CHARACTERIZATIONS OF SUPERCOMPACT SPACES

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1. Following DE GROOT [6], a topological space X is called *supercompact* if X has a so-called *binary* subbase, that is a subbase S such that if $S' \subset S$ and $\cap S' = \emptyset$ then there are S_1 and S_2 in S' with $S_1 \cap S_2 = \emptyset$. (Here and in the sequel each subbase is supposed to be a subbase for the closed sets.) Using Alexander's subbase lemma it is clear that each supercompact space is compact. Also it is easy to prove that the product of supercompact spaces is again supercompact. Not every compact Hausdorffspace is supercompact, since BELL [1] proved that $\beta\mathbb{N}$ is not supercompact (cf. [4]).
Examples of supercompact spaces:
(1) Compact orderable spaces (binary subbase: the collection of closed intervals); and more generally:
(2) Compact lattice spaces (binary subbase: the collection of closed intervals);
(3) Compact treelike spaces (a space X is *treelike* if X is connected and for each two different points there is a point separating them; binary subbase: the collection of closed connected subsets; cf. [3,9,10]);
(4) Compact metric spaces (STROK & SZYMAŃSKI [11]).
Also products of the examples give supercompact spaces.

2. A first characterization of supercompactness uses the notion of an interval structure. An *interval structure* on a set X is a function $I\colon X \times X \to P(X)$ such that:

(i) $\forall x,y \in X$: $x,y \in I(x,y)$,
(ii) $\forall x,y,u,v \in X$: $u,v \in I(x,y)$ implies $I(u,v) \subset I(x,y)$,
(iii) $\forall x,y,z \in X$: $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$.

A subset S of X is called *I-convex* if $I(x,y) \subset S$ for all $x,y \in S$. Using a result of GILMORE [5] (cf. [2] p.396) the characterization is as follows.

THEOREM. *A space* X *is supercompact if and only if* X *is compact and there is an interval structure* I *on* X *and a subbase S for* X *such that each set in S is* I-*convex*.

In the first three examples of section 1 we can take as interval structure the obvious intervals:

in (1) : $I(x,y) = [x,y]$ if $x \le y$,
$I(x,y) = [y,x]$ if $x > y$;
in (2) : $I(x,y) = [x \wedge y, x \vee y]$;
in (3) : $I(x,y) = \{x,y\} \cup \{z \mid z$ is a point seperating x and y$\}$.

3. The second characterization needs the notion of a graph. A *graph* G is a pair (V,E), where V is a set and

$$E \subset \{\{v,w\} \mid v,w \in V,\ v \neq w\}$$

(cf. [2]).

A subset V' of V is *stable* if $\{v,w\} \notin E$ for all $v,w \in V'$, and *maximal stable* if V' is stable and not contained in another stable subset of V (by Zorn's lemma each stable subset is contained in a maximal stable subset).

Now define successively:

$$\mathcal{I}(G) = \{V' \subset V \mid V' \text{ maximal stable}\};$$

$$B_v = \{V' \in \mathcal{I}(G) \mid v \in V'\}, \text{ for each } v \in V;$$

$$\mathcal{B}(G) = \{B_v \mid v \in V\}.$$

$\mathcal{B}(G)$ is a collection of subsets of $\mathcal{I}(G)$. Let T(G) be the space with point set $\mathcal{I}(G)$ and subbase $\mathcal{B}(G)$. We call T(G) the *stability space* of G.

The following theorem is due to DE GROOT [7].

THEOREM. *A space* X *is* T_1 *and supercompact if and only if* X *is (homeomorphic to) the stability space of a graph.*

Special classes of supercompact T_1-spaces can be characterized by being the stability space of special graphs:

(1) *A space* X *is compact orderable if and only if* X *is (homeomorphic to) the stability space of a connected comparable graph* (a graph $G = (V,E)$ is *connected* if for each two $v,w \in V$ there are $v_1,\ldots,v_n$ in V such that

$$\{v,v_1\},\{v_1,v_2\},\ldots,\{v_n,w\} \in E;$$

G is *comparable* if from

$$\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_4,v_5\} \in E$$

it follows that $\{v_1,v_4\} \in E$ or $\{v_2,v_5\} \in E$).
A space X *is a product of compact orderable spaces if and only if* X *is (homeomorphic to) the stability of a comparable graph* (this follows easily from the foregoing characterizations; cf. [8]).

(2) *A space* X *is a compact lattice space if and only if* X *is (homeomorphic to) the stability space of a bipartite graph* (a graph $G = (V,E)$ is *bipartite* if V is the disjoint union of two sets V_1 and V_2 such that

$$E \subset \{\{v,w\} \mid v \in V_1, w \in V_2\}).$$

(3) For a graphical characterization of compact treelike spaces see [10].

(4) *A Hausdorff space* X *is compact metrizable if and only if* X *is (homeomorphic to) the stability space of a countable graph* (this result of DE GROOT [7] is based on STROK & SZYMAŃSKI's theorem [11] that each compact metric space is supercompact).
For proofs and more details we refer to [10].

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