THE LINKING OF MATROIDS BY LINKING SYSTEMS

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With the help of the concept of a linking system theorems relating
matroids with bipartite and directed graphs can be deduced. In this way
one obtains natural generalizations of theorems of Brualdi, Edmonds &
Fulkerson, Mason, Perfect, Pym and Rado. (For a survey on theorems on
"matroids induced by directed graphs" see Brualdi [3].) Here I give
one such generalization.

First I give the definition of a matroid and I mention one result from
matroid theory, namely Edmonds' intersection theorem [5], since this
theorem is used in the proofs of Theorems 1 and 2.

A matroid is a pair \((X, \mathcal{I})\), where \(X\) is a finite set and \(\emptyset \neq \mathcal{I} \subseteq \mathcal{P}(X)\), such that

(i) if \(X' \subseteq X'' \in \mathcal{I}\) then \(X'' \in \mathcal{I}\), and

(ii) if \(X' \in \mathcal{I}, X'' \in \mathcal{I}\) and \(|X'| < |X''|\) then \(X' \cup \{x\} \in \mathcal{I}\) for
some \(x \in X'' \setminus X'\).

The elements of \(\mathcal{I}\) are called the independent sets of the matroid. The
matroid \((X, \mathcal{S})\) determines a rank function \(\rho: \mathcal{P}(X) \rightarrow \mathbb{Z}\), where for each subset \(X'\) of \(X\) the rank \(\rho(X')\) equals the maximal cardinality of an independent subset contained in \(X'\).

Matroids can be obtained, inter alia, from graphs (in some different ways) and linear spaces (see, e.g., Welsh [20]); matroid theory has applications in graph theory, transversal theory, network analysis, operations research. One of the matroid theoretical results is Edmonds’ intersection theorem [5]: let \((X, \mathcal{S}_1)\) and \((X, \mathcal{S}_2)\) be matroids, with rank functions \(\rho_1\) and \(\rho_2\), respectively. Then the maximal cardinality of a common independent set (that is a set in \(\mathcal{S}_1 \cap \mathcal{S}_2\)) equals

\[
\min_{X' \subseteq X} (\rho_1(X') + \rho_2(X \setminus X')).
\]

For a proof of this theorem we refer to Welsh [19].

A second structure we need is that of a linking system, defined as follows.

A linking system is a triple \((X, Y, \Lambda)\), where \(X\) and \(Y\) are finite sets and \(\phi \neq \Lambda \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)\), such that:

(i) if \((X', Y') \in \Lambda\) then \(|X'| = |Y'|\);

(ii) if \((X', Y') \in \Lambda\) and \(X'' \subseteq X'\), then \((X'', Y'') \in \Lambda\) for some \(Y'' \subseteq Y'\);

(iii) if \((X', Y') \in \Lambda\) and \(Y'' \subseteq Y'\), then \((X'', Y'') \in \Lambda\) for some \(X'' \subseteq X'\);

(iv) if \((X_1, Y_1) \in \Lambda\) and \((X_2, Y_2) \in \Lambda\) then there is a \((X', Y') \in \Lambda\) such that \(X_1 \subseteq X' \subseteq X_1 \cup X_2\) and \(Y_2 \subseteq Y' \subseteq Y_1 \cup Y_2\).

A linking system determines a linking function \(\lambda: \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathbb{Z}\), where, for \(X' \subseteq X\) and \(Y' \subseteq Y\),

\[
\lambda(X', Y') = \max \{|X''| \mid X'' \subseteq X', \ Y'' \subseteq Y' \text{ and } (X'', Y'') \in \Lambda\}.
\]

This notion is comparable with that of the rank function of a matroid.

Examples of linking systems may be obtained as follows.
(a) Let \((X, Y, E)\) be a bipartite graph and let \(\Lambda\) be the set of all pairs \((X', Y')\), such that \(X' \subseteq X\), \(Y' \subseteq Y\) and \(X'\) and \(Y'\) are matched in the bipartite graph (that is, there is a bijection \(f: X' \rightarrow Y'\) such that \((x, f(x)) \in E\) for all \(x\) in \(X'\)). Then \((X, Y, \Lambda)\) is a linking system. The axioms (i), (ii) and (iii) for a linking system are verified easily; axiom (iv) is implied by a theorem of Ore [10] (cf. Perfect & Pym [13]). By König's theorem the linking function \(\lambda\) of this linking system is given, for \(X' \subseteq X\) and \(Y' \subseteq Y\), by

\[
\lambda(X', Y') = \min_{Y'' \subseteq Y} (|E(Y'') \cap X'| + |Y' \setminus Y''|),
\]

where \(E(Y'') = \{x \in X\mid (x, y) \in E\text{ for some } y \in Y''\}\), for \(Y'' \subseteq Y\).

(b) Let \((Z, E)\) be a directed graph and let \(X \subseteq Z\) and \(Y \subseteq Z\). Let furthermore \(\Lambda\) be the collection of all pairs \((X', Y')\) such that \(X' \subseteq X\), \(Y' \subseteq Y\), \(|X'| = |Y'|\) and there are \(|X'|\) pairwise vertex-disjoint paths starting in \(X'\) and ending in \(Y'\). (Note that a path may consist of a single vertex.) Then \((X, Y, \Lambda)\) is a linking system. Again, the axioms (i), (ii) and (iii) follow straightforwardly; axiom (iv) follows from the "linkage theorem" of Pym [14] (cf. Brualdi & Pym [4]). Let \(\lambda\) be the linking function of this linking system. Then, by Menger's theorem, for \(X' \subseteq X\) and \(Y' \subseteq Y\), \(\lambda(X', Y')\) equals the minimal cardinality of a subset of \(Z\) intersecting each path from \(X'\) to \(Y'\). Also it is true that

\[
\lambda(X', Y') = \min_{Z' \subseteq Z \setminus Y'} (|E(Z') \cup X'| - |Z'|),
\]

where \(E(Z') = Z' \cup \{z \in Z\mid (z', z) \in E\text{ for some } z' \in Z'\}\), for \(Z' \subseteq Z\).

(c) Let \(M\) be a matrix over some field, with collection of rows \(X\) and collection of columns \(Y\). Let \(\Lambda\) be the collection of all pairs \((X', Y')\) such that \(X' \subseteq X\), \(Y' \subseteq Y\) and the submatrix of \(M\) generated by the rows \(X'\) and the columns \(Y'\) is regular. Then, using simple linear algebraic methods, one proves that \((X, Y, \Lambda)\) is a linking system. Clearly, its linking function \(\lambda\) is such that \(\lambda(X', Y')\) equals the rank of the submatrix of \(M\) generated by the rows \(X'\) and the columns \(Y'\) \((X' \subseteq X, Y' \subseteq Y)\).
Linking systems and matroids have close relations; one of these relations is shown here in two theorems.

Theorem 1. Let \((X, \mathcal{S})\) be a matroid, with rank function \(\rho\), and let \((X, Y, \Lambda)\) be a linking system, with linking function \(\lambda\). Define

\[
\mathcal{S} \ast \Lambda = \{Y' \subseteq Y \mid (X', Y') \in \Lambda \text{ for some } X' \in \mathcal{S}\}.
\]

Then \((Y, \mathcal{S} \ast \Lambda)\) is again a matroid. The rank function \(\rho \ast \lambda\) of this matroid is given by

\[
(\rho \ast \lambda)(Y') = \min_{X' \subseteq X} (\rho(X \setminus X') + \lambda(X', Y')).
\]

Proof. We first prove that \((Y, \mathcal{S} \ast \Lambda)\) is a matroid. Since \(\phi \in \mathcal{S}\) and \((\phi, \phi) \in \Lambda\), it follows that \(\mathcal{S} \ast \Lambda \neq \emptyset\). Furthermore, if \(Y'' \subseteq Y' \in \mathcal{S} \ast \Lambda\), then \((X', Y') \in \Lambda\) for some \(X' \in \mathcal{S}\). By axiom (iii) of the definition of a linking system there exists an \(X'' \subseteq X'\) such that \((X'', Y'') \in \Lambda\). Since \(X'' \subseteq X'\), also \(X'' \in \mathcal{S}\), which implies \(Y'' \in \mathcal{S} \ast \Lambda\).

If \(Y' \in \mathcal{S} \ast \Lambda\), \(Y'' \in \mathcal{S} \ast \Lambda\) and \(|Y'| < |Y''|\), we have to prove that \(Y' \cup \{y\} \in \mathcal{S} \ast \Lambda\) for some \(y \in Y'' \setminus Y'\). Let \(X' \in \mathcal{S}\) and \(X'' \in \mathcal{S}\), such that \((X', Y') \in \Lambda\), \((X'', Y'') \in \Lambda\) and \(|X' \cap X''|\) is as large as possible. As \(|X'| = |Y'| < |Y''| = |X''|\), there is an \(x \in X'' \setminus X'\), such that \(X' \cup \{x\} \in \mathcal{S}\). Now \((X' \cap X'') \cup \{x\} \subseteq X''\), hence, by axiom (ii) of the definition of a linking system, there exists a subset \(Y''\) of \(Y''\) such that \(((X' \cap X'') \cup \{x\}, Y'') \in \Lambda\). By axiom (iv) there is a pair \((X_0, Y_0)\) in \(\Lambda\) such that:

\[
(X' \cap X'') \cup \{x\} \subseteq X_0 \subseteq X' \cup \{x\} \quad \text{and} \quad Y' \subseteq Y_0 \subseteq Y' \cup Y''.
\]

Now it is not possible that \(Y_0 = Y'\), for in this case \((X_0, Y') \in \Lambda\), \(X_0 \in \mathcal{S}\) and \(|X_0 \cap X''| \geq |X' \cap X''|\), contradicting the maximality of \(|X' \cap X''|\). Therefore \(|Y_0| > |Y'|\), hence \(X_0 = X' \cup \{x\}\) and \(Y_0 = Y' \cup \{y\}\) for some \(y \in Y'' \setminus Y'\). It follows that \(Y' \cup \{y\} \in \mathcal{S} \ast \Lambda\). Therefore, \((Y, \mathcal{S} \ast \Lambda)\) is a matroid.

Next we show that the rank \((\rho \ast \lambda)(Y')\) of a set \(Y' \subseteq Y\) equals

\[
\min_{X' \subseteq X} (\rho(X \setminus X') + \lambda(X', Y')).
\]
This is done using Edmonds' intersection theorem. By the same method as in the first part of this proof one shows that, if \( \mathcal{J} = \{X' \subset X\} \) \((X', Y'') \in \Lambda \) for some \( Y'' \subset Y' \), the pair \((X, \mathcal{J})\) is a matroid. Here one uses the matroid on \( Y \) where each subset of \( Y' \) is independent. It is clear that in the matroid \((X, \mathcal{J})\) the rank of a subset \( X' \) of \( X \) equals \( \lambda(X', Y') \). Now by Edmonds' intersection theorem:

\[
(\rho \cdot \lambda)(Y') = \max \{ |X'| | (X', Y'') \in \Lambda \text{ for some } Y'' \subset Y' \text{ and } X' \in \mathcal{J} \} = \\
\max \{ |X'| | X' \in \mathcal{J} \text{ and } X' \in \mathcal{J} \} = \\
\min_{X' \subset X} (\rho(X \setminus X') + \lambda(X', Y')),
\]

which was to be demonstrated.

Besides results of Edmonds & Fulkerson [6] (if \((X, Y, E)\) is a bipartite graph and \( \mathcal{J} = \{Y' \subset Y\} \) there is a matching in \( E \) between some subset of \( X \) and \( Y' \)), then \((Y, \mathcal{J})\) is a matroid) and Perfect [11] and Py m [15] (if \((Z, E)\) is a digraph, \( X, Y \subset Z \) and \( \mathcal{J} = \{Y' \subset Y\} \) there are \(|Y'|\) pairwise vertex-disjoint paths in \( E \) starting in \( X \) and ending in \( Y' \)), then \((Y, \mathcal{J})\) is a matroid), we have the following corollaries.

**Corollary 1a** (Perfect [12], Rado [16]). Let \((X, Y, E)\) be a bipartite graph and let \((X, \mathcal{J})\) be a matroid, with rank function \( \rho \). Let \( \mathcal{J} \) be the collection of all subsets of \( Y \), which are matched with some \( X' \in \mathcal{J} \). Then \((Y, \mathcal{J})\) is again a matroid. The rank \( \sigma(Y') \) of a subset \( Y' \) of \( Y \) equals \( \min_{Y'' \subset Y'} (\rho(E(Y'')) + |Y' \setminus Y''|) \).

**Proof.** Straightforward by applying Theorem 1 to example (a). Observe that

\[
\min_{X' \subset X} (\rho(X \setminus X') + |E(Y'') \cap X'|) = \rho(E(Y'')) \quad \text{for} \quad Y'' \subset Y.
\]

**Corollary 1b** (Brualdi [2], Mason [9]). Let \((Z, E)\) be a directed graph and \( X, Y \subset Z \). Let \((X, \mathcal{J})\) be a matroid, with rank function \( \rho \). Let \( \mathcal{J} \) be the set of all subsets \( Y' \) of \( Y \) such that there are \(|Y'|\) pairwise vertex-disjoint paths starting in some independent sub-
set \( X' \) of \( X \) and ending in \( Y' \). Then \( (Y, \mathcal{F}) \) is again a matroid. The rank of a subset \( Y' \) of \( Y \) equals
\[
\min_{Z \subseteq Z \setminus Y'} (\rho(X \setminus E(Z')) + |E(Z')| - |Z'|).
\]

**Proof.** This is again clear in the light of example (b) and Theorem 1. In this case we need the following obvious identity:
\[
\min_{X' \subseteq X} (\rho(X \setminus X') + |E(Z') \cup X'|) = \\
= \rho(X \setminus E(Z')) + |E(Z')|, \quad \text{for} \quad Z' \subseteq Z. \]

**Corollary 1c.** Let \( M \) be a matrix over some field, with row collection \( X \) and column collection \( Y \). Let furthermore \((X, \mathcal{F})\) be a matroid. Define \( \mathcal{F} \) as the collection of all subsets \( Y' \) of \( Y \) such that, for some \( X' \in \mathcal{F} \), the submatrix of \( M \) generated by the rows \( X' \) and the columns \( Y' \) is regular. Then \((Y, \mathcal{F})\) is again a matroid.

**Proof.** Apply Theorem 1 to example (c).

The second theorem generalizes a theorem of B r u a l d i [1] (see Corollary 2a).

**Theorem 2.** Let \((X, Y, \Lambda)\) be a linking system, with linking function \( \lambda \), and let \((X, \mathcal{F})\) and \((Y, \mathcal{G})\) be matroids, with rank function \( \rho \) and \( \sigma \), respectively. Then the maximal cardinality of a set \( Y' \in \mathcal{G} \) such that for some \( X' \in \mathcal{F} \) the pair \((X', Y')\) is in \( \Lambda \), equals
\[
\min_{X' \subseteq X, Y' \subseteq Y} (\rho(X \setminus X') + \lambda(X', Y') + \sigma(Y \setminus Y')).
\]

**Proof.** If a pair \((X', Y') \in \Lambda\) is such that \( X' \in \mathcal{F} \) and \( Y' \in \mathcal{G} \), then \( Y' \in \mathcal{F} \ast \Lambda \) (cf. Theorem 1). Hence the maximal cardinality of a set \( Y' \in \mathcal{G} \) such that \((X', Y') \in \Lambda\) for some \( X' \in \mathcal{F} \) equals
\[
\max \{|Y'| \mid Y' \in \mathcal{G} \text{ and } Y' \in \mathcal{F} \ast \Lambda\}; \quad \text{by Edmonds’ intersection theorem this is}
\]
\[
\min_{Y' \subseteq Y} ((\rho \ast \lambda)(Y') + \sigma(Y \setminus Y')),
\]
and by Theorem 1 this equals

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\begin{equation*}
\min_{X' \subseteq X, Y' \subseteq Y} (\rho(X \setminus X') + \lambda(X', Y') + \sigma(Y \setminus Y')).
\end{equation*}

This theorem has the following corollaries.

**Corollary 2a** (Brualdi [1]). Let \((X, Y, E)\) be a bipartite graph and let \((X, \mathcal{F})\) and \((Y, \mathcal{F}')\) be matroids, with rank functions \(\rho\) and \(\sigma\), respectively. Then the maximal cardinality of a set \(Y' \in \mathcal{F}'\) which is matched in \(E\) with some \(X' \in \mathcal{F}\) equals

\begin{equation*}
\min_{Y' \subseteq Y} (\rho(E(Y'))) + \sigma(Y \setminus Y')).
\end{equation*}

**Proof.** Apply Theorem 2 to example (a) and observe that

\begin{equation*}
\min_{X' \subseteq X} (\rho(X \setminus X') + |E(Y' \cap X'|) = \rho(E(Y'))),
\end{equation*}

and

\begin{equation*}
\min_{Y' \subseteq Y'} (|Y' \setminus Y''| + \sigma(Y \setminus Y')) = \sigma(Y \setminus Y''),
\end{equation*}

for \(Y'' \subseteq Y\). \(\blacksquare\)

**Corollary 2b.** Let \((Z, E)\) be a directed graph and let \(X\) and \(Y\) be subsets of \(Z\). Let \((X, \mathcal{F})\) and \((Y, \mathcal{F}')\) be matroids, with rank functions \(\rho\) and \(\sigma\), respectively. Then the maximal cardinality of a set \(Y' \in \mathcal{F}'\) such that there are \(|Y'|\) pairwise vertex-disjoint paths starting in an \(X'\) in \(\mathcal{F}\) and ending in \(Y'\) equals

\begin{equation*}
\min_{Z' \subseteq Z} (\rho(X \setminus E(Z')) + |E(Z')| - |Z'| + \sigma(Z' \cap Y')).
\end{equation*}

**Proof.** Apply Theorem 2 to example (b) and observe that

\begin{equation*}
\min_{X' \subseteq X} (\rho(X \setminus X') + |E(Z') \cup X'|) = \rho(X \setminus E(Z')) + |E(Z')|).
\end{equation*}

Other theorems of Mason [7], [8] and Brualdi [2] on matroids and graphs have also their generalizations to theorems on matroids and linking systems. For more details see [17] and [18].
REFERENCES


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