LINKING SYSTEMS, MATROIDS AND BIPARTITE GRAPHS

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In this talk the notion of a "linking system" is defined, a notion closely related to matroid theory. With this concept theorems on bipartite graphs and directed graphs, in relation to matroids, can be generalized. Linking systems can be interpreted as a special case of the "tabloids" of S. Hocquenghem [3].

DEFINITION. A *linking system* is a triple (X,Y,Λ) where X and Y are finite sets and $\emptyset \neq \Lambda \subset P(X) \times P(Y)$, such that:

(i) if $(X',Y') \in \Lambda$, then $|X'| = |Y'|$;

(ii) if $(X',Y') \in \Lambda$ and $X'' \subset X'$, then $(X'',Y'') \in \Lambda$ for some $Y'' \subset Y'$;

(iii) if $(X',Y') \in \Lambda$ and $Y'' \subset Y'$, then $(X'',Y'') \in \Lambda$ for some $X'' \subset X'$;

(iv) if $(X_1,Y_1) \in \Lambda$ and $(X_2,Y_2) \in \Lambda$, then there is an $(X',Y') \in \Lambda$ such that $X_1 \subset X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$.

Examples of linking systems may be obtained as follows.

(a) Let (X,Y,E) be a bipartite graph (i.e. $E \subset X \times Y$) and $\Lambda = \Delta_E = \{(X',Y') \mid$ there exists a matching in E between $X' \subset X$ and $Y' \subset Y\}$. Then (X,Y,Λ) is a linking system. Axiom (iv) was proved by H. Perfect and J.S. Pym [6]. A linking system constructed in this way is called a *deltoid linking system*.

(b) Let (Z,Γ) be a directed graph and $X,Y \subset Z$. Let furthermore: $\Lambda = \{(X',Y') \mid$ there are pairwise vertex-disjoint paths in Γ between $X' \subset X$ and $Y' \subset Y$, such that in each vertex of X' starts a path and in each vertex of Y' ends a path$\}$. Then (X,Y,Λ) is a linking system. Axiom (iv) was proved by J.S. Pym [7]. Linking systems constructed in this way are called *gammoid linking systems*.

(c) Let (X,Y,ϕ) be a matrix over a field $\mathbb{F}$ (i.e. $\phi : X \times Y \longrightarrow \mathbb{F}$), and let $\Lambda = \{(X',Y') \mid$ the submatrix generated by $X' \subset X$ and $Y' \subset Y$ is

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regular}. Then (X,Y,Λ) is a linking system. Such a linking system is called *representable over* $\mathbb{F}$.

Of course, example (a) is a special case of example (b): *each deltoid linking system is a gammoid linking system.*

There exist close relations between linking systems and matroids. In fact each linking system may be understood as a matroid with a fixed base (a *based matroid*).

THEOREM 1. *Let* X *and* Y *be disjoint finite sets. Then there exists a one-to-one relation between:*

(1) *linking systems* (X,Y,Λ), *and*

(2) *matroids* $(X \cup Y,\mathcal{B})$ *with* $X \in \mathcal{B}$ *($\mathcal{B}$ is the collection of bases), given by:*

$$(X',Y') \in \Lambda \qquad \textit{iff } (X\backslash X') \cup Y' \in \mathcal{B}.$$

The correspondence is such that the linking system is a deltoid linking system iff the matroid is a deltoid; likewise, this correspondence exists between gammoid linking systems and gammoids and between linking systems representable over a field $\mathbb{F}$ and matroids representable over $\mathbb{F}$.

A second relation between linking systems and matroid theory gives the following theorem.

THEOREM 2. *Let* $(X,\mathcal{I})$ *be a matroid ($\mathcal{I}$ is the collection if independent subsets of* X*) and let* (X,Y,Λ) *be a linking system. Let furthermore:*
$\mathcal{I} * \Lambda = \{Y' \subset Y \mid$ *there exists an* $X' \in \mathcal{I}$ *such that* $(X',Y') \in \Lambda\}$.
Then $(Y,\mathcal{I} * \Lambda)$ *is a matroid.*

The proof of this theorem makes use of theorem 1 and the fact that the union of two matroids is again a matroid.

As corollaries we have:

(1) (J. Edmonds & D.R. Fulkerson [2]) if (X,Y,E) is a bipartite graph and $\mathcal{J} = \{Y' \subset Y \mid Y'$ is matched with some subset of $X\}$, then $(Y,\mathcal{J})$ is a matroid;

(2) (H. Perfect [4]) if (Z,Γ) is a digraph, $X,Y \subset Z$ and $\mathcal{J} = \{Y' \subset Y \mid$ there are $|Y'|$ pairwise vertex-disjoint paths start-

ing in X and ending in Y'}, then $(Y,\mathcal{J})$ is a matroid;

(3) (H. Perfect [5]) if $(X,\mathcal{I})$ is a matroid, (X,Y,E) a bipartite graph and $\mathcal{J} = \{Y' \subset Y \mid Y'$ is matched with some $X' \in \mathcal{I}\}$, then $(Y,\mathcal{J})$ is a matroid;

(4) (R.A. Brualdi [1]) is $(X,\mathcal{I})$ is a matroid, (Z,Γ) a digraph, $X,Y \subset Z$ and $\mathcal{J} = \{Y' \subset Y \mid$ there are $|Y'|$ pairwise vertex-disjoint paths starting in X' and ending in Y'}, then $(Y,\mathcal{J})$ is a matroid.

Of course, corollaries (1), (2) and (3) are also consequences of corollary (4).

Theorem 2 gave a kind of product of a matroid and a linking system. The next theorem gives in an analogue way a product of two linking systems.

THEOREM 3. *Let* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *be linking systems. Let furthermore* $\Lambda_1 * \Lambda_2 = \{(X',Z') \mid$ *there is a* $Y' \subset Y$ *such that* $(X',Y') \in \Lambda_1$ *and* $(Y',Z') \in \Lambda_2\}$. *Then* $(X,Z,\Lambda_1 * \Lambda_2)$ *is again a linking system.*

Again, the proof of this theorem uses theorem 1 and the union-theorem of matroids.

A linking system is partially determined by its "underlying" bipartite graph, as defined in the following theorem.

THEOREM 4. *Let* (X,Y,Λ) *be a linking system and let* (X,Y,E) *be the bipartite graph with:* $(x,y) \in E$ *iff* $(\{x\},\{y\}) \in \Lambda$. *Then:*

(1) *if there is exactly one matching in* E *between* $X' \subset X$ *and* $Y' \subset Y$, *then* $(X',Y') \in \Lambda$;

(2) *if* $(X',Y') \in \Lambda$, *then there exists at least one matching in* E *between* X' *and* Y'.

(2) means: $\Lambda \subset \Delta_E$ (as defined in example (a)). Thus the maximum of all linking systems with the same underlying bipartite graph (X,Y,E) is the deltoid linking system (X,Y,Δ_E), since this last linking system has also (X,Y,E) as underlying bipartite graph.

Proofs and more details can be found in [8] and [9].

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