Asymptotic Expansion of a Special Integral

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The integral

$$I = \int_a^b \left[\frac{x+a}{x-a}\right]^{2ai} \left[\frac{b-x}{b+x}\right]^{2bi} \frac{dx}{x}$$

is considered for large positive values of a and b; the parameters tend to infinity in such a manner that the quotient $b/a = c$ is a constant greater than 1. In a recent paper, Mahler showed that the integral tends more rapidly to 0 than any finite negative power of a and he gives an upper bound of the integral. As Mahler admitted, his results do not imply estimates of the form $I = \mathcal{O}(e^{-a})$. Our results give $I = \mathcal{O}(e^{-2\pi a})$. Mahler's technique is based on integration by parts. Here we use a different technique, based on complex variables, and we construct the leading term and the first terms in the asymptotic expansion.

1. Mahler's approach

Quite recently the physicist J. Lekner of the Victoria University of Willington, New Zealand presented K. Mahler of the Australian National Univeristy of the following integral

$$I = \int_a^b \left[\frac{x+a}{x-a}\right]^{2ai} \left[\frac{b-x}{b+x}\right]^{2bi} \frac{dx}{x}. \tag{1.1}$$

Lekner was interested in the behaviour of I for large values of the parameters a, b. He used this integral for describing the Rayleigh approximation for a reflection amplitude in the theory of electromagnetic and particle waves. See formula 6.64 in [2]. Mahler [3] proved the following two theorems.

Theorem 1. *Assume that the two positive parameters a and b tend tp $+\infty$ in such a manner that the quptient $b/a = c$ remains equal to a constant $c > 1$. Then the integral I in (1.1) tends more rapidly to 0 than any finite, negative, power of a*

In other words, the integral I is asymptotically equal to zero with respect to the scale $\{a^{-n}\}$, as $a \to \infty$. That is,

$$I \sim 0 \quad \{a^{-n}\}, \text{ as } a \to \infty.$$

For this notation we refer to Lauwerier [1].

Another theorem appeared in the second part of [3]:

THEOREM 2. *Denote by $A(a)$ a monotone, increasing, positive-valued continuous function of a which tends arbitrarily slowly to $+\infty$ as a tends to $+\infty$; further let $\Gamma>0$ be an arbitrarily large positive constant. Then for all sufficiently large positive a the integral (1.1) satisfies the inequality*

$$|I| < \exp\left[\frac{-a\Gamma}{A(a)}\right] \tag{1.2}$$

Mahler suggests to take for A the m-times iterated logarithm

$$\ln_m(a) = \ln(\ln(...(\ln a)...)), \ (m \ \mathit{log's}),$$

where m is any positive integer. Observe that in (1.2) $\Gamma/A(a)$ tends to 0. As Mahler admits, his result does not quite imply the estimate $|I| = \mathcal{O}(e^{-a})$, and he concludes his paper with the words *'and I do not know whether it is true'*. From our results it follows that $|I| = \mathcal{O}(e^{-2\pi a})$, and the purpose of the paper is to give a complete description of the asymptotic behaviour of I.

Mahler proved his result by considering

$$I = \int_1^c F(u)^{2ai}\frac{du}{u}, \quad F(u) = \frac{u+1}{u-1}\left[\frac{c-u}{c+u}\right]^c. \tag{1.3}$$

This representation easily follows by using $b = ac$ and introducing a new variable of integration u by writing $x = au$. Then Mahler used integration by parts to prove the theorems.

2. COMPLEX VARIABLE APPROACH

As remarked earlier, the above theorems give only a partial result, since no information is given on the leading term of the asymptotic estimate and of the terms in the asymptotic expansion. It does not seem possible that one can obtain these leading terms by using only real integration variables. Therefore we replace (1.1) by a loop integral in the complex plane, which gives the required information. But first we transform (1.1) into an integral on the interval $[0,\infty)$. It is easier to handle then the various branch-points of the integrand, since one of them is sent to ∞ by this transformation.

Let us write $t = (x-a)/(b-x)$. Then we obtain

$$I = A\int_0^\infty \frac{t^{-2ai}(t+t_1)^{2ai}\,dt}{(t+t_2)^{2bi}(1+t)(1+ct)}, \tag{2.1}$$

where

$$t_1 = \frac{2}{c+1}, \quad t_2 = \frac{c+1}{2c} \tag{2.2}$$

and

$$A = (2c)^{-2bi}(c+1)^{2ai}(c-1)^{2bi-2ai+1}. \tag{2.3}$$

Recall that $c>1$. Hence we have the inequalities

$$0<\frac{1}{c}<\frac{2}{c+1}<\frac{c+1}{2c}<1.$$

This shows that the five singular points $0, -1/c, -t_1, -t_2, -1$ of the above integral satisfy $-1<-t_2<-t_1<-1/c<0$. Since c is assumed to be fixed, no confluence of singularities can happen. Otherwise the problem would be much more difficult.

Before choosing a proper loop integral based on (2.1), we compute the stationary point(s) of the integrand.

We write

$$\frac{t^{-2ai}(t+t_1)^{2ai}}{(t+t_2)^{2bi}} = e^{-2ai\phi(t)},$$

introducing the function

$$\phi(t) = \ln t - \ln(t+t_1) + c\ln(t+t_2). \tag{2.4}$$

For $t>0$ we assume real values of the logarithms. It is straightforward to verify that

$$\phi'(t) = \frac{c(t+1/c)^2}{t(t+t_1)(t+t_2)}. \tag{2.5}$$

Hence ϕ has a (double) stationary point at $-1/c$. It follows that we can expand

$$\phi(t) = \phi(-1/c) + \frac{1}{6}\phi'''(-1/c)(t+1/c)^3 + O(t+1/c)^4. \tag{2.6}$$

A few computations give

$$\phi'''(-1/c) = -\frac{4c^4(c+1)}{(c-1)^2}. \tag{2.7}$$

We observe that (2.1) has a double stationary point outside the interval of integration, and that this point coincides with a single pole of the integrand. From an asymptotic point of view, this combination of phenomena is not just trivial. However, in Lauwerier's book the theory needed to handle this problem is presented for an analogue case.

3. A loop integral

The final preparatory step is to introduce a suitable loop integral of which the path of integration can be shifted to the stationary point at $-1/c$. We introduce

$$J = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{(-t)^{-2ai}(t+t_1)^{2ai}\,dt}{(t+t_2)^{2bi}(1+t)(1+ct)}, \tag{3.1}$$

where δ is a positive number satisfying $0<\delta<1/c$. It is not difficult to verify that the integral is convergent at ∞. The phase of the complex parameter t is, initially, between $\pi/2$ and $3\pi/2$. The minus-sign in $(-t)$ is interpreted as $e^{-\pi i}$ (this choice is irrelevant, but it brings a nice symmetry in the relation between I and J). Hence we assume that for negative values of t the phase of $-t$ equals zero. The branches of the remaining many-valued functions are chosen in the normal way: we assume that the phases of $t+t_1$ and $t+t_2$ are zero for positive values of t.

Our procedure is as follows. First we show, by modifying the vertical path of integration, that J equals I (up to a simple function of a and b). On the other hand, we can shift the vertical path to the left until it meets the real negative axis at $-1/c$; that is, we let $\delta \to 1/c$. Then we apply the method of stationary phase from asymptotics.

To recover I from the complex integral, we bend the vertical path around the interval $[0,\infty)$. At the upper side of this interval, where $\arg t = 0$, we have

$$(-t)^{-2ai} = (e^{-\pi i}|t|)^{-2\pi a}|t|^{-2ai}.$$

At the lower side, where $\arg t = 2\pi i$, we have

$$(-t)^{-2ai} = (e^{-\pi i}|t|e^{2\pi i})^{-2ai} = e^{2\pi a}|t|^{-2ai}.$$

The integration near the origin gives no problems. So we arrive at the result

$$J = -2\sinh(2\pi a)I/A, \tag{3.2}$$

where A is given in (2.3).

4. Asymptotic expansion

We slightly change the phase function introduced in (2.4) by writing

$$\phi(t) = \ln(e^{-\pi i}t) - \ln(t+t_1) + c\ln(t+t_2). \tag{4.1}$$

The formulas (2.5), (2.6) and (2.7) also hold for this new ϕ. We have

$$J = \int_{\mathfrak{L}} e^{-2ia\phi(t)} \frac{dt}{(1+t)(1+ct)},$$

where $\mathfrak{L}$ is the above introduced vertical, now with $\delta = 1/c$ and with a small semi-circle at the right of the pole at $-1/c$. We introduce the transformation of variables (see (2.6))

$$w^3 = \frac{\phi(t)-\phi(-1/c)}{\phi'''(-1/c)/6}, \tag{4.2}$$

and we choose the branch that satisfies $w \sim t+1/c$ in a neighborhood of the stationary point $-1/c$.

On using (4.1), (4.2) in (3.1), we obtain

$$J = B\int e^{\frac{1}{3}i\mu a w^3} f(w)\frac{dw}{w}, \tag{4.3}$$

where

$$B = e^{-2ai\phi(-1/c)} = (c-1)/A, \tag{4.4}$$

$$f(w) = \frac{w}{(1+t)(1+ct)} \frac{dt}{dw}, \tag{4.5}$$

$$\mu = -\phi'''(-1/c). \tag{4.6}$$

Since μ is positive (see (2.7)), the 'best' path in (4.3) is the steepest descent path defined by the rays

$$\arg w = -\pi/2, \; \arg w = \pi/6. \tag{4.7}$$

Locally, the same holds for the t-plane near $t = -1/c$. In order to avoid the pole, the path in the w-plane has a small circular arc near the origin. The integration runs from $-i\infty$ to $\infty\exp(\pi i/6)$, and the pole at the origin is at the left hand side of the contour.*)

We substitute the MacLaurin series

$$f(w) = \sum_{k=0}^{\infty} c_k w^k \tag{4.8}$$

in (4.3), and we interchange summation and integration. The result is the asymptotic expansion

$$J \sim B \sum_{k=0}^{\infty} c_k F_k, \quad F_k = \int e^{\frac{1}{3} i\mu a w^3} w^{k-1} dw, \tag{4.9}$$

as $a \to \infty$. To compute F_k we use the path described by (4.7). F_0 needs some special care. We write

$$F_0 = \left[\int_{-i\infty}^{-ir} + \int_{r\exp(i\pi/6}^{\infty\exp(i\pi/6)} \right] e^{\frac{1}{3} i\mu a w^3} \frac{dw}{w} + i \int_{-\pi/2}^{\pi/6} e^{\frac{1}{3} i\mu a r^3} e^{3i\theta} d\theta$$

for any positive number r. The first two integrals cancel. The third one assumes in the limit $r \to 0$ the value $2\pi i/3$, which is 1/3 of the residue of the pole. Hence $F_0 = 2\pi i/3$. The remaining integrals follow straightforwardly:

$$F_k = [e^{i\pi k/6} - e^{-i\pi k/2}] \int_0^{\infty} e^{-\frac{1}{3}\mu a w^3} w^{k-1} dw$$

$$= \frac{2i}{3} e^{-i\pi k/6} \sin(k\pi/3) \Gamma(k/3) (\mu a/3)^{-k/3}, \quad k = 1,2,3,....$$

Observe that this result can also be interpreted for $k = 0$. Combining (3.2), (4.4) and (4.9) we obtain the final result

$$I \sim -\frac{c-1}{2\sinh(2\pi a)} \sum_{k=0}^{\infty} c_k F_k, \quad \text{as } a \to \infty. \tag{4.10}$$

The dominant term in this expansion reads very simple. We have from (4.5) by

*) We do not prove that the function f admits such a contour; to do so, we should examine the mapping (4.2) more globally. For the construction of the asymptotic expansion we only need a local analysis around the origin.

using l'Hôpital's rule

$$c_0 = f(0) = \frac{dt}{dw}|_{w=0}\frac{c}{c-1}\lim_{w\to 0}\frac{w}{1+ct} = \frac{1}{c-1}.$$

This gives

$$I \sim -\frac{\pi i}{3\sinh 2\pi a}, \quad \text{as } a\to\infty.$$

We conclude this section by giving the first few coefficients c_k. They are obtained by using (4.2), (4.5), and (4.8). Thus we obtain

$$c_0 = \frac{1}{c-1}, \quad c_1 = 0, \quad c_2 = -\frac{2c^2(c^2+1)}{5(c-1)^3}, \quad c_3 = \frac{4c^3(c^2+1)}{5(c-1)^4},$$
$$c_4 = -\frac{2c^4(3c^4+17c^2+24)}{35(c-1)^5}, \quad c_5 = \frac{8c^5(3c^4+3c^2+10)}{35(c-1)^6}.$$

5. A Fourier integral

When we take in (1.3)

$$t = -\ln F(u) = \ln(u-1)-\ln(u+1)+c\ln(c+u)-c\ln(c-u) \quad (5.1)$$

as a new variable of integration, we obtain the Fourier integral

$$I = \int_{-\infty}^{\infty} e^{-2ait}g(t)dt, \quad g(t) = \frac{1}{u}\frac{dt}{du} = \frac{(u^2-1)(c^2-u^2)}{2u^3(c^2-1)}. \quad (5.2)$$

By considering the mapping $u\mapsto t$ in more detail, we see that it is one-to-one on $[1,c]$, and that, consequently, g is a C^∞-function on $\mathbb{R}$. Moreover, g is exponentially small at $\pm\infty$. That is,

$$g(t) = O(e^{t/c}), \text{ as } t\to-\infty, \quad g(t) = O(e^{-t}), \text{ as } t\to+\infty.$$

This easily follows from (5.1) and (5.2). By using these properties, Theorem 1 can be proved immediately.

The function g is singular at $u = 0$, that is, at $t = -i\pi$. Hence, we can shift the contour of integration in (5.2) downwards to this point, and we can expand the function g at this singularity. Observe that the exponential function in (5.2) assumes the value $\exp(-2\pi a)$ at this point. This dominant factor also occurs in (4.10), and we expect that (5.2) can be used to obtain the same or a similar expansion.

References

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