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THE LINKING OF MATROIDS BY LINKING SYSTEMS

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With the help of the concept of a linking system theorems relating matroids with bipartite and directed graphs can be deduced. In this way one obtains natural generalizations of theorems of Brualdi, Edmonds & Fulkerson, Mason, Perfect, Pym and Rado. (For a survey on theorems on "matroids induced by directed graphs" see Brualdi [3].) Here I give one such generalization.

First I give the definition of a matroid and I mention one result from matroid theory, namely Edmonds' intersection theorem [5], since this theorem is used in the proofs of Theorems 1 and 2.

A *matroid* is a pair $(X, \mathscr{I})$, where X is a finite set and $\phi \neq \mathscr{I} \subset \subset \mathscr{P}(X)$, such that

(i) if $X'' \subset X' \in \mathscr{I}$ then $X'' \in \mathscr{I}$, and

(ii) if $X' \in \mathscr{I}$, $X'' \in \mathscr{I}$ and $|X'| < |X''|$ then $X' \cup \{x\} \in \mathscr{I}$ for some $x \in X'' \setminus X'$.

The elements of $\mathscr{I}$ are called the *independent* sets of the matroid. The matroid $(X, \mathscr{I})$ determines a *rank function* $\rho: \mathscr{P}(X) \to Z$, where for each subset X' of X the *rank* $\rho(X')$ equals the maximal cardinality of an independent subset contained in X'.

Matroids can be obtained, inter alia, from graphs (in some different ways) and linear spaces (see, e.g., Welsh [20]); matroid theory has applications in graph theory, transversal theory, network analysis, operations research. One of the matroid theoretical results is Edmonds' intersection theorem [5]: *let* $(X, \mathscr{I}_1)$ *and* $(X, \mathscr{I}_2)$ *be matroids, with rank functions* ρ_1 *and* ρ_2, *respectively. Then the maximal cardinality of a common independent set (that is a set in* $\mathscr{I}_1 \cap \mathscr{I}_2$*) equals*

$$\min_{X' \subset X} (\rho_1(X') + \rho_2(X \setminus X')).$$

For a proof of this theorem we refer to Welsh [19].

A second structure we need is that of a linking system, defined as follows.

A *linking system* is a triple (X, Y, Λ), where X and Y are finite sets and $\phi \neq \Lambda \subset \mathscr{P}(X) \times \mathscr{P}(Y)$, such that:

(i) if $(X', Y') \in \Lambda$ then $|X'| = |Y'|$;

(ii) if $(X', Y') \in \Lambda$ and $X'' \subset X'$, then $(X'', Y'') \in \Lambda$ for some $Y'' \subset Y'$;

(iii) if $(X', Y') \in \Lambda$ and $Y'' \subset Y'$, then $(X'', Y'') \in \Lambda$ for some $X'' \subset X'$;

(iv) if $(X_1, Y_1) \in \Lambda$ and $(X_2, Y_2) \in \Lambda$ then there is a $(X', Y') \in \Lambda$ such that $X_1 \subset X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$.

A linking system determines a *linking function* $\lambda: \mathscr{P}(X) \times \mathscr{P}(Y) \to Z$, where, for $X' \subset X$ and $Y' \subset Y$,

$$\lambda(X', Y') = \max \{|X''| \mid X'' \subset X',\ Y'' \subset Y' \text{ and } (X'', Y'') \in \Lambda\}.$$

This notion is comparable with that of the rank function of a matroid.

Examples of linking systems may be obtained as follows.

(a) Let (X, Y, E) be a bipartite graph and let Λ be the set of all pairs (X', Y'), such that $X' \subset X$, $Y' \subset Y$ and X' and Y' are matched in the bipartite graph (that is, there is a bijection $f\colon X' \to Y'$ such that $(x, f(x)) \in E$ for all x in X'). Then (X, Y, Λ) is a linking system. The axioms (i), (ii) and (iii) for a linking system are verified easily; axiom (iv) is implied by a theorem of Ore [10] (cf. Perfect & Pym [13]). By König's theorem the linking function λ of this linking system is given, for $X' \subset X$ and $Y' \subset Y$, by

$$\lambda(X', Y') = \min_{Y'' \subset Y'} (|E(Y'') \cap X'| + |Y' \setminus Y''|),$$

where $E(Y'') = \{x \in X \mid (x, y) \in E \text{ for some } y \in Y''\}$, for $Y'' \subset Y$.

(b) Let (Z, E) be a directed graph and let $X \subset Z$ and $Y \subset Z$. Let furthermore Λ be the collection of all pairs (X', Y') such that $X' \subset X$, $Y' \subset Y$, $|X'| = |Y'|$ and there are $|X'|$ pairwise vertex-disjoint paths starting in X' and ending in Y'. (Note that a path may consist of a single vertex.) Then (X, Y, Λ) is a linking system. Again, the axioms (i), (ii) and (iii) follow straightforwardly; axiom (iv) follows from the "linkage theorem" of Pym [14] (cf. Brualdi & Pym [4]). Let λ be the linking function of this linking system. Then, by Menger's theorem, for $X' \subset X$ and $Y' \subset Y$, $\lambda(X', Y')$ equals the minimal cardinality of a subset of Z intersecting each path from X' to Y'. Also it is true that

$$\lambda(X', Y') = \min_{Z' \subset Z \setminus Y'} (|E(Z') \cup X'| - |Z'|),$$

where $E(Z') = Z' \cup \{z \in Z \mid (z', z) \in E \text{ for some } z' \in Z'\}$, for $Z' \subset Z$.

(c) Let M be a matrix over some field, with collection of rows X and collection of columns Y. Let Λ be the collection of all pairs (X', Y') such that $X' \subset X$, $Y' \subset Y$ and the submatrix of M generated by the rows X' and the columns Y' is regular. Then, using simple linear algebraic methods, one proves that (X, Y, Λ) is a linking system. Clearly, its linking function λ is such that $\lambda(X', Y')$ equals the rank of the submatrix of M generated by the rows X' and the columns Y' ($X' \subset X$, $Y' \subset Y$).

Linking systems and matroids have close relations; one of these relations is shown here in two theorems.

Theorem 1. *Let* $(X, \mathscr{I})$ *be a matroid, with rank function* ρ, *and let* (X, Y, Λ) *be a linking system, with linking function* λ. *Define*

$$\mathscr{I} * \Lambda = \{Y' \subset Y \mid (X', Y') \in \Lambda \text{ for some } X' \in \mathscr{I}\}.$$

Then $(Y, \mathscr{I} * \Lambda)$ *is again a matroid. The rank function* $\rho * \lambda$ *of this matroid is given by*

$$(\rho * \lambda)(Y') = \min_{X' \subset X} (\rho(X \setminus X') + \lambda(X', Y')).$$

Proof. We first prove that $(Y, \mathscr{I} * \Lambda)$ is a matroid. Since $\phi \in \mathscr{I}$ and $(\phi, \phi) \in \Lambda$ it follows that $\mathscr{I} * \Lambda \neq \phi$. Furthermore, if $Y'' \subset Y' \in \mathscr{I} * \Lambda$ then $(X', Y') \in \Lambda$ for some $X' \in \mathscr{I}$. By axiom (iii) of a linking system there exists an $X'' \subset X'$ such that $(X'', Y'') \in \Lambda$. Since $X'' \subset X'$, also $X'' \in \mathscr{I}$, which implies $Y'' \in \mathscr{I} * \Lambda$.

If $Y' \in \mathscr{I} * \Lambda$, $Y'' \in \mathscr{I} * \Lambda$ and $|Y'| < |Y''|$, we have to prove that $Y' \cup \{y\} \in \mathscr{I} * \Lambda$ for some $y \in Y'' \setminus Y'$. Let $X' \in \mathscr{I}$ and $X'' \in \mathscr{I}$, such that $(X', Y') \in \Lambda$, $(X'', Y'') \in \Lambda$ and $|X' \cap X''|$ is as large as possible. As $|X'| = |Y'| < |Y''| = |X''|$, there is an $x \in X'' \setminus X'$, such that $X' \cup \{x\} \in \mathscr{I}$. Now $(X' \cap X'') \cup \{x\} \subset X''$, hence, by axiom (ii) of the definition of a linking system, there exists a subset Y''' of Y'' such that $((X' \cap X'') \cup \{x\}, Y''') \in \Lambda$. By axiom (iv) there is a pair (X_0, Y_0) in Λ such that:

$$(X' \cap X'') \cup \{x\} \subset X_0 \subset X' \cup \{x\} \quad \text{and} \quad Y' \subset Y_0 \subset Y' \cup Y'''.$$

Now it is not possible that $Y_0 = Y'$, for in this case $(X_0, Y') \in \Lambda$, $X_0 \in \mathscr{I}$ and $|X_0 \cap X''| > |X' \cap X''|$, contradicting the maximality of $|X' \cap X''|$. Therefore $|Y_0| > |Y'|$, hence $X_0 = X' \cup \{x\}$ and $Y_0 = Y' \cup \{y\}$ for some $y \in Y''' \setminus Y'$. It follows that $Y' \cup \{y\} \in \mathscr{I} * \Lambda$. Therefore, $(Y, \mathscr{I} * \Lambda)$ is a matroid.

Next we show that the rank $(\rho * \lambda)(Y')$ of a set $Y' \subset Y$ equals $\min_{X' \subset X} (\rho(X \setminus X') + \lambda(X', Y'))$.

This is done using Edmonds' intersection theorem. By the same method as in the first part of this proof one shows that, if $\mathscr{J} = \{X' \subset \subset X \mid (X', Y'') \in \Lambda$ for some $Y'' \subset Y'\}$, the pair $(X, \mathscr{J})$ is a matroid. Here one uses the matroid on Y where each subset of Y' is independent. It is clear that in the matroid $(X, \mathscr{J})$ the rank of a subset X' of X equals $\lambda(X', Y')$. Now by Edmonds' intersection theorem:

$$(\rho * \lambda)(Y') = \max\{|X'| \mid (X', Y'') \in \Lambda \text{ for some } Y'' \subset Y' \text{ and } X' \in \mathscr{I}\} =$$
$$= \max\{|X'| \mid X' \in \mathscr{J} \text{ and } X' \in \mathscr{I}\} =$$
$$= \min_{X' \subset X} (\rho(X \setminus X') + \lambda(X', Y')),$$

which was to be demonstrated.∎

Besides results of Edmonds & Fulkerson [6] (if (X, Y, E) is a bipartite graph and $\mathscr{J} = \{Y' \subset Y \mid$ there is a matching in E between some subset of X and $Y'\}$, then $(Y, \mathscr{J})$ is a matroid) and Perfect [11] and Pym [15] (if (Z, E) is a digraph, $X, Y \subset Z$ and $\mathscr{J} = \{Y' \subset Y \mid$ there are $|Y'|$ pairwise vertex-disjoint paths in E starting in X and ending in $Y'\}$, then $(Y, \mathscr{J})$ is a matroid), we have the following corollaries.

Corollary 1a (Perfect [12], Rado [16]). *Let (X, Y, E) be a bipartite graph and let $(X, \mathscr{I})$ be a matroid, with rank function ρ. Let $\mathscr{J}$ be the collection of all subsets of Y, which are matched with some $X' \in \mathscr{I}$. Then $(Y, \mathscr{J})$ is again a matroid. The rank $\sigma(Y')$ of a subset Y' of Y equals* $\min_{Y'' \subset Y'} (\rho(E(Y'')) + |Y' \setminus Y''|)$.

Proof. Straightforward by applying Theorem 1 to example (a). Observe that

$$\min_{X' \subset X} (\rho(X \setminus X') + |E(Y'') \cap X'|) = \rho(E(Y'')), \quad \text{for} \quad Y'' \subset Y. ∎$$

Corollary 1b (Brualdi [2], Mason [9]). *Let (Z, E) be a directed graph and $X, Y \subset Z$. Let $(X, \mathscr{I})$ be a matroid, with rank function ρ. Let $\mathscr{J}$ be the set of all subsets Y' of Y such that there are $|Y'|$ pairwise vertex-disjoint paths starting in some independent sub-*

set X' *of* X *and ending in* Y'. *Then* $(Y, \mathscr{J})$ *is again a matroid. The rank of a subset* Y' *of* Y *equals* $\min\limits_{Z' \subset Z \setminus Y'} (\rho(X \setminus E(Z')) + |E(Z')| - - |Z'|)$.

Proof. This is again clear in the light of example (b) and Theorem 1. In this case we need the following obvious identity:

$$\min_{X' \subset X} (\rho(X \setminus X') + |E(Z') \cup X'|) =$$

$$= \rho(X \setminus E(Z')) + |E(Z')|, \quad \text{for} \quad Z' \subset Z. \blacksquare$$

Corollary 1c. *Let* M *be a matrix over some field, with row collection* X *and column collection* Y. *Let furthermore* $(X, \mathscr{I})$ *be a matroid. Define* $\mathscr{J}$ *as the collection of all subsets* Y' *of* Y *such that, for some* $X' \in \mathscr{I}$, *the submatrix of* M *generated by the rows* X' *and the columns* Y' *is regular. Then* $(Y, \mathscr{J})$ *is again a matroid.*

Proof. Apply Theorem 1 to example (c).$\blacksquare$

The second theorem generalizes a theorem of Brualdi [1] (see Corollary 2a).

Theorem 2. *Let* (X, Y, Λ) *be a linking system, with linking function* λ, *and let* $(X, \mathscr{I})$ *and* $(Y, \mathscr{J})$ *be matroids, with rank function* ρ *and* σ, *respectively. Then the maximal cardinality of a set* $Y' \in \mathscr{J}$ *such that for some* $X' \in \mathscr{I}$ *the pair* (X', Y') *is in* Λ, *equals*

$$\min_{X' \subset X,\, Y' \subset Y} (\rho(X \setminus X') + \lambda(X', Y') + \sigma(Y \setminus Y')).$$

Proof. If a pair $(X', Y') \in \Lambda$ is such that $X' \in \mathscr{I}$ and $Y' \in \mathscr{J}$, then $Y' \in \mathscr{I} * \Lambda$ (cf. Theorem 1). Hence the maximal cardinality of a set $Y' \in \mathscr{J}$ such that $(X', Y') \in \Lambda$ for some $X' \in \mathscr{I}$ equals $\max\{|Y'| \mid Y' \in \mathscr{J}$ and $Y' \in \mathscr{I} * \Lambda\}$; by Edmonds' intersection theorem this is

$$\min_{Y' \subset Y} ((\rho * \lambda)(Y') + \sigma(Y \setminus Y')),$$

and by Theorem 1 this equals

$$\min_{X' \subset X,\, Y' \subset Y} (\rho(X \setminus X') + \lambda(X', Y') + \sigma(Y \setminus Y')).\blacksquare$$

This theorem has the following corollaries.

Corollary 2a (Brualdi [1]). *Let (X, Y, E) be a bipartite graph and let $(X, \mathscr{I})$ and $(Y, \mathscr{J})$ be matroids, with rank functions ρ and σ, respectively. Then the maximal cardinality of a set $Y' \in \mathscr{J}$ which is matched in E with some $X' \in \mathscr{I}$ equals*

$$\min_{Y' \subset Y} (\rho(E(Y')) + \sigma(Y \setminus Y')).$$

Proof. Apply Theorem 2 to example (a) and observe that

$$\min_{X' \subset X} (\rho(X \setminus X') + |E(Y'') \cap X'|) = \rho(E(Y'')),$$

and

$$\min_{Y' \supset Y''} (|Y' \setminus Y''| + \sigma(Y \setminus Y')) = \sigma(Y \setminus Y''),$$

for $Y'' \subset Y.\blacksquare$

Corollary 2b. *Let (Z, E) be a directed graph and let X and Y be subsets of Z. Let $(X, \mathscr{I})$ and $(Y, \mathscr{J})$ be matroids, with rank functions ρ and σ, respectively. Then the maximal cardinality of a set Y' in $\mathscr{J}$ such that there are $|Y'|$ pairwise vertex-disjoint paths starting in an X' in $\mathscr{I}$ and ending in Y' equals*

$$\min_{Z' \subset Z} (\rho(X \setminus E(Z')) + |E(Z')| - |Z'| + \sigma(Z' \cap Y)).$$

Proof. Apply Theorem 2 to example (b) and observe that

$$\min_{X' \subset X} (\rho(X \setminus X') + |E(Z') \cup X'|) = \rho(X \setminus E(Z')) + |E(Z')|.\blacksquare$$

Other theorems of Mason [7], [8] and Brualdi [2] on matroids and graphs have also their generalizations to theorems on matroids and linking systems. For more details see [17] and [18].

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