LINKING SYSTEMS, MATROIDS AND BIPARTITE GRAPHS

by

Lex Schrijver
Mathematisch Centrum
Tweede Boerhaavestraat 49
Amsterdam - 1005, Holland

In this talk the notion of a "linking system" is defined, a notion closely related to matroid theory. With this concept theorems on bipartite graphs and directed graphs, in relation to matroids, can be generalized. Linking systems can be interpreted as a special case of the "tabloids" of S. Hocquenghem [3].

DEFINITION. A linking system is a triple \((X,Y,\Lambda)\) where \(X\) and \(Y\) are finite sets and \(\emptyset \neq \Lambda \subset P(X) \times P(Y)\), such that:

(i) if \((X',Y') \in \Lambda\), then \(|X'| = |Y'|\);

(ii) if \((X',Y') \in \Lambda\) and \(X'' \subset X'\), then \((X'',Y'') \in \Lambda\) for some \(Y'' \subset Y'\);

(iii) if \((X',Y') \in \Lambda\) and \(Y'' \subset Y'\), then \((X'',Y'') \in \Lambda\) for some \(X'' \subset X'\);

(iv) if \((X_1,Y_1) \in \Lambda\) and \((X_2,Y_2) \in \Lambda\), then there is a \((X',Y') \in \Lambda\) such that \(X_1 \subset X' \subset X_1 \cup X_2\) and \(Y_2 \subset Y' \subset Y_1 \cup Y_2\).

Examples of linking systems may be obtained as follows.

(a) Let \((X,Y,E)\) be a bipartite graph (i.e. \(E \subset X \times Y\)) and
\[
\Lambda = \Lambda_E = \{(X',Y') \mid \text{there exists a matching in } E \text{ between } X' \subset X \text{ and } Y' \subset Y\}.
\] Then \((X,Y,\Lambda)\) is a linking system. Axiom (iv) was proved by H. Perfect and J.S. Pym [6]. A linking system constructed in this way is called a deltoid linking system.

(b) Let \((Z,\Gamma)\) be a directed graph and \(X,Y \subset Z\). Let furthermore:
\[
\Lambda = \{(X',Y') \mid \text{there are pairwise vertex-disjoint paths in } \Gamma \text{ between } X' \subset X \text{ and } Y' \subset Y\}.
\] Then \((X,Y,\Lambda)\) is a linking system. Axiom (iv) was proved by J.S. Pym [7]. Linking systems constructed in this way are called gammoid linking systems.

(c) Let \((X,Y,\phi)\) be a matrix over a field \(\mathbb{F}\) (i.e. \(\phi : X \times Y \to \mathbb{F}\)), and let \(\Lambda = \{(X',Y') \mid \text{the submatrix generated by } X' \subset X \text{ and } Y' \subset Y\} is
Then \((X,Y,A)\) is a linking system. Such a linking system is called representable over \(\mathbb{F}\).

Of course, example (a) is a special case of example (b): each deltoid linking system is a gammoid linking system.

There exist close relations between linking systems and matroids. In fact each linking system may be understood as a matroid with a fixed base (a based matroid).

**Theorem 1.** Let \(X\) and \(Y\) be disjoint finite sets. Then there exists a one-to-one relation between:

1. linking systems \((X,Y,A)\), and
2. matroids \((X \cup Y, B)\) with \(X \in B\) (\(B\) is the collection of bases),

where given by:

\[(X',Y') \in A \iff (X \setminus X') \cup Y' \in B.\]

The correspondence is such that the linking system is a deltoid linking system iff the matroid is a deltoid; likewise, this correspondence exists between gammoid linking systems and gammoids and between linking systems representable over a field \(\mathbb{F}\) and matroids representable over \(\mathbb{F}\).

A second relation between linking systems and matroid theory gives the following theorem.

**Theorem 2.** Let \((X,I)\) be a matroid (\(I\) is the collection of independent subsets of \(X\)) and let \((X,Y,A)\) be a linking system. Let furthermore:

\[I * A = \{Y' \subseteq Y \mid \text{there exists an } X' \subseteq I \text{ such that } (X',Y') \in A\}.\]

Then \((Y,I * A)\) is a matroid.

The proof of this theorem makes use of theorem 1 and the fact that the union of two matroids is again a matroid.

As corollaries we have:

1. (J. Edmonds & D.R. Fulkerson [2]) if \((X,Y,E)\) is a bipartite graph and \(J = \{Y' \subseteq Y \mid Y' \text{ is matched with some subset of } X\}\), then \((Y,J)\) is a matroid;
2. (H. Perfect [4]) if \((Z,J')\) is a digraph, \(X,Y \subseteq Z\) and \(J = \{Y' \subseteq Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths start-}

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(3) (H. Perfect [5]) if \((X, I)\) is a matroid, \((X, Y, E)\) a bipartite graph and \(J = \{Y' \subset Y \mid Y' \text{ is matched with some } X' \in I\}\), then \((Y, J)\) is a matroid;

(4) \((\text{R.A. Brualdi [1]})\) if \((X, I)\) is a matroid, \((Z, \Gamma)\) a digraph, \(X, Y \subset Z\) and \(J = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths starting in } X' \text{ and ending in } Y'\}\), then \((Y, J)\) is a matroid.

Of course, corollaries (1), (2) and (3) are also consequences of corollary (4).

Theorem 2 gave a kind of product of a matroid and a linking system. The next theorem gives in an analogue way a product of two linking systems.

**THEOREM 3.** Let \((X, Y, \Lambda_1)\) and \((Y, Z, \Lambda_2)\) be linking systems. Let furthermore \(\Lambda_1 \ast \Lambda_2 = \{(X', Z') \mid \text{there is a } Y' \subset Y \text{ such that } (X', Y') \in \Lambda_1 \text{ and } (Y', Z') \in \Lambda_2\}\). Then \((X, Z, \Lambda_1 \ast \Lambda_2)\) is again a linking system.

Again, the proof of this theorem uses theorem 1 and the union-theorem of matroids.

A linking system is partially determined by its "underlying" bipartite graph, as defined in the following theorem.

**THEOREM 4.** Let \((X, Y, \Lambda)\) be a linking system and let \((X, Y, E)\) be the bipartite graph with: \((x, y) \in E \text{ iff } (\{x\}, \{y\}) \in \Lambda\). Then:

1. If there is exactly one matching in \(E\) between \(X' \subset X\) and \(Y' \subset Y\), then \((X', Y') \in \Lambda\);
2. If \((X', Y') \in \Lambda\), then there exists at least one matching in \(E\) between \(X'\) and \(Y'\).

(2) means: \(\Lambda \subset \Lambda_E\) (as defined in example (a)). Thus the maximum of all linking systems with the same underlying bipartite graph \((X, Y, E)\) is the deltoid linking system \((X, Y, \Lambda_E)\), since this last linking system has also \((X, Y, E)\) as underlying bipartite graph.

Proofs and more details can be found in [8] and [9].
REFERENCES


