-4 above may be translated as follows.

1. $\alpha(\overline{K(n,k)}) = \lfloor n/k \rfloor$,
2. $\alpha(K(n,k)) = \binom{n-1}{k-1}$,
3. $\gamma(\overline{K(n,k)}) = \left\lceil \binom{n}{k} / \lfloor \frac{n}{k} \rfloor \right\rceil$,
4. $\gamma(K(n,k)) = n-2k+2$.

In particular, if k divides n, the inequalities in (4), for $G = \overline{K(n,k)}$

become equalities.

In this chapter we shall discuss the above mentioned and related problems. In Sections 1,2,3 and 4 we go further into the problems 1,2,3 and 4, respectively.

1. COLLECTIONS OF PAIRWISE DISJOINT SETS

Let n and k be natural numbers such that $k \le n$. Let X be an n-set. In this section we consider problems asking for the maximum size of collections of disjoint or "almost" disjoint sets in $P_k(X)$, and in some derived collections. The first question to arise is easy to answer: what is the maximum number of pairwise disjoint sets in $P_k(X)$? Answer: $\lfloor \frac{n}{k} \rfloor$. However, this question has some more difficult and more interesting generalizations.

Our first generalization is to investigate the maximum number $D(t,k,n)$ of k-subsets of X such that no two of them intersect in t or more elements. So $D(1,k,n) = \lfloor n/k \rfloor$. The problem of determining $D(t,k,n)$ is a genuine packing problem: $D(t,k,n)$ is the maximum number of pairwise disjoint sets $P_t(Y)$ for $Y \in P_k(X)$. Its covering counterpart is the problem of determining the minimum number $C(t,k,n)$ of k-subsets of X such that each t-subset is contained in at least one of them. So $C(t,k,n)$ is the minimum number of collections $P_t(Y)$ (for $Y \in P_k(X)$) covering the collection $P_t(X)$.

It is easy to see that $D(t,k,n) = C(t,k,n)$ *if and only if there exists a Steiner system* $S(t,k,n)$ (i.e., a collection of k-subsets of X such that each t-subset is in *exactly* one of them).

The investigations into the functions $C(t,k,n)$ and $D(t,k,n)$, and their design-theoretical aspects have assumed such large proportions that they will be dealt with in Chapter 5 ("The Wilson theory") and 6 ("Packing and covering of $\binom{k}{t}$-sets"). In Chapter 6, when considering $C(t,k,n)$-problems, t and k are assumed to be fixed, while the behaviour of $C(t,k,n)$ as a function of n is viewed. Now $C(n-\ell,n-k,n)$ is the minimum number of (n-k)-subsets of X covering each $(n-\ell)$-subset. Passing to complements, one can view this as Turán's problem: what is the minimum number $T(n,k,\ell)$ of k-subsets of X such that each ℓ-subset contains one of them as a subset? So

$$(1) \qquad C(n-\ell,n-k,n) = T(n,k,\ell).$$

The distinction between the investigations into C and into T does not rest on any analytical basis but is simply a difference in approach: $T(n,k,\ell)$ will be considered mainly as a function of n (fixing k and ℓ).

We may view the problems of determining $D(2,k,n)$, $C(2,k,n)$ and $T(n,2,\ell)$ as graph-theoretical problems: $D(2,k,n)$ is the maximum number of pairwise edge-disjoint complete graphs K_k in K_n; $C(2,k,n)$ is the minimum number of complete subgraphs K_k in K_n covering all edges of K_n; and $T(n,2,\ell)$ is the minimum number of edges in a graph on n vertices containing no ℓ pairwise nonadjacent points. So $\binom{n}{2} - T(n,2,\ell)$ is the maximum number of edges in a graph on n vertices containing no clique of size ℓ.

The Turán-like problems will be considered more extensively in Chapter 7 ("Turán theory and the Lotto problem").

Now look at a second generalization of our main problem. Call a subset $Y_1 \times \ldots \times Y_d$ of $X \times \ldots \times X = X^d$ a k-*hypercube* if $|Y_1| = \ldots = |Y_d| = k$. Now we may ask for the maximum number $H(d,k,n)$ of pairwise disjoint k-hypercubes in X^d. So $H(1,k,n) = \lfloor n/k \rfloor$ and $H(d,k,n) = 1$ if $k > \frac{1}{2}n$. Furthermore

PROPOSITION 1. $H(d+1,k,n) \le \lfloor \frac{n}{k} \cdot H(d,k,n) \rfloor$.

PROOF. Suppose there are h pairwise disjoint k-hypercubes in X^{d+1}. The number of points contained in the union of these k-hypercubes equals $h.k^{d+1}$. For any $x \in X$, the number of points contained in $X^d \times \{x\}$ is at most $k^d.H(d,k,n)$. So the total number $h.k^{d+1}$ is at most $n.k^d.H(d,k,n)$, which implies that $h \le \lfloor \frac{n}{k}.H(d,k,n) \rfloor$. □

COROLLARY 2. $H(d,k,n) \le \underbrace{\lfloor \frac{n}{k} \lfloor \frac{n}{k} \ldots \lfloor \frac{n}{k}}_{d \text{ times}} \rfloor \ldots \rfloor \rfloor$.

By a straightforward construction one sees that, if k divides n, $H(d,k,n) = (\frac{k}{n})^d$, so in those cases the inequality passes into equality. This happens also if $d = 2$.

THEOREM 2. $H(2,k,n) = \lfloor \frac{n}{k} \lfloor \frac{n}{k} \rfloor \rfloor$.

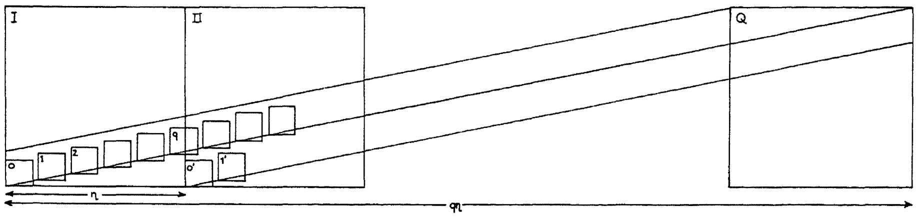
PROOF. Suppose $X = \{0,\ldots,n-1\}$, and let $C = \mathbb{R}/n\mathbb{Z}$ be the circle of length n; so C^2 is a torus. We identify C with the interval $[0,n)$, in which we count modulo n. Let $n = qk + r$, where q and r are integers such that $0 \le r \le k-1$. Let

$$p = \lfloor \tfrac{n}{k} \lfloor \tfrac{n}{k} \rfloor \rfloor = q^2 + \lfloor \tfrac{qr}{k} \rfloor. \tag{2}$$

Choose in C^2 the squares $[x,x+k) \times [y,y+k)$ with

(3) $$(x,y) = (0,0),\ (\frac{qn}{p},\frac{n}{p}),\ 2(\frac{qn}{p},\frac{n}{p}),\ldots,(p-1)(\frac{qn}{p},\frac{n}{p}),$$

respectively. That is, the vertices (x,y) lie equidistantly on a spiral of the torus with q rotations. In the following figure q copies of the torus are unrolled and glued together:



Inspection of the figure yields that disjointness of the squares follows from

(5) $$\text{(i)}\ \frac{qn}{p} \geq k, \quad \text{and} \quad \text{(ii)}\ q.\frac{qn}{p} \leq n.$$

(i) implies that square numbered 1 is disjoint from square numbered 0. (ii) implies that square numbered q still has points in torus copy I. (i) again gives that square numbered q is "high" enough to be disjoint from square numbered 0'.

Now we have p disjoint squares, of side k, in C^2. Since $X^2 \subset C^2$, the intersection $S \cap X^2$ is a k-hypercube in X^2, for any square S. So the intersections of the squares with X^2 from a packing of p k-hypercubes in X^2. □

Again, problems of dimension 2 can be formulated in the language of graphs. $H(2,k,n)$ can be regarded as the maximum number of edge-disjoint $K_{k,k}$'s in $K_{n,n}$. BEINEKE [8] showed that the maximum number of edge-disjoint subgraphs $K_{k,\ell}$ of $K_{m,n}$ (such that the "k-sides" of $K_{k,\ell}$ coincide with the "m-side" of $K_{m,n}$) equals

(5) $$\min\{\lfloor\frac{m}{k}\lfloor\frac{n}{\ell}\rfloor\rfloor,\ \lfloor\frac{n}{\ell}\lfloor\frac{m}{k}\rfloor\rfloor\};$$

that is, the maximum number of disjoint $k\times\ell$-rectangles (i.e., sets $Y_1 \times Y_2$

such that $|Y_1| = k$ and $|Y_2| = \ell$) in a set $X_1 \times X_2$ with $|X_1| = m$ and $|X_2| = n$, is equal to expression (5). This can be proved in a manner similar to the proof of Theorem 3.

Theorem 3 proves equality in Corollary 2 for $d = 2$. This cannot be generalized to arbitrary d, since it can be shown that $H(4,2,5) < 30 = \lfloor\frac{5}{2}\lfloor\frac{5}{2}\lfloor\frac{5}{2}\lfloor\frac{5}{2}\rfloor\rfloor\rfloor\rfloor$ (note that $H(3,2,5) = 12$). In fact it seems that if k is not a divisor of n, then the inequality of Corollary 2 is strict for some d.

It is straightforward to see that $H(d,k,n) = \alpha(\overline{K(n,k)}^d)$, where the product graph is defined in Section 4 of Chapter 3 ("Eigenvalue methods"). So

$$\sup_d \sqrt[d]{H(d,k,n)} = \sup_d \sqrt[d]{\alpha(\overline{K(n,k)}^d)} = \Theta(\overline{K(n,k)}) \tag{6}$$

equals the Shannon-capacity of $\overline{K(n,k)}$. In Chapter 3 an upper bound of $\frac{n}{k}$ for $\Theta(\overline{K(n,k)})$ is given (this upper bound also follows from Corollary 2), but it is still an open problem whether this upper bound can be actually reached; so we have the

PROBLEM. Is $\sup_d \sqrt[d]{H(d,k,n)} = \frac{n}{k}$, for $k \le \frac{1}{2}n$?

The answer is obviously "yes" if k divides n, but for no other values of k and n do we know an answer. For $k = 2$, $n = 5$, the simplest unknown case, $\overline{K(n,k)}$ is the complement of the Petersen-graph. To calculate (6) in this case we cannot adapt the construction of the proof of Theorem 3 straightforwardly: that construction yields "connected" k-hypercubes of $\{0,\ldots,n-1\}^d$ (i.e., the projections onto the components are connected intervals in the cyclic ordering). The maximum number of disjoint connected 2-hypercubes in $\{0,\ldots,n-1\}^d$ is equal to $\alpha(C_n^d)$, where C_n is the circuit on n vertices. LOVÁSZ [66] (cf. Chapter 3) showed that, for odd n,

$$\Theta(C_n) := \sup_d \sqrt[d]{\alpha(C_n^d)} \le \frac{n.\cos(\pi/n)}{1+\cos(\pi/n)} < \frac{n}{2}, \tag{7}$$

whence $\Theta(C_5) = \sqrt{5}$. Since this number is smaller than 5/2 we cannot use the construction of Theorem 3 to answer the problem affirmatively for $k = 2$, $n = 5$ (for some calculations of $\alpha(C_n^d)$ see BAUMERT, et al. [7]).

2. INTERSECTING FAMILIES

2.1. The Erdős-Ko-Rado theorem

Let k and n be natural numbers such that $2k \le n$, and let X be an n-set. The following theorem of ERDŐS, KO & RADO [33] is fundamental to this section.

THEOREM 1. (The Erdős-Ko-Rado theorem) *The maximal number of pairwise intersecting* k*-subsets of an* n*-set is* $\binom{n-1}{k-1}$.

PROOF. Evidently, the value $\binom{n-1}{k-1}$ can be reached. Let $\mathcal{A}$ be a subset of $\mathcal{P}_k(X)$ such that no two sets in $\mathcal{A}$ are disjoint. Let $\mathcal{C}$ be the collection of all cyclic orderings of the set X; so $|\mathcal{C}| = (n-1)!$. Make a (0,1)-matrix M, with rows indexed by $\mathcal{C}$ and columns indexed by $\mathcal{A}$, as follows. The entry of M in the (C,A)-position is a one if and only if the set A occurs consecutively in the cyclic ordering C; that is, if and only if A induces a (cyclic) interval on C ($C \in \mathcal{C}$, $A \in \mathcal{A}$).

It is easy to see that the sum of the entries in any column of M equals $k!(n-k)!$. So the total number of ones in M is equal to $|\mathcal{A}|.k!(n-k)!$. We are finished once we have proved that the number of ones in each row is at most k, since it then follows that the total number of ones is at most $k.|\mathcal{C}| = k.(n-1)!$, which yields

$$|\mathcal{A}|.k!(n-k)! \le k.(n-1)!,$$

i.e., $|\mathcal{A}| \le \binom{n-1}{k-1}$.

So let $C \in \mathcal{C}$ be the index of an arbitrary row. We may suppose that $X = \{1,\dots,n\}$ and that C represents the usual cyclic ordering of $\{1,\dots,n\}$ modulo n. We have to prove that there are at most k sets in $\mathcal{A}$ occurring as an interval in C. To this end, underline any number from $1,\dots,n$ which is the first element (in C) of an interval (of length k) belonging to $\mathcal{A}$. Moreover, encircle any number j whenever $j-k \pmod n$ is underlined; thus encircled numbers are numbers directly following the last element of an interval in $\mathcal{A}$. So no number will be both underlined and encircled, since $\mathcal{A}$ contains no disjoint sets ($n \ge 2k$).

Now consider any encircled number, say, j. Then the n-2k subsequent numbers $j+1,\dots,j+n-2k \pmod n$ cannot be underlined since any interval starting in one of these points is disjoint from the interval starting in j-k (which is in $\mathcal{A}$). So there exists an encircled number j such that the n-2k numbers following j are neither underlined nor encircled. Since the number of underlined numbers is equal to the number of encircled numbers, there cannot be more than k underlined numbers, i.e., the sum of the entries in the row indexed with C is at most k. □

This method of proof is due to KATONA [58,60] (for a generalization, see GREENE, KATONA & KLEITMAN [48]; for a proof using the "Kruskal-Katona theorem", see DAYKIN [23]; for a proof using eigenvalues, see LOVÁSZ [66] (cf. Chapter 3)). The proof may be easily adapted to show that we may replace the condition $\mathcal{A} \subset \mathcal{P}_k(n)$ by: all sets in $\mathcal{A}$ have at most k elements, and no two of these sets are contained in each other.

FRANKL [36] generalized the above proof to obtain $|\mathcal{A}| \le \binom{n-1}{k-1}$ whenever $\mathcal{A} \subset \mathcal{P}_k(X)$, $ik/(i-1) \le n$, and any i sets in $\mathcal{A}$ have nonempty intersection.

2.2. Sharper bounds

Elaboration of the proof also shows that, in case $2k < n$, the bound $\binom{n-1}{k-1}$ can be achieved only by "stars", i.e., by collections consisting of all k-subsets of C containing a fixed element of X. HILTON & MILNER [55] (answering a question of ERDÖS, KO & RADO [33]) proved that collections $\mathcal{A}$ of pairwise intersecting k-subsets of X which are not a star (that is, $\cap\mathcal{A} = \emptyset$), have at most $1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1}$ elements (this bound can easily seen to be attained; Hilton & Milner also showed that all collections achieving the bound are isomorphic).

MEYER [69] asked for the minimum size of a maximal (under inclusion) collection of pairwise intersecting k-subsets of X; he conjectured that the set of lines in a finite projective plane achieves this minimum.

2.3. Larger intersections

ERDÖS, KO & RADO [33] also proved the following extension of Theorem 1. Let $0 \le t \le k$. The maximum number of k-subsets of X such that any two of them intersect in at least t elements, is equal to $\binom{n-t}{k-t}$, provided that n is large enough (with respect to k and t). Let $n(k,t)$ be the smallest number such that for all $n \ge n(k,t)$ the maximum is attained only by collections of k-subsets of X containing a fixed t-subset of X. So $n(k,1) = 2k+1$.

After earlier estimates given by ERDÖS, KO & RADO [33] and HSIEH [56], FRANKL [38] determined $n(k,t)$ for $t \ge 15$; he found that $n(k,t)$ is about $(k-t+1)(t+1)+1$ if $t \ge 19$, and that, for all t, $(k-t+1)(t+1)+1 \le n(k,t) \le 2(k-t+1)(t+1)+1$.

A related conjecture of Erdös, Ko and Rado is that, if k is even and $n = 2k$, the maximum number of k-subsets of X which pairwise intersect in at least two elements is equal to $\frac{1}{2}(\binom{n}{k}-\binom{k}{\frac{1}{2}k}^2)$. FRANKL [38] extended this to the conjecture that for each n-set X the maximum size of a collection of k-subsets pairwise intersecting in at least t elements always is attained by a collection $\mathcal{A}$ of the form

$$\mathcal{A} = \{A \subset X \mid |A| = k \text{ and } |A \cap X'| \geq t+r\}$$

for some $r = 0,\dots,\lfloor\frac{1}{2}(n-t)\rfloor$ and some $(t+2r)$-subset X' of X.

KATONA [60] observed that if a t-(n,k,1)-design exists (i.e. a collection $\mathcal{D}$ of k-subsets of X such that each t-subset of X is in exactly one set of $\mathcal{D}$; cf. Chapter 5), then certainly the maximum cardinality of a collection of k-subsets, pairwise intersecting in at least t elements, is $\binom{n-t}{k-t}$. For let $\mathcal{A}$ be such a collection and let $\mathcal{D}$ be a t-(n,k,1)-design. So

$$|\mathcal{D}| = \frac{n.\ \dots\ .(n-t+1)}{k.\ \dots\ .(k-t+1)}\ .$$

For each permutation π of X let $\pi\mathcal{D}$ be the design $\{\pi A \mid A \in \mathcal{D}\}$, where $\pi A = \{\pi x \mid x \in A\}$.

So $\mathcal{A}\cap\pi\mathcal{D}$ contains at most one set, for any permutation π, since any two sets in $\pi\mathcal{D}$ have intersection at most t-1; hence

$$n! \geq \sum_{\pi} |\mathcal{A}\cap\pi\mathcal{D}|,$$

where π ranges over the set of permutations of X. The right hand side of this inequality is equal to the number of triples $A \in \mathcal{A}$, $D \in \mathcal{D}$, π permutation, such that $\pi D = A$. For fixed A and D the number of permutations π such that $\pi D = A$, is equal to $k!(n-k)!$. Therefore

$$n! \geq |\mathcal{A}|.|\mathcal{D}|.k!(n-k)! = |\mathcal{A}|.\ \frac{n.\ \dots\ .(n-t+1)}{k.\ \dots\ .(k-t+1)}.k!(n-k)!,$$

and the required upper bound for $\mathcal{A}$ follows. (This result also follows from Delsarte's linear programming bound (Theorem 15 of Chapter 3).)

The following question was asked by FRANKL [36]: does there exists an $\varepsilon > 0$ such that if $k \leq (\frac{1}{2}+\varepsilon)n$, $\mathcal{A} \subset \mathcal{P}_k(n)$ and $|A\cap B\cap C| \leq 2$ whenever $A,B,C \in \mathcal{A}$, then $|\mathcal{A}| \leq \binom{n-2}{k-2}$?

FRANKL [37] investigated the following problem of Erdős, Rothschild and Szemerédi: given t and $0 < c < 1$, what is the maximum cardinality of a collection $\mathcal{A}$ of k-subsets of X such that $|A \cap B| \geq t$, whenever $A, B \in \mathcal{A}$, and for all $x \in X$:

$$|\{A \in \mathcal{A} | x \in A\}| < c.|\mathcal{A}|?$$

2.4. The Hajnal-Rothschild generalization

HAJNAL & ROTHSCHILD [52] generalized the Erdős-Ko-Rado theorem as follows. Let $\mathcal{A}$ be a collection of k-subsets of X such that each subcollection $\mathcal{A}'$ of $\mathcal{A}$ with more than r elements, contains two sets which intersect in at least t elements; then

$$|\mathcal{A}| \leq \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \binom{n-it}{k-it},$$

provided that n is large enough with respect to k,r,t, i.e., $n \geq n(k,r,t)$. Clearly, in case $r = 1$, this result reduces to the Erdős-Ko-Rado theorem. If we put $t = 1$, Hajnal and Rothschild's theorem becomes: if $\mathcal{A} \subset \mathcal{P}_k(X)$ contains no r+1 pairwise disjoint sets then

$$|\mathcal{A}| \leq \binom{n}{k} - \binom{n-r}{k},$$

provided that $n \geq n(k,r,1)$. ERDŐS [28] conjectures that for all n

$$|\mathcal{A}| \leq \max\{\binom{rk+k-1}{k}, \binom{n}{k}-\binom{n-r}{k}\};$$

this was proved for $k = 2$ by ERDŐS & GALLAI [31].

ERDŐS [28] showed that $n(k,r,1) \leq c_k.r$, and KATONA [60] conjectured that $n(k,2,1) = 3k+1$ (taking all k-subsets of a fixed (3k-1)-subset of X in case $n = 3k$, shows that 3k+1 is the smallest number we may hope for).

2.5. A relation with Turán's theorem

CHVÁTAL [20] has designed the following framework generalizing both the Erdős-Ko-Rado theorem and Turán's theorem (cf. Chapter 7). Call a collection $\mathcal{A}$ of sets *m-intersecting* if any m sets in $\mathcal{A}$ have nonempty intersection. Let $f(n,k,m)$ be the maximum cardinality of a collection $\mathcal{A}$ of k-subsets of X such that for all $\mathcal{A}' \subset \mathcal{A}$: $\mathcal{A}'$ is m-intersecting implies $\mathcal{A}'$ is (m+1)-intersecting.

So $f(n,k,1) = \binom{n-1}{k-1}$, for $n \geq 2k$, is equivalent to the Erdős-Ko-Rado theorem; $f(n,2,2) = \lfloor \frac{1}{4}n^2 \rfloor$, is the content of TURÁN's theorem [76,77] and TURÁN [78] asked (in another terminology) for the number $f(n,k,k)$.

CHVÁTAL [20] proved that $f(n,k,k-1) = \binom{n-1}{k-1}$ if $n \geq k+2$. ERDŐS [29] wondered whether $f(n,k,2) = \binom{n-1}{k-1}$ if $k > 2$ and $n \geq \frac{3}{2}k$; CHVÁTAL [20] extended Erdős' question to the conjecture that $f(n,k,m) = \binom{n-1}{k-1}$ whenever $k > m$ and $n \geq \frac{m+1}{m}.k$. So this has been proved for $k = m+1$, and for $m = 1$. For some more results see BERMOND & FRANKL [13].

2.6. Some further related problems and results

HILTON [54] showed that, if $1 \leq h \leq k \leq n$, $h+k \leq n$, and $\mathcal{A}$ consists of pairwise intersecting subsets A of X with $h \leq |A| \leq k$, then

$$|\mathcal{A}| \leq \sum_{i=h}^{k} \binom{n-1}{i-1}.$$

KLEITMAN [61] proved that if $h+k \leq n$ and $\mathcal{A}$ and $\mathcal{B}$ consists of k-subsets and h-subsets, respectively, of X such that $A \cap B \neq \emptyset$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|\mathcal{A}| \geq \binom{n-1}{k-1}$ implies $|\mathcal{B}| \leq \binom{n-1}{h-1}$; HILTON [53] generalized this result.

KATONA [59] (cf. LOVÁSZ [64]) proved the following conjecture of Ehrenfeucht and Mycielski: let $A_1,\ldots,A_m$ be k-subsets of X, and let $B_1,\ldots,B_m$ be h-subsets of X, such that $A_i \cap B_j \neq \emptyset$ iff $i \neq j$; then $m \leq \binom{h+k}{k}$. This result was generalized by T. Tarján - see KATONA [60].

ERDŐS & RADO [34] proved that, given natural numbers c and k, there is a number $\phi_c(k)$ such that if $\mathcal{A}$ is a collection of k-sets with $\phi_c(k)$ elements, then $\mathcal{A}$ has a subcollection $\mathcal{A}'$ of cardinality c with the property: if $A,B \in \mathcal{A}'$ then $A \cap B = \cap\mathcal{A}'$. They conjectured that one can take $\phi_c(k) < (cc')^k$ for a certain absolute constant c'. SPENCER [74] proved an upper bound for $\phi_c(k)$ of order about $c^k.k!$ (cf. ERDŐS [30]).

FRANKL [39] proved that if $A_1,\ldots,A_m$ are k-subsets of X such that $|A_i \cap A_j| \neq 1$ then $m \leq \binom{n-2}{k-2}$ if $k \geq 4$ and n large enough with respect to k. See FRANKL [41] for extensions.

2.7. Permutations

An analogue of the Erdős-Ko-Rado theorem, due to FRANKL & DEZA [42] is: let Π be a collection of permutations of X such that for all $\pi_1, \pi_2 \in \Pi$ there is at least one $x \in X$ such that $\pi_1 x = \pi_2 x$; then $|\Pi| \leq (n-1)!$. A generalization has been conjectured by Deza and Frankl: if for any two $\pi_1, \pi_2 \in \Pi$

there are at least t distinct elements $x_1,\dots,x_t$ in X such that $\pi_1 x_i = \pi_2 x_i$, for $i = 1,\dots,t$, then $|\Pi| \le (n-t)!$.

In a way similar to Katona's method using t-designs mentioned above, one can derive this bound for $t = 2$ from the existence of a collection P of permutations of X such that for all distinct $x_1,x_2 \in X$ and for all distinct $y_1,y_2 \in X$ there is exactly one permutation ρ in P such that $\rho x_1 = y_1$ and $\rho x_2 = y_2$. The existence of such a collection P is easily seen to be equivalent to the existence of a set of n-1 mutually orthogonal latin squares of order n; so the conjecture is true, in case $t = 2$, for prime powers n. (See also BANDT [1].)

In this section we have considered mainly intersection problems for collections of sets with a fixed size. For a more extensive survey of (also more general) intersection problems and results we refer to ERDŐS & KLEITMAN [32], KATONA [60], GREENE & KLEITMAN [49], BOLLOBÁS [14].

For a more general approach to intersection problems - see DEZA, ERDŐS & FRANKL [26]. Such problems can be handled with eigenvalue techniques within the theory of association schemes (using Eberlein polynomials) - see DELSARTE [24], SCHRIJVER [73], and Chapter 3.

Often one may replace expressions like "k-subsets of an n-set" by "k-dimensional flats in an n-dimensional projective space", and binomial coefficients by Gaussian coefficients (cf. [47]), and so on, to obtain analogous results - see DELSARTE [25], LOVÁSZ [64,67].

3. BARANYAI'S THEOREM AND EDGE COLOURING OF UNIFORM HYPERGRAPHS

3.1. Partitioning into partitions

Let X be a fixed n-set, In this section we consider partitions of $\mathcal{P}_k(X)$ into classes of disjoint sets, and some generalizations. BARANYAI [3] showed that the minimum possible number of classes in such a partition is equal to

$$\left\lceil \binom{n}{k} \Big/ \left\lfloor \frac{n}{k} \right\rfloor \right\rceil.$$

In the Introduction we saw already that proving this consists of showing that this minimum can be achieved. Before going further into the general problem we prove a special but nevertheless interesting case of Baranyai's theorem, namely the case when n is a multiple of k. Then the theorem becomes

THEOREM 1. (BARANYAI [3]) *Let* n *be a multiple* k. *Then there exist* $\binom{n-1}{k-1}$ *partitions of* X *into* k*-sets such that each* k*-subset of* X *occurs in exactly one of these partitions.*

(This was proved for $k = 3$ by PELTESOHN [70] and for $k = 4$ by J.-C. Bermond.) In order to prove Theorem 1 we prove a corollary of this theorem which contains Theorem 1 as a special case. To this end let $n = mk$ and $M = \binom{n-1}{k-1}$. Call an ordered m-tuple $(Y_1,\dots,Y_m)$ an m-*partition* of a set Y if $Y_i \cap Y_j = \emptyset$ whenever $i \neq j$, and $Y = \cup Y_i$. (So the empty set may occur once or more times in an m-partition.) Moreover we assume $X = \{1,\dots,n\}$.

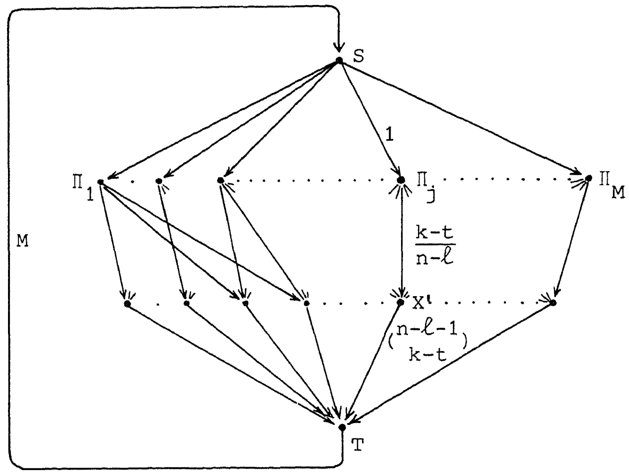
Now suppose we have, as in Theorem 1, m-partitions $\Pi_1,\dots,\Pi_M$ of X such that each k-subset of X occurs in exactly one of these partitions as a class. Let $0 \le \ell \le n$. Then we have also m-partitions $\Pi_1',\dots,\Pi_M'$ of $\{1,\dots,\ell\}$ such that, for $t = 0,\dots,k$, each t-subset of $\{1,\dots,\ell\}$ occurs exactly $\binom{n-\ell}{k-t}$ times among these partitions. This can be seen by taking $\Pi_j' = (X_1 \cap X',\dots,X_m \cap X')$ where $\Pi_j = (X_1,\dots,X_m)$ and $X' = \{1,\dots,\ell\}$. So Theorem 2 is equivalent to Theorem 1, since taking $\ell = n$ reduces Theorem 2 to Theorem 1.

THEOREM 2. *Let* $n = mk$, $M = \binom{n-1}{k-1}$ *and* $0 \le \ell \le n$. *Then there are* m*-partitions* $\Pi_1,\dots,\Pi_M$ *of* $\{1,\dots,\ell\}$ *such that each* t*-subset of* $\{1,\dots,\ell\}$ *occurs exactly* $\binom{n-\ell}{k-t}$ *times among these partitions, for* $t = 0,\dots,k$.

A basis for the proof of Theorem 2 is Ford & Fulkerson's integer flow theorem (cf. Chapter 13).

INTEGER FLOW THEOREM. *Let* $D = (V,A)$ *be a directed graph, and let* $f: A \to \mathbb{R}$ *be a flow function (i.e., for each vertex* $v \in V$ *the sum of the values* $f(a)$ *of arrows* a *with head* v, *is equal to the sum of the values* $f(a)$ *of arrows* a *with tail* v). *Then there exists a flow function* $g: A \to \mathbb{Z}$ *such that for each arrow* a *we have*: $g(a) = \lfloor f(a) \rfloor$ *or* $g(a) = \lceil f(a) \rceil$.

PROOF OF THEOREM 2. We proceed by induction on ℓ. For $\ell = 0$ the theorem is trivial; we can take $\Pi_1 = \dots = \Pi_M = (\emptyset,\dots,\emptyset)$. Suppose we have proved the theorem for some fixed $\ell < n$. Let $\Pi_1,\dots,\Pi_M$ be partitions of $\{1,\dots,\ell\}$ such that, for $t = 0,\dots,k$, each t-subset of $\{1,\dots,\ell\}$ occurs exactly $\binom{n-\ell}{k-t}$ times among these partitions. Make a directed graph with vertices: S, T (two new objects), the partitions $\Pi_1,\dots,\Pi_M$, and all subsets of $\{1,\dots,\ell\}$ with cardinality k or less. There are arrows from S to any partition Π_j, from any subset of $\{1,\ldots,\ell\}$ to T, and from T to S. Furthermore there is an arrow from Π_j to subset X' iff X' occurs in Π_j as a class.



Now let $f: A \to \mathbb{R}$ be given by:

$$(1) \quad f(a) = \begin{cases} 1, & \text{if } a = (S,\Pi_j) \text{ for some } j; \\ \binom{n-\ell-1}{k-t-1}, & \text{if } a = (X',T) \text{ for some } X' \subset \{1,\ldots,\ell\} \text{ with } |X'| = t; \\ M, & \text{if } a = (T,S); \\ \frac{k-t}{n-\ell}, & \text{if } a = (\Pi_j,X') \text{ and } |X'| = t > 0; \\ \lambda\frac{k}{n-\ell}, & \text{if } a = (\Pi_j,\emptyset) \text{ and } \emptyset \text{ occurs } \lambda \text{ times in } \Pi_j. \end{cases}$$

It is straightforward to check that f is a flow function. By the integer flow theorem there is an integer-valued flow function g and A such that g coincides with f on the arrows given in the first three lines of (1). Furthermore for the two remaining possibilities for a we have $0 \le f(a) \le 1$ since the total amount of flow on arrows with tail Π_j is equal to 1. Hence we can take g(a) to be 0 or 1 on those arrows.

So for each $j = 1,\ldots,M$ there is a unique X' in Π_j such that $g(\Pi_j,X') = 1$. Now let Π'_j arise from Π_j by replacing this unique X' by $X' \cup \{\ell+1\}$ (for $j = 1,\ldots,M$). Then $\Pi'_1,\ldots,\Pi'_M$ are m-partitions of $\{1,\ldots,\ell+1\}$ such that each t-subset of $\{1,\ldots,\ell+1\}$ occurs exactly $\binom{n-\ell-1}{k-t}$ times among these partitions (for $t = 0,\ldots,k$). $\square$

3.2. Colourings

Let $H = (X,\mathcal{E})$ be a hypergraph with vertex set X and edge set $\mathcal{E}$. A (vertex) p-*colouring* of H is a partition $\mathcal{C} = \{C_i | i \le p\}$ of X into p (possibly empty) subsets ('colours'). We consider four successively stronger requirements on the colouring.

(i) $\mathcal{C}$ is called *proper* if no edge containing more than one point is monochromatic, i.e. $E \in \mathcal{E}$ and $|E| > 1$ imply $E \not\subset C_i$ for all $i = 1,\dots,p$.

(ii) $\mathcal{C}$ is called *good* if each edge E has as many colours as it can possibly have, i.e., $|\{i | E \cap C_i \neq \emptyset\}| = \min(|E|, p)$.

(iii) $\mathcal{C}$ is called *fair* or *equitable* if on each edge E the colours are represented as fairly as possible, i.e.,

$$\lfloor \frac{|E|}{p} \rfloor \le |E \cap C_i| \le \lceil \frac{|E|}{p} \rceil \qquad \text{for } i = 1,\dots,p.$$

(iv) $\mathcal{C}$ is called *strong* if on each edge E all colours are different i.e., $|E \cap C_i| \le 1$ for $i = 1,\dots,p$.

(This is just the special case of a good or fair colouring with p colours when $p \ge \max\{|E| \, | E \in \mathcal{E}\}$.) Instead of asking for an equal partition over the edges one may ask for an equal partition of colours over the points:

(v) A proper colouring is called *equipartite* if for $i = 1,\dots,p$ we have

$$\lfloor \frac{|X|}{p} \rfloor \le |C_i| \le \lceil \frac{|X|}{p} \rceil .$$

Dually one defines a (proper, good, fair, strong, equipartite) edge p-colouring of H as such a p-colouring of $H^* = (\mathcal{E},X)$, the dual of H (where $x \in X$ is identified with $\mathcal{E}_x = \{E \in \mathcal{E} | x \in E\}$).

EXAMPLE 0. For $p \ge |X|$ the partition of X into singletons is an equipartite and strong p-colouring. Hence any H has a proper, good, fair, strong and equipartite p-colouring for some p.

In the case of proper or strong colourings the only interesting question is to ask for the minimum number of colours needed (which number is usually called $\chi(H)$ resp. $\gamma(H)$ in case of vertex-colourings and ?(H) resp. $q(H)$ in case of edge-colourings) since here adding unused colours does not change the property. In the case of good, fair or equipartite colourings we really want to know for which p such a colouring exists.

EXAMPLE 1. Let $H = (X, E)$ be a simple (undirected) graph (i.e. $E \subset P_2(X)$). By VIZING's theorem [80] if

$$p \geq \max_{x \in X} \delta(x) + 1$$

then H has a good (hence fair & strong) edge p-colouring. By GUPTA's theorem [50,51] if

$$p \leq \max_{x \in X} \delta(x) - 1$$

then H has a good edge p-colouring (but not necessarily a fair one, and certainly no strong one).
[Here (and below) $\delta(x) = |E_x| = |\{x | x \in E \in E\}|$.]

EXERCISE 1. Determine the minimal p for which there exists a proper edge p-colouring of K_n^k. [$K_n^k = (X, P_k(x))$ where $|X| = n$.]

EXERCISE 2. Verify that the complete graph $K_7 (=K_7^2)$ has a fair edge p-colouring unless p = 2 or 6, a good edge p-colouring unless p = 6 and an equipartite edge p-colouring unless p = 1.

EXERCISE 3. (FOURNIER [35]) Let $H = (X, E)$ be a graph. Then H has a good edge 2-colouring iff no component of H is an odd cycle.

3.3. Baranyai's theorem

Let $|X| = n$. The hypergraph $H = (X, P_k(X))$ is called the *complete* k-*uniform hypergraph*, written K_n^k. In this case BARANYAI [3] provided a complete solution for the edge-colouring problems by proving

THEOREM 3. *Let* $H = K_n^k$ *and write* $N = \binom{n}{k}$, *the number of edges of* H. *Then*

(i) H *has a good edge* p-*colouring iff it is not the case that*

$$N / \lceil \tfrac{n}{k} \rceil < p < N / \lfloor \tfrac{n}{k} \rfloor,$$

i.e. iff

$$\frac{N}{p} \leq \lfloor \tfrac{n}{k} \rfloor \quad \text{or} \quad \frac{N}{p} \geq \lceil \tfrac{n}{k} \rceil$$

(ii) H *has a fair edge* p-*colouring iff*

$$\left\lceil \left\lfloor \frac{\Delta}{p} \right\rfloor \frac{n}{k} \right\rceil \le \frac{N}{p} \le \left\lfloor \left\lceil \frac{\Delta}{p} \right\rceil \frac{n}{k} \right\rfloor$$

where $\Delta = \frac{Nk}{n}$ *is the degree (valency) of each point.*

(iii) $q(H) = \lceil N / \lfloor \frac{n}{k} \rfloor \rceil$.

Note that (iii) generalizes Theorem 1. For the moment we restrict ourselves to proving necessity.

PROOF OF NECESSITY. This part of the proof will be valid for any regular k-uniform hypergraph on n points with N edges. Let $\mathcal{C}$ be any edge p-colouring of H and define for $x \in X$

$$c(x) := |\{i \mid E_x \cap C_i \neq \emptyset\}|,$$

the number of colours found at point x.

(i) $p < N/\lfloor \frac{n}{k} \rfloor$, i.e., $\lfloor \frac{n}{k} \rfloor < \frac{N}{p}$ means that there exist two non-disjoint edges with the same colour i.e., $c(x) < \delta(x) = \Delta$ for some x.
$p > N/ \lceil \frac{n}{k} \rceil$, i.e., $\lceil \frac{n}{k} \rceil > \frac{N}{p}$ means that not every colour occurs at each point, i.e., $c(x) < p$ for some x.
But for a good edge p-colouring we have $\forall x$: $c(x) = \min(\delta(x), p)$.

(ii) By definition of a fair edge colouring we have for each i

$$\left\lfloor \frac{\Delta}{p} \right\rfloor \le \frac{k}{n} |C_i| \le \left\lceil \frac{\Delta}{p} \right\rceil,$$

and hence

$$\left\lceil \left\lfloor \frac{\Delta}{p} \right\rfloor \frac{n}{k} \right\rceil \le |C_i| \le \left\lfloor \left\lceil \frac{\Delta}{p} \right\rceil \frac{n}{k} \right\rfloor.$$

Averaging over i we find the stated condition.

(iii) $q(H) \ge \lceil N / \lfloor \frac{n}{k} \rfloor \rceil$ immediately follows from (i). □

REMARK. (i) and (iii) can be formulated more generally as follows. For a regular hypergraph $H = (X, E)$ let $\nu(H)$ be the maximum cardinality of a set of pairwise disjoint edges in H, and let $\rho(H)$ be the minimum cardinality of a set of edges covering all vertices.

(i) can be stated as: if

$$\nu(H) < \frac{|E|}{p} < \rho(H),$$

then H does not have a good edge p-colouring.

(iii) can be stated as:

$$q(H) \geq \left\lceil \frac{|E|}{\nu(H)} \right\rceil .$$

Concerning the sufficiency half of Theorem 3 we shall in fact prove slightly more, since we need it later. Let s be a positive integer, and $H = (X, E)$ be a hypergraph. Then define $sH = (X, sE)$ to be the hypergraph with the same vertices as H, but with each edge from H taken with multiplicity s. Obviously $\nu(sH) = \nu(H)$ and $\rho(sH) = \rho(H)$. A colouring of sH with p colours is sometimes called a *fractional colouring* of H with $q = \frac{p}{s}$ colours. We show here that sK_n^k has a good or fair edge p-colouring iff p satisfies the conditions (i) resp. (ii), where now $N = s\binom{n}{k}$.

A hypergraph (X, E) is called *almost regular* if for all $x, y \in X$ we have $|\delta(x) - \delta(y)| \leq 1$. Now we have

<u>THEOREM 4</u>. (BARANYAI [3]) *Let* $a_1, \ldots, a_t$ *be natural numbers such that* $\sum_{i=1}^{t} a_i = N := \binom{n}{k} s$. *Then the edges of* sK_n^k *can be partitioned in almost regular hypergraphs* (X, E_j) *such that* $|E_j| = a_j$ $(1 \leq j \leq t)$.

It is easily verified that Theorem 3 follows from Theorem 4:

(i) If $p \leq N/\lceil \frac{n}{k} \rceil$ then use Theorem 4 with $s = 1$, $t = p$ and $a_1 = \ldots = a_{t-1} = \lceil \frac{n}{k} \rceil$, $a_t = N-(t-1)\lceil \frac{n}{k} \rceil$.
If $p \geq N/\lfloor \frac{n}{k} \rfloor$ then use Theorem 4 with $t = \lceil N/\lfloor \frac{n}{k} \rfloor \rceil$ and $a_1 = \ldots = a_{t-1} = \lfloor \frac{n}{k} \rfloor$, $a_t = N-(t-1) \lfloor \frac{n}{k} \rfloor$.
This also proves (iii).

(ii) Write $f_0 = \lceil \lfloor \frac{\Delta}{k} \rfloor \frac{n}{k} \rceil$ and $f_1 = \lfloor \lceil \frac{\Delta}{p} \rceil \frac{n}{k} \rfloor$. If $pf_0 \leq N \leq pf_1$ then use Theorem 4 with $s = 1$, $t = p$ and $a_1 = \ldots = a_g = \lfloor \frac{N}{p} \rfloor + 1$ and $a_{g+1} = \ldots = a_t = \lfloor \frac{N}{p} \rfloor$ where $g = N - p \lfloor \frac{N}{p} \rfloor$.
$\forall_i\ f_0 \leq a_i \leq f_1$ guarantees that we get a fair colouring.

Theorem 4 will be proved in subsection 3.6 as a consequence of much more general theorems.

3.4. <u>Normal, balanced and unimodular hypergraphs</u>

The results mentioned in this subsection are treated more extensively in Chapter 13.

DEFINITION. A hypergraph $H = (X,\mathcal{E})$ is called *balanced* if for any odd cycle

$$a_0, E_0, a_1, E_1, \ldots, E_{2p}, a_{2p+1} = a_0$$

(where $a_i, a_{i+1} \in E_i \in \mathcal{E}$ $(0 \le i \le 2p)$) there is an i $(0 \le i \le 2p)$ such that E_i contains at least three vertices of the cycle.

Note that for graphs balanced means the same as bipartite (no odd circuits).

EXAMPLE 2. $X = \mathbb{R}$, $\mathcal{E} = \{E \subset \mathbb{R} \mid E \text{ connected}\}$ yields a balanced hypergraph.

PROPOSITION 1. *The dual of a balanced hypergraph is balanced.* □

PROPOSITION 2. $H = (X,\mathcal{E})$ *is balanced iff for each* $A \subset X$ *the subhypergraph* $H_A = (A,\{E \cap A \mid E \in \mathcal{E}\})$ *has* $\chi(H_A) \le 2$.

PROOF. (if) Obvious from the definitions. (only if) Induction on $|X|$. Let $(X,\mathcal{E})$ be a balanced hypergraph, and let $\mathcal{G} = \mathcal{E} \cap \mathcal{P}_2(X)$. Let $a \in X$ be a non-cut point of the bipartite graph $(X,\mathcal{G})$. $H_{X\setminus\{a\}}$ is balanced, hence by induction it has a proper bicolouring: $X\setminus\{a\} = C_1+C_2$. Since $(X,\mathcal{G})$ is bipartite and a is not a cut point all neighbours of a in this graph have the same colour, say C_1. But then $X = C_1 + (C_2 \cup \{a\})$ is a proper bicolouring of $(X,\mathcal{E})$. □

THEOREM 15. (BERGE [9]) *Let* $H = (X,\mathcal{E})$ *be balanced. Then* H *has a good vertex* p*-colouring for each* p.

PROOF. Let $\mathcal{C} = \{C_i \mid i \le p\}$ be a best possible vertex p-colouring, i.e., one with maximal $\sum_{E\in\mathcal{E}} c(E)$ (where $c(E)$ is the number of colours of edge E).
If $\mathcal{C}$ is not good then for some $E \in \mathcal{E}$ we have $c(E) < \min(|E|,p)$.
Since $c(E) < |E|$ there is a colour i with $|C_i \cap E| \ge 2$.
Since $c(E) < p$ there is a colour j with $|C_j \cap E| = 0$.
Since H is balanced $H_{C_i \cup C_j}$ has a good 2-colouring $(C_i \cup C_j) = C_i' + C_j'$.
Replacing C_i and C_j by C_i' and C_j' we obtain a colouring with larger value of $\sum_{E\in\mathcal{E}} c(E)$. Contradiction. □

COROLLARY. *Let* H *be balanced. Then* H *has an edge* p*-colouring for each* p.

COROLLARY. *Let* H *be balanced. Then*

$$\gamma(H) = \max_{E\in\mathcal{E}} |E|,$$

$$q(H) = \max_{x\in X} \delta(x),$$

$$H \text{ has } \min_{E\in\mathcal{E}} |E| \textit{ disjoint transversals,}$$

$$H \text{ has } \min_{x\in X} \delta(x) \textit{ disjoint point covers.}$$

DEFINITION. A hypergraph $H = (X,\mathcal{E})$ is called *normal* if for each partial hypergraph $H' = (X,\mathcal{E}')$ of H [i.e. $\mathcal{E}' \subset \mathcal{E}$] we have $q(H') = \Delta(H')$ [where $\Delta(H)$ denotes the maximal degree of a hypergraph H: $\Delta(H) = \max_{x\in X} \delta(x)$]. By the second line of the second corollary a balanced hypergraph is normal.

PROPOSITION 3. (LOVÁSZ [63]) *Let* $H = (X,\mathcal{E})$ *be normal and* $E \in \mathcal{E}$. *Then* $H' = (X,\mathcal{E}+\{E\})$ *is normal too. That is, increasing the multiplicity of edges leaves a normal hypergraph normal.*

THEOREM 4. (LOVÁSZ [63]) $H = (X,\mathcal{E})$ *is normal iff for each partial hypergraph* H' *we have* $\nu(H') = \tau(H')$. *[Where* $\nu(H)$ *is the maximum cardinality of a set of pairwise disjoint edges and* $\tau(H)$ *is the minimum cardinality of a transversal (set of points meeting every edge).]*

COROLLARY. (BERGE & LAS VERGNAS [12]) *Let* $H = (X,\mathcal{E})$ *be balanced. Then* $\nu(H) = \tau(H)$.

COROLLARY. $H = (X,\mathcal{E})$ *is balanced iff for all* $H' = (X',\mathcal{E}')$ *with* $X' \subset X$, $\mathcal{E}' \subset \{E \cap X' \mid E \in \mathcal{E}\}$ *we have* $\nu(H') = \tau(H')$ *(or:* $\gamma(H') = \max_{E\in\mathcal{E}'} |E|$*; or:* $q(H') = \max_{x\in X} \delta'(x)$*; or:* H' *has* $\min_{E\in\mathcal{E}'} |E|$ *disjoint transversals; or:* H' *has* $\min_{x\in X} \delta'(x)$ *disjoint point covers).*

DEFINITION. A hypergraph $H = (X,\mathcal{E})$ is called *unimodular* if its incidence matrix is totally unimodular (i.e. each square submatrix has determinant 0 or ± 1).

THEOREM 7. (GHOUILA-HOURI [46]) H *is unimodular iff for each* $A \subset X$ *the sub-hypergraph* H_A *has a fair vertex 2-colouring.*

COROLLARY. *A unimodular hypergraph is balanced.*

Note that for (multi)graphs unimodular is equivalent to bipartite. If a hypergraph is unimodular, then so is its dual and any partial sub-hypergraph.

THEOREM 8. (BERGE [9]) *Let* $H = (X,E)$ *be unimodular. Then* H *has a fair vertex* p-*colouring for each* p.

PROOF. Similar to the analogous one in the balanced case. □

3.5. The r-partite case

Let X be partitioned into r subsets: $X = \sum_{i=1}^{r} X_i$, and let $n = |X|$, $n_i = |X_i|$. The hypergraph $H = (X,E)$ with $E = \{E \in P_k(X) | \forall_i: |E \cap X_i| \le 1\}$ is called a *complete* r-*partite* k-*uniform hypergraph*, written $K^k_{n_1,\dots,n_r}$. When $n_1 = \dots = n_r = m$ then H is written $K^k_{r\times m}$. Here the problems are not yet solved, but the following is known.

- For $K^k_{r\times m}$ BARANYAI [4] proved the analogue of Theorem 1 and Theorem 3. The results are exactly the same when we read there $n = mr$, $N = \binom{r}{k} m^k$, $\Delta = \binom{r-1}{j-1} m^{k-1}$.
- For $k = r$ BERGE [10] showed that $K^r_{n_1,\dots,n_r}$ has the edge-colouring property (ECP), that is $q(H) = \max_{x\in X} \delta(x)$.
  In this case, when $n_1 \ge n_2 \ge \dots \ge n_r$ this means that $q(H) = \Pi_{i=1}^{r-1} n_i$.
  Then MEYER [68] showed that $K^r_{n_1,\dots,n_r}$ has a good p-colouring for any $p \ge 1$ (explicitly constructing one).
- Finally BARANYAI & BROUWER [6] showed that $K^r_{n_1,\dots,n_r}$ has a fair p-colouring for any $p \ge 1$ as a corollary of the theory in the previous sections and the fact that the $1\times r$ matrix $(11\dots1)$ is totally unimodular:

The arguments proving this run along the following lines. Let $R = \{1,2,\dots,r\}$ and let a hypergraph $H = (R,E)$ be given. Define $H(n_1,\dots,n_r) = (X,E(n_1,\dots,n_r))$ where $X = \sum_{i=1}^{r} X_i$, $n_i = |X_i|$ and

$$E(n_1,\dots,n_r) = \{E \in P(X) | \forall i: |X_i \cap E| \le 1 \;\&\; \{i | |X_i \cap E| \ne 0\} \in E\}.$$

Define $H^0(n_1,\dots,n_r)$ to be the hypergraph with vertices R and edges E but each edge $E \in E$ with multiplicity $\Pi_{i\in E} n_i$.

With this notation we have for $H = K^k_r$ that $H(n_1,\dots,n_r) = K^k_{n_1,\dots,n_r}$.

THEOREM 9. *If* $H^0(n_1,\dots,n_r)$ *has a fair edge* p-*colouring then* $H(n_1,\dots,n_r)$ *has one too.*

COROLLARY. *If* H *is unimodular then* $H(n_1,\dots,n_r)$ *has a fair* p-*colouring for any* $p \ge 1$.

COROLLARY. *If* H *has a fair edge* p-*colouring and* $\Pi_{i\in E} n_i$ *does not depend on* E *(e.g. when* $n_1 = \dots = n_r$ *and* H *is* k-*uniform) then* $H(n_1,\dots,n_r)$ *has a fair edge* p-*colouring.*

Hence all above mentioned results on $K^k_{n_1,\dots,n_r}$ follow from Theorem 9 (and Theorem 3).

EXERCISE 4. (Brouwer.) Show that $q(K^2_{p,q,r}) = p+q+\varepsilon$ when $p \geq q \geq r$ and $\varepsilon = 0$ unless $p = q = r \equiv 1 \pmod 2$ or $p - 1 = q = r \equiv 0 \pmod 2$ in which case $\varepsilon = 1$.

3.5. Parallelisms

A *parallelism* or 1-*factorization* of a hypergraph $H = (X,\mathcal{E})$ is a partition $\mathcal{E} = \sum_{i=1}^{q} \mathcal{F}_i$ where each $\mathcal{F}_i$ is a *parallel class* or 1-*factor*, that is, a partition of X. In other words, a parallelism of H is a strong edge-colouring of H with $\delta(H)$ colours.

REMARK. Let $\omega(H)$ be the maximum cardinality of a set of pairwise intersecting edges (clique) in H. Obviously $\Delta(H) \leq \omega(H) \leq q(H)$ for any H. V. Chvátal conjectured that if H is *hereditary*, i.e. if $E' \subset E \subset \mathcal{E}$ implies $E' \in \mathcal{E}$, then $\Delta(H) = \omega(H)$, i.e. some maximum clique is a star.

Concerning the edge-colouring property for hereditary hypergraphs we have:

THEOREM 10. (BROUWER & TIJDEMAN [18]) *Let* $H = \hat{K}^k_n = (X, \mathcal{P}_{\leq k}(X))$ *where* $|X| = n$. *Then* H *has the edge-colouring property (and hence a fair* p-*colouring for any* p*) iff*

(i) $n \leq 2k$ *and* $\hat{K}^{n-k-1}_n$ *has the edge-colouring property,*

or

(ii) $n > 2k$ *and*

either $n \equiv 0 \pmod k$ *and* $n \geq k(k-2)$
or $n \equiv -1 \pmod k$ *and* $n \geq \frac{1}{2}k(k-2)-1$.

When $\hat{K}^k_n$ does not have the edge-colouring property not much is known. J.-C. Bermond proved for $k = 3$ and $n \equiv 1 \pmod 3$, $n \geq 7$ that

$$q(\hat{K}^3_n) = \Delta(\hat{K}^3_n) + \lceil \tfrac{n-4}{4} \rceil .$$

BERGE & JOHNSON [11] showed that for $k = 4$ and $n \geq 9$ that

$$\text{if } n \equiv 1 \pmod 4 \text{ then } q(\hat{K}^4_n) = \Delta(\hat{K}^4_n) + \left\lceil \frac{n(n-5)}{9} \right\rceil ,$$

$$\text{if } n \equiv 2 \pmod 4 \text{ then } q(\hat{K}^4_n) = \Delta(\hat{K}^4_n) + \left\lceil \frac{n(n-7)}{6} \right\rceil .$$

They also showed that $\hat{K}^r_{n_1,\ldots,n_r}$ has the edge-colouring property.

When parallelisms exist we may study them as geometrical objects, or look for parallelisms with special properties (cf. CAMERON [19]). Let $\{F_i \mid i \leq q\}$ be a fixed parallelism on (X,E). We say that Y is a *subspace* of X when $Y \subset X$ and for each i the collection $\{F \mid F \in F_i \text{ and } F \subset Y\}$ is either empty or a partition of Y. In this case the non-empty ones among these collections form a parallelism on (Y,E_Y) where $E_Y = \{E \mid E \in E \text{ and } E \subset Y\}$. (In geometrical terms: Y is a subspace of X when for $y \in Y$ and $E \subset Y$ the unique line F containing y and parallel to E is contained entirely within Y.)

Now let $(X,E) = K^k_n$. By Theorem 1 a parallelism exists iff $k \mid n$. Let Y be a proper subspace, and $|Y| = m$. CAMERON [19] showed that $m \leq \frac{1}{2}n$ (since the $\binom{m-1}{k-1}$ colours used to colour $P_k(Y)$ colour $\frac{n-m}{k}\binom{m-1}{k-1}$ k-subsets of $X \setminus Y$, so that $\frac{n-m}{k}\binom{m-1}{k-1} \leq \binom{n-m}{k}$, hence $\binom{m-1}{k-1} \leq \binom{n-m-1}{k-1}$ and consequently $m \leq n-m$). Conversely it seems to be true that $2|Y| \leq |X|$ and $|X| - |Y| \equiv 0 \pmod k$ suffices to guarantee the existence of a parallelism on (the k-subsets of) X with subspace Y. BARANYAI & BROUWER [6] proved this for $k \leq 3$ and for arbitrary k, when $n \geq mk$ or $m \mid n$. In case $m \mid n$ there even exists a parallelism on X with $\frac{n}{m}$ disjoint subspaces of size m.

EXERCISE 5. (WILSON [81]) Show that for $k = 2$ the existence of a parallelism on K_n with a subparallelism on K_m for $n \geq 2m$ is equivalent to the fact (proved by CRUSE [22]) that any symmetric Latin square of order m can be embedded in a symmetric Latin square of order n iff $n \geq 2m$.

EXAMPLE. An interesting example of a parallelism on 24 points is obtained from the Steiner system $S(5,8,24)$. Take as parallel classes all partitions of the 24 points into 6 4-sets with the property that the union of any two of the 4-sets is a block in the Steiner system. There are $\binom{23}{3}$ such partitions, and they form a parallelism. Each block of the Steiner system is a subspace of this parallelism.

3.6. Baranyai's method

Baranyai (see BARANYAI [3],[4],[5] and BROUWER [16]) proved a large number of very general theorems (sometimes so general as to be almost unintelligible) all to the effect that if certain matrices exist then hypergraphs exist of which the valency pattern and cardinalities are described by those matrices. An example is

THEOREM 11. *Let* $|X| = n$, $H = (X,E)$ *where* $E = \sum_{i=1}^{s} P_{k_i}(X)$ *(the* k_i *not necessarily different). Let* $A = (a_{ij})$ *be an* $s\times t$*-matrix with nonnegative integral entries such that for its row sums* $\sum_{j=1}^{t} a_{ij} = \binom{n}{k_i}$ *holds. (For* $k < 0$ *or* $k > n$ *we read* $\binom{n}{k} = 0$.)

Then there exist hypergraphs $H_{ij} = (X,E_{ij})$ *such that*

(i) $|E_{ij}| = a_{ij}$,

(ii) $P_{k_i}(X) = \sum_{j=1}^{t} E_{ij}$ $(1 \le i \le s)$,

(iii) $(X,\sum_{i=1}^{s} E_{ij})$ *is almost regular* $(1 \le j \le t)$.

Note that for $k_1 = \ldots = k_s = k$ this implies Theorem 4. If ℓ is an integer, let $\ell \approx d$ (and $d \approx \ell$) denote that either $\ell = \lfloor d \rfloor$ or $\ell = \lceil d \rceil$ holds. We first give some lemmas.

LEMMA 1. *For integral* A *we have*

$$\left\lfloor \frac{A}{n} \right\rfloor = \left\lfloor \frac{A-\lceil A/n \rceil}{n-1} \right\rfloor \quad \textit{and} \quad \left\lceil \frac{A}{n} \right\rceil = \left\lceil \frac{A-\lfloor A/n \rfloor}{n-1} \right\rceil.$$

Lemma 1 is an easy exercise in calculus.

LEMMA 2. *Let* $H = (X,E)$ *and* $a \in X$. *Then* H *is almost regular iff* $H_{X\setminus\{a\}}$ *is almost regular and* $\delta_H(a) \approx \frac{1}{n} \sum_{E\in F} |E|$.

This can be proved by using Lemma 1.

LEMMA 3. *Let* (ε_{ij}) *be a matrix with real entries. Then there exists a matrix* (e_{ij}) *with integral entries such that*

(i) $e_{ij} \approx \varepsilon_{ij}$ for all i,j,

(ii) $\sum_i e_{ij} \approx \sum_i \varepsilon_{ij}$ for all j,

(iii) $\sum_j e_{ij} \approx \sum_j \varepsilon_{ij}$ for all i,

(iv) $\sum_{i,j} e_{ij} \approx \sum_{i,j} \varepsilon_{ij}$.

PROOF. This follows straightforwardly from Ford & Fulkerson's Integer flow theorem (subsection 3.1). □

PROOF OF THEOREM 11. By induction on $n = |X|$. If $n = 0$ the theorem is true. The induction step consists of one application of Lemma 3. We may suppose that for $i \le s$ we have $0 \le k_i \le n$. Let $\varepsilon_{ij} = \frac{k_i}{n} a_{ij}$, the average degree of the hypergraph (X, E_{ij}) we want to construct.

By Lemma 3 there exist nonnegative integers e_{ij} with $\sum_j e_{ij} = \binom{n-1}{k_i-1}$, $\sum_j (a_{ij}-e_{ij}) = \binom{n-1}{k_i}$ and $\sum_i e_{ij} \approx \frac{1}{n} \sum_i k_i a_{ij}$.

Let $a \in X$ and apply the induction hypothesis to $X' = X\setminus\{a\}$ with $s' = 2s$, $t' = t$, $k'_i = k_i$, $k'_{i+s} = k_i-1$ $(1 \le i \le s)$, $a'_{ij} = a_{ij} - e_{ij}$, $a'_{(i+s)j} = e_{ij}$.

(That this is the proper thing to do is seen by reasoning backward: when we have E_{ij} and then remove the point a, E_{ij} is split up into the class of edges that remain of size k_i and the class of edges that have now size k_i-1. The latter class has cardinality ε_{ij} on the average.)

By the induction hypothesis we find hypergraphs F_{ij} and G_{ij} such that

$$|F_{ij}| = a_{ij}-e_{ij}, \qquad |G_{ij}| = e_{ij},$$

$$\textstyle\sum_j F_{ij} = P_{k_i}(X), \qquad \sum_j G_{ij} = P_{k_i-1}(X),$$

$$\textstyle\sum_i (F_{ij}+G_{ij}) \text{ is almost regular.}$$

Defining $E_{ij} = F_{ij} \cup \{G\cup\{a\} \mid G \in G_{ij}\}$ we are done (using Lemma 2). □

SKETCH OF THE PROOF OF THEOREM 8.

(i) The 'only if' part rests on estimates of (sums of) binomial coefficients. E.g., if $n > 3k$ and $n \not\equiv 0$ or $-1 \pmod k$ then a parallelism cannot exist since each parallel class (colour) must contain at least one edge of size at most $k-2$ but $\sum_{i\le k-2}\binom{n}{i} < \binom{n-1}{k-1}$, so that there are not enough small sets.

(ii) The 'if' part follows from Theorem 11: Let $\Delta = \sum_{i\le k}\binom{n-1}{i-1}$ be the degree of $\hat{K}^k_n$. If there exists a $\Delta \times k$-matrix D such that

(i) D has nonnegative integral entries,

(ii) $\sum_{j=1}^k d_{ij} j = n$ for all $i \le \Delta$,

(iii) $\sum_{i=1}^{\Delta} d_{ij} = \binom{n}{j}$ for all $j \le k$, then $\hat{K}_n^k$ has a parallelism (the proof is an exercise). It turns out that in all cases a suitable matrix D can be found (or at least it can be proved to exist). □

A more general multipartite version (see BROUWER [16] for the regular case, BARANYAI [5] for the almost regular case) is:

THEOREM 12. *Let* $n_1,\ldots,n_r$ *be positive integers, and let* $K = (k_{tj})_{t\le r, j\le s}$ *be a matrix of integers, where* $0 \le k_{tj} \le n_t$ $(t \le r)$. *Let* $\mathcal{Q} = \{Q_1,\ldots,Q_p\}$ *be a partition of* $\{1,2,\ldots,s\}$, *and suppose that*

$$\#\{j \mid j \in Q_i,\ (k_{1j},k_{2j},\ldots,k_{rj}) = (k_1,k_2,\ldots,k_r)\} \le \Pi_{i=1}^{r}\binom{n_i}{k_i}$$

for all $i \le p$ *and all integer vectors* $(k_1,k_2,\ldots,k_r)$.

Then there exist (0,1)-*matrices* $(e_{tj\ell})_{j\le s,\ell\le n_t}$ *for* $t \le r$ *such that*

(i) $\sum_{\ell=1}^{n_t} e_{tj\ell} = k_{tj}$ *for all* t,j,

(ii) *the vectors* $(e_{tj\ell})_{t\le r,\ell\le n_t}$ *are different for* $j \in Q_i$,

(iii) *the matrices* $(e_{tj\ell})_{\ell\le n_t, j\le s}$ *are almost regular for all* t, *that is,* $|\sum_{j=1}^{s} e_{tj\ell} - \sum_{j=1}^{s} e_{tj\ell'}| \le 1$ *for* $\ell,\ell' \le n_t$.

Even more generally, for each t let $\mathcal{F}_t$ be a *forest* (or *laminar*) *hypergraph* on the set $\{1,2,\ldots,s\}$ (i.e. a hypergraph such any two of its edges are disjoint or comparable). Then we may also require that all matrices $(e_{tj\ell})_{\ell\le n_t, j\in F}$ are almost regular, for all $F \in \mathcal{F}_t$, $t \le r$.

The proof is similar to that of Theorem 11 (use induction on r). The results about the existence of parallelism with subspaces of a given size follow as corollaries of this theorem.

4. PARTITIONING INTO INTERSECTING FAMILIES

Let n and k be natural numbers such that $n \ge 2k$, and let X be an n-set. Call a subset $\mathcal{A}$ of $\mathcal{P}_k(X)$ a *clique* if any two elements of $\mathcal{A}$ intersect. This section is concerned with the question of determining the minimal number of cliques needed to cover $\mathcal{P}_k(X)$, and with related questions.

As stated in the Introduction to this chapter, the minimal number of cliques needed to cover $\mathcal{P}_k(X)$ must be at least $\lceil n/k \rceil$ and at most $n-2k+2$. KNESER's conjecture [62] is that $n-2k+2$ indeed is the minimal number. This problem can be visualized by considering the Kneser-graph $K(n,k)$ (cf. the

Introduction): Kneser conjectured that the chromatic number $\gamma(K(n,k))$ of $K(n,k)$ is equal to $n-2k+2$.

For $k = 1$ or 2, Kneser's conjecture is easy to prove; GAREY & JOHNSON [44] proved the conjecture for $k = 3$. In 1977 LOVÁSZ [65] was able to prove Kneser's conjecture for general k, using algebraic topology and Borsuk's antipodal theorem; also in 1977 BÁRÁNY [2] showed that Kneser's conjecture immediately follows from Borsuk's theorem and a theorem of Gale from 1956. Below we give Bárány's proof. First we give the two ingredients of the proof.

Let S^d be the d-dimensional sphere, i.e. $S^d = \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$. Borsuk's antipodal theorem [15] says that *if* S^d *is covered with* $d+1$ *closed subsets, then one of these subsets contains two antipodal points* (for a proof see DUGUNDJI [27]). Simple topological arguments show that in Borsuk's theorem we may replace "closed" by "open". [Borsuk's theorem is also equivalent to the assertion that for each $\varepsilon > 0$, the chromatic number of the Borsuk-graph $B(d,\varepsilon)$ is at least $d+2$, where the *Borsuk-graph* $B(d,\varepsilon)$ has vertex-set S^d, two vertices being adjacent iff their euclidean distance is at least $2-\varepsilon$ (in fact $\gamma(B(d,\varepsilon)) = d+2$ if ε is small enough).]

GALE's theorem [43] states that *one can choose* $2k+d$ *points on* S^d *such that each open hemisphere contains at least* k *of these points*. PETTY [71] (cf. SCHRIJVER [72]) found that one can take these points to be $w_1,\ldots,w_{2k+d} \in S^d$, where

$$w_i = \frac{v_i}{\|v_i\|}, \quad \text{and} \quad v_i = (-1)^i(i^0,i^1,\ldots,i^d) \in \mathbb{R}^{d+1},$$

for $i = 1,2,3,\ldots$ (The proof consists of showing that for each non-zero real polynomial $p(x)$ of degree at most d there exist n distinct natural numbers i between 1 and $2k+d$ such that $(-1)^i p(i) > 0$, which is not hard.)

We now prove Lovász's Kneser-theorem with Bárány's method.

THEOREM 1. (LOVÁSZ [65]) *The minimal number of clique needed to cover* $P_k(X)$ *is equal to* $n-2k+2$.

PROOF. Let $d = n-2k$. Suppose we could divide $P_k(X)$ into $n-2k+1 = d+1$ cliques, say $A_1,\ldots,A_{d+1}$. We may assume that X is embedded in S^d so that any open hemisphere of S^d contains at least k points of X (Gale's theorem). Define the open subsets $U_1,\ldots,U_{d+1}$ of S^d by

$$U_i = \{x \in S^d \mid \text{the open hemisphere with centre } x \text{ contains a } k\text{-subset of } X \text{ which is an element of } A_i\}.$$

So $S^d = U_1 \cup \ldots \cup U_{d+1}$ and hence by Borsuk's theorem one of the sets, say U_i, contains two antipodal points. But these antipodal points are the centres of disjoint open hemispheres, each containing a k-subset in A_i. These k-sets are necessarily disjoint, contradicting the fact that A_i is a clique. □

Using Bárány's method SCHRIJVER [72] showed that the set of all stable k-subsets of a circuit with n vertices (a subset is *stable* if it contains no two neighbours) constitutes a minimal subcollection of $\mathcal{P}_k(X)$ which cannot be divided into n-2k+1 cliques (identifying X with the set of vertices of the circuit); in other words, the subgraph of K(n,k) induced by the stable subsets is (n-2k+2)-vertex-critical.

An interesting extension of Kneser's conjecture was raised by STAHL [75]. Define for each graph G and for each natural number ℓ the *ℓ-chromatic number* $\gamma_\ell(G)$ by

> $\gamma_\ell(G)$ is the minimal number of colours needed to give each vertex of G ℓ colours such that no colour occurs at two adjacent vertices.

Otherwise stated, $\gamma_\ell(G)$ is the minimal number of stable subsets of the vertex set of G such that each vertex occurs in at least ℓ of them.

First observe that $\gamma_\ell(G) \le n$ if and only if

$$G \to K(n,\ell),$$

where the (ad hoc) notation $G \to H$ stands for: there is a function ϕ from the vertex set V(G) of G into the vertex set V(H) of H such that if v and w are adjacent vertices of G then $\phi(v)$ and $\phi(w)$ are adjacent in H (in particular, $\phi(v) \neq \phi(w)$).

Stahl showed that

$$K(n,k) \to K(n-2,k-1),$$

for each n and k, from which it follows that for any graph G

$$\gamma_k(G) \ge \gamma_{k-1}(G) + 2. \tag{1}$$

(Stahl showed $K(n,k) \to K(n-2,k-1)$ as follows. Assume K(n,k) (K(n-2,k-1), respectively) has vertices all k-subsets ((k-1)-subsets, respectively) of

$\{1,\ldots,n\}$ ($\{1,\ldots,n-2\}$, respectively). Now define

$$\phi(A) = \{i \in \{1,\ldots,n-2\} \mid j \in A \text{ for all } j = i+1,\ldots,n, \text{ or } i \in A \text{ and } j \in A \text{ for some } j > i\},$$

for all k-subsets A of $\{1,\ldots,n\}$. Then ϕ has the required properties.) Since $\gamma_1(K(n,k)) = n-2k+2$ (Kneser's conjecture) and $\gamma_k(K(n,k) = n$ (since, by the Erdös-Ko-Rado theorem, each colour class contains at most $\binom{n-1}{k-1}$ vertices), it follows from (1) that, for $1 \leq \ell \leq k$,

$$\gamma_\ell(K(n,k)) = n-2k+2\ell.$$

STAHL [75] conjectures that, in general,

$$\gamma_\ell(K(n,k)) = \left\lceil \frac{\ell}{k} \right\rceil (n-2k) + 2\ell. \tag{2}$$

Again by using the Erdös-Ko-Rado theorem one can prove the validity of (2) if ℓ is a multiple of k. By (1) the right hand side of (2) is an upper bound for $\gamma_\ell(K(n,k))$. Also by (1) it is sufficient to show (2) for $\ell \equiv 1 \pmod k$.

Stahl proved (2) in case $n = 2k$ or $n = 2k+1$ (cf. also GELLER & STAHL [45]); moreover GAREY & JOHNSON [44] proved (2) for $k = 3$, $\ell = 4$.

Some asymptotic results were also obtained. Stahl showed that if ℓ is large with respect to n and k then $\gamma_{\ell+k}(K(n,k)) = n + \gamma_\ell(K(n,k))$, so for fixed n and k we have to prove (2) for only a finite number of ℓ. CHVÁTAL, GAREY & JOHNSON [21] showed (using Hilton and Milner's result of subsection 2.2) that if n is large with respect to k then $\gamma_{k+1}(K(n,k)) = \gamma_{k+1}(K(n-1,k))+2$ so for fixed k and $\ell = k+1$ it is sufficient to prove (2) for only a finite number of n.

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