STRONG T-PERFECTION OF BAD-$K_4$-FREE GRAPHS*

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Abstract. We show that each graph not containing a bad subdivision of $K_4$ as a subgraph is strongly t-perfect. Here a graph $G = (V,E)$ is strongly t-perfect if, for each weight function $w : V \to \mathbb{Z}_+$, the maximum weight of a stable set is equal to the minimum (total) cost of a family of vertices, edges, and circuits covering any vertex $v$ at least $w(v)$ times. By definition, the cost of a vertex or edge is 1, and the cost of a circuit $C$ is $|C|$. A subdivision of $K_4$ is called bad if each triangle has become an odd circuit and if it is not obtained by making the edges in a 4-circuit of $K_4$ evenly subdivided, while the other two edges are not subdivided.


Key words. t-perfect, graph, stable set, polytope

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1. Introduction. A graph $G = (V,E)$ is called t-perfect if the stable set polytope of $G$ (= the convex hull of the incidence vectors in $\mathbb{R}^V$ of stable sets) is determined by

\begin{align}
(1.1) \quad \text{(i)} & \quad 0 \leq x_v \leq 1 \quad \text{for each } v \in V; \\
\quad \text{(ii)} & \quad x_u + x_v \leq 1 \quad \text{for each edge } uv \in E; \\
\quad \text{(iii)} & \quad x(VC) \leq \frac{1}{2} |VC| \quad \text{for each odd circuit } C.
\end{align}

Here $x(U) := \sum_{v \in U} x_v$ for any $U \subseteq V$, $V$, and $E$. A circuit $C$ is odd (even) if $|VC|$ is odd (even).

A motivation for the concept of t-perfection lies in the fact that a linear function $w^T x$ can be maximized over (1.1) in strongly polynomial time (with the ellipsoid method, since the separation problem over (1.1) is polynomial-time solvable). Hence a maximum-weight stable set in a t-perfect graph can be found in strongly polynomial time.

$G$ is called strongly t-perfect if system (1.1) is totally dual integral—that is, if for each weight function $w : V \to \mathbb{Z}_+$, the linear program of maximizing $w^T x$ over (1.1) has an integer optimum dual solution. This implies that it also has an integer optimum primal solution. In particular, all vertices of the polytope determined by (1.1) are integer, and hence the polytope is the stable set polytope. So strong t-perfection implies t-perfection.

Strong t-perfection can be characterized equivalently as follows. For any $w : V \to \mathbb{Z}_+$, let $\alpha_w(G)$ denote the maximum weight of a stable set in $G$. Define a $w$-cover as a family of vertices, edges, and odd circuits such that each vertex $v$ is covered at least $w(v)$ times. (In a family, repetition is allowed.) By definition, the cost of a vertex or edge is 1, the cost of a circuit $C$ is $|C|$, and the cost of a $w$-cover is the sum

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of the costs of its elements (counting multiplicities). Let $\tilde{p}_w(G)$ denote the minimum cost of a $w$-cover. Then

a graph $G$ is strongly $t$-perfect if and only if $\alpha_w(G) = \tilde{p}_w(G)$ for each $w : V \to \mathbb{Z}_+$.  
(1.2)

The classes of $t$-perfect and strongly $t$-perfect graphs are closed under taking induced subgraphs. However, no characterization is known in terms of forbidden induced subgraphs.

If we also take noninduced subgraphs, the situation is clearer (although it does not yield a characterization). Here subdivisions of $K_4$ come in. A $K_4$-subdivision $H$ is called odd, or just an odd $K_4$, if each triangle of $K_4$ has become an odd circuit in $H$. It was shown by Gerards [6] that

(1.3) each graph without odd $K_4$ is strongly $t$-perfect.

(By “a graph without” odd $K_4$ we mean a graph not containing an odd $K_4$ as subgraph.) It extends an earlier result of Gerards and Schrijver [7] that such graphs are $t$-perfect.

There exist, however, odd $K_4$’s that are $t$-perfect. Following Gerards and Shepherd [8], we call an odd $K_4$-subdivision a bad $K_4$ if it does not have the following property:

(1.4) the edges of $K_4$ that have become an even path form a 4-cycle in $K_4$,
while the two other edges of $K_4$ are not subdivided.

This name is motivated by the fact, shown by Barahona and Mahjoub [1], that a subdivision of $K_4$ is $t$-perfect if and only if it is not a bad $K_4$. Gerards and Shepherd [8] proved that

(1.5) each graph without bad $K_4$ is $t$-perfect.

(Gerards and Shepherd [8] also showed that graphs without bad $K_4$ can be recognized in polynomial time.)

In the present paper, we show more strongly that these graphs are strongly $t$-perfect. This generalizes (1.3) and (1.5), and implies for any graph $G$ that

(1.6) each subgraph of $G$ is $t$-perfect
$\iff$ each subgraph of $G$ is strongly $t$-perfect
$\iff$ $G$ has no bad $K_4$ as subgraph.

On the other hand, there exist strongly $t$-perfect graphs that contain a bad $K_4$; see Figure 1.1.

Our proof method was inspired by a method of Geelen and Guenin [5] for proving a special case of a theorem of Seymour [12] on packing the edge sets of odd circuits in odd-$K_4$-free graphs.

The above results contain the strong $t$-perfection of series-parallel graphs, which are, as is well known, those graphs not containing any $K_4$-subdivision (Boulala and Uhry [2]), and of almost bipartite graphs—graphs $G$ having a vertex $v$ with $G - v$ bipartite (Fonlupt and Uhry [4], Sbihi and Uhry [10]).
A related theorem was proved by Sewell and Trotter [11]. A $K_4$-subdivision is called a totally odd $K_4$ if it arises from $K_4$ by replacing each edge by an odd path. The theorem says that a graph $G$ without totally odd $K_4$ satisfies $\alpha_1(G) = \tilde{\rho}_1(G)$, where $1$ denotes the all-one weight function. This result does not follow from our methods.

The totally odd $K_4$'s are precisely those $K_4$-subdivisions $G$ with $\alpha_1(G) < \tilde{\rho}_1(G)$. So the theorem of Sewell and Trotter and the theorem presented in this paper suggest the question of whether, for each graph $G$ and each $w : VG \rightarrow \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ with $\alpha_w(G) \leq \tilde{\rho}_w(G)$, $G$ contains a $K_4$-subdivision $H$ as subgraph such that $\alpha_{w'}(H) < \tilde{\rho}_{w'}(H)$, where $w' := w|VH$. The answer is unknown.

To complete the picture, it was shown by Zang [15] and Thomassen [13] that $\chi(G) \leq 3$ for any graph $G$ without totally odd $K_4$. This was conjectured by Toft [14], and was proved by Hadwiger [9] for series-parallel graphs, by Catlin [3] for odd-$K_4$-free graphs, and by Gerards and Shepherd [8] for bad-$K_4$-free graphs. (However, there exist strongly $t$-perfect graphs $G$ with $\chi(G) > 3$.)

A.M.H. Gerards and P.D. Seymour proved in 1991 (personal communication) that if $G$ contains no odd $K_4$, then the stable set polytope of $G$ has the integer decomposition property. In other words, any $w : VG \rightarrow \mathbb{Z}_+$ is the sum of the incidence vectors of $k$ stable sets, where $k$ is the minimum integer for which $\frac{1}{k}w$ belongs to the stable set polytope. It implies the result of Catlin mentioned above.

2. Graphs without bad $K_4$. In this section we prove a technical lemma on bad-$K_4$-free graphs. Let $G$ be graph without bad $K_4$, and let $C$ be an even circuit in $G$. Let $e_1, \ldots, e_n$ be chords of $C$ such that $e_i$ has ends $s_i$ and $s_{i+1}$ (say) (for $i = 1, \ldots, n$), such that $s_1, \ldots, s_{2n}$ are distinct and occur in this order clockwise along $C$, and such that, for each $i = 1, \ldots, 2n$, the clockwise $s_{i-1} - s_i$ path $R_i$ along $C$ has even length. (We take indices mod $2n$ and set $e_{n+i} := e_i$ for $i = 1, \ldots, n$.) Define $D := \{e_1, \ldots, e_n\}$.

Call a path $B$ in $G$ a bow if $B$ is simple, has length at least 2, and intersects $C$ precisely in its end vertices. We call a bow an odd bow if it forms with a subpath of $C$ an odd circuit and an even bow if it forms with a subpath of $C$ an even circuit. (So an odd (even) bow need not be an odd (even) path. To avoid confusion, we therefore do not use the more familiar term "ear."")

We will study in particular the occurrence of odd bows. We say that a bow $B$ crosses an edge $e \in D$ (and conversely) if $e$ is disjoint from the ends $a, b$ (say) of $B$.
and connects distinct components of the graph $C - a - b$. Then

\begin{equation}
\tag{2.1}
\text{an odd bow } B \text{ does not cross any edge } e \text{ in } D.
\end{equation}

Otherwise, $C$, $B$, and $e$ form a bad $K_4$, a contradiction.

Equation (2.1) implies that the ends of any odd bow belong to $VR_j$ for some $j = 1, \ldots , 2n$. Define

\begin{equation}
\tag{2.2}
J := \{ j \in \{1, \ldots , 2n\} | \text{there exists an odd bow with ends in } VR_j \}.
\end{equation}

We prove the following lemma.

**Lemma 2.1.** There exists an $i \in \{1, \ldots , 2n\}$ such that $i+1, i+2, \ldots , i+n-1 \notin J$.

**Proof.** Consider a counterexample with $n$ as small as possible. Define $L := \{i, i+2, \ldots , i+n-1 \notin J\}$. Then, for each $i$.

\begin{equation}
\tag{2.3}
i \in L \text{ or } i+n \in L.
\end{equation}

To see this, by symmetry it suffices to show this for $i = n$. Delete $e_n$. By the minimality of $n$, the lemma holds for the new structure. In the new structure, the paths $R_n$ and $R_{n+1}$ have merged to one path, and similarly the path $R_{2n}$ and $R_1$ have merged to one path. If (2.3) does not hold for the original structure, then, for some $i \in \{2, \ldots , n-1\}$, there is no odd bow with ends in one of $VR_i - 1, VR_{2} \cup VR_{n+1}, VR_{n+2}, \ldots , VR_{i+n-1}$ or there is no odd bow with ends in one of $VR_{i+n-1}, VR_{2n}, UV R_1, VR_{2}, \ldots , VR_{i-1}$. Either case implies the lemma for the original structure, a contradiction. So we have (2.3).

We derive from this that $n = 2$. As the lemma does not hold, we know that $i \notin L$ or $i+1 \notin L$ for each $i$. Hence, by (2.3), $i \in L$ or $i+1 \in L$ for each $i$. So the indices $i$ are alternatingly in and out of $L$. If $n \geq 4$, then we can assume that each even $i$ belongs to $L$, and hence, by the definition of $L$, $J = \emptyset$, a contradiction.

So $n \leq 3$. Suppose $n = 3$. We may assume $J = \{1, 2, 3\}$. For $j = 1, 2, 3$, let $B_j$ be an odd bow with ends in $VR_j$. Then $B_1, B_3, B_5$ are pairwise disjoint, for suppose that (say) $B_1$ and $B_3$ have a vertex in common. Choose an end $a$ of $B_1$ with $a \neq s_1$. Follow $B_3$ from $a$ until we reach $B_3$. We can continue along $B_3$ so as to create an odd bow $B$ (as $B_3$ is an odd bow). As $B$ crosses $e_1$, this contradicts (2.1).

So $B_1, B_3, B_5$ are pairwise disjoint. Let $R'_j$ be obtained from $R_j$ by replacing part of $R_j$ by $B_j$. Then $R_1, R_2, R_3, R_4, R_5$ and $e_1, e_2, e_3$ form a bad $K_4$, a contradiction.

So $n = 2$. As the lemma does not hold, we know $J = \{1, 2, 3, 4\}$. For $j = 1, \ldots , 4$, let $B_j$ be an odd bow with ends in $VR_j$. If the $B_j$ are pairwise internally vertex-disjoint, we obtain a bad $K_4$, a contradiction. So at least two of the $B_j$ have an internal vertex in common. Define $S := \{s_1, s_2, s_3, s_4\}$. To analyze this, we first prove the following:

\begin{equation}
\tag{2.4}
\text{Let } B \text{ be a bow with ends } a, b \text{ and } a \in VR_1 \setminus S \text{ and } b \notin VR_1.
\end{equation}

Then $a$ and $b$ are equal to the middle vertices of $R_1$ and $R_3$, respectively.

By (2.1), $B$ is an even bow. By symmetry, we can assume that $b \in VR_2 \cup VR_3 \setminus \{s_1, s_3\}$. Let $C'$ be the (even) circuit obtained from $C$ by replacing the $a - b$ path $P$ along $C$. Let $e'$ be the extension of $e_1$ with the $s_1 - a$ part of $R_1$. So $e'$ is an odd bow of $C'$. If $b \in VR_2$, then $e_2$ is a chord of $C'$ that crosses $e'_1$, contradicting (2.1). So $b \in VR_3 \setminus S$.

Let $e'_2$ be the extension of $e_2$ with the $s_3 - b$ part of $R_3$. Again, $e'_2$ is an odd bow of $C'$. Then $C', e'_1, e'_2$ form an odd $K_4$-subdivision $H$, with trivalent vertices $a, b, s_3,$
and $s_4$. As $H$ is not bad, and as $s_4$ is nonadjacent (in $H$) to $b$ and $s_3$, we know that $s_4$ is adjacent (in $H$) to $a$. By symmetry, $a$ is adjacent to $s_1$, and $b$ to $s_2$ and to $s_3$. This gives (2.4).

From this we derive the following:

Let $T$ be a tree with three end vertices $a, b, c$, and trivalent vertex $v$ such that $T$ has only its end vertices in common with $C$ and such that $a, b, c$ do not all belong to some $VR_i$ ($i = 1, \ldots, 4$). Then for some $i$, \{a, b, c\} = \{s_{i-1}, s_i, s_{i+1}\}, $s_i$ is adjacent to $v$, and the $v - s_{i-1}$ and $v - s_{i+1}$ paths along $T$ are even.

We first show that $a, b, c \in S$. Suppose not. Then we can assume $a \in VR_1 \setminus S$. Since $a, b, c$ not all belong to $VR_1$, we can assume that $b \notin VR_1$. Then by (2.4), $a$ and $b$ are the middle vertices of $R_1$ and $R_3$, respectively. By symmetry of $a$ and $b$, we can assume that $c \notin VR_1$, implying similarly that $c = b$, a contradiction. So $a, b, c \in S$.

Next we can assume that \{a, b, c\} = \{s_1, s_2, s_3\}. Let $P_i$ be the $v - s_i$ path in $T$ (for $i = 1, 2, 3$) (cf. Figure 2.1(a)). As $P_1$ and $P_3$ form a bow connecting $s_1$ and $s_3$, it is an even bow and we have $|EP_1| \equiv |EP_3| \pmod 2$. If, moreover, $|EP_1| \equiv |EP_2| \pmod 2$, then $P_1, P_2, P_3, R_1, R_4, e_1, e_2$ form a bad $K_4$. So $|EP_1| \not\equiv |EP_2| \pmod 2$. Then $P_1, P_2, P_3, R_2, R_3,$ and $e_1$ form an odd $K_4$. As it is not bad and as $e_1$ has length 1, we have $|EP_2| = 1$, implying (2.5).

This implies that

\begin{equation}
(2.6) \quad G - VC \text{ has no component } K \text{ with } s_1, s_2, s_3, s_4 \in N(K).
\end{equation}

Otherwise, there is a tree $T$ intersecting $VC$ only in its end vertices $s_1, s_2, s_3, s_4$. By (2.5), the neighbor $v_i$ of any $s_i$ in $T$ has degree at least 3 (by considering a subtree with ends $s_{i-1}, s_i, s_{i+1}$). It also follows from (2.5) that $v_i \neq v_{i+1}$ for each $i$. So $v_1 = v_4$, contradicting (2.5) (by considering a subtree with ends $s_1, s_2, s_3$). This gives (2.6).

This implies that $B_1$ and $B_3$ are disjoint. Otherwise, by (2.5), the ends of $B_1$ and $B_3$ are $s_1, s_2, s_3, s_4$, contradicting (2.6). Similarly, $B_2$ and $B_4$ are disjoint.

So we can assume that $B_2$ and $B_3$ have a vertex in common, and hence, by (2.5), that there is a vertex $v \notin VC$ adjacent to $s_2$ and a $v - s_1$ path $Q_2$ and a $v - s_3$ path $Q_3$ such that, for $i = 2, 3$, $B_i$ is the concatenation of the edge $s_2v$ and $Q_i$ (cf. Figure 2.1(b)).
By (2.6), neither $B_1$ nor $B_4$ has an internal vertex in common with $B_2$ and $B_3$. If $B_1$ and $B_4$ are internally vertex-disjoint, then $B_1, B_4, e_1, e_2, vs_2, Q_1, Q_2$, and parts of $R_1$ and $R_4$ form a bad $K_4$.

So $B_1$ and $B_4$ are not internally vertex-disjoint. Hence, by (2.5), there is a vertex $u \not\in VC$ adjacent to $s_4$ and a $u-s_1$ path $Q_1$ and a $u-s_3$ path $Q_4$ such that, for $i = 1, 4$, $B_i$ is the concatenation of the edge $s_4u$ and $Q_i$ (cf. Figure 2.1(c)). Then $Q_1, \ldots, Q_4, vs_2, us_4, e_2, e_1$ form a bad $K_4$, a contradiction.

3. Strong $t$-perfection of bad-$K_4$-free graphs. We now prove our main theorem.

**Theorem 3.1.** A graph without bad $K_4$ is strongly $t$-perfect.

**Proof.** Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimum. For any weight function $w : V \rightarrow \mathbb{Z}_+$, denote $\alpha_w := \alpha_w(G)$ and $\tilde{\rho}_w := \tilde{\rho}_w(G)$. For any subset $U$ of $V$ let $\chi^U$ be the incidence vector of $U$. So for an edge $e = uv$, $\chi^e$ is the 0,1 vector in $\mathbb{R}^V$ having 1's in positions $u$ and $v$.

We first show the following claim.

**Claim 1.** There is a $w : V \rightarrow \mathbb{Z}_+$ and an edge $f$ such that

\[ \tilde{\rho}_w + \chi^f = \alpha_w + 1 = \tilde{\rho}_w \]

and such that

\[ \alpha_w - \chi^C = \tilde{\rho}_w - \chi^C \]

for each odd circuit $C$.

**Proof of Claim 1.** Choose a vertex $u$. For any $w : V \rightarrow \mathbb{Z}_+$ with $\alpha_w < \tilde{\rho}_w$ one has

\[ w(u) < w(N(u)) \]

(where $N(u)$ denotes the set of neighbors of $u$). Otherwise, by the minimality of $G$, setting $G' := G - u - N(u)$ and $w' := w|VG'$,

\[ \alpha_w(G) = w(u) + \alpha_w'(G') = w(u) + \tilde{\rho}_w'(G') \geq \tilde{\rho}_w(G), \]

since $G'[\{u\} \cup N(u)]$ has a $w|N(u) \cup \{u\}$-cover of cost $w(u)$ (as $w(u) \geq w(N(u))$). Equation (3.4) contradicts our assumption, which proves (3.3).

By (3.3), we can choose $w$ such that $\alpha_w < \tilde{\rho}_w$ and such that $w(V \setminus \{u\}) - w(u)$ is as small as possible. Then

\[ \text{there exists a } z \in \mathbb{Z}^+ \text{ such that for } \tilde{w} := w + \sum_{e \in \delta(u)} z_e \chi^e \text{ we have } \alpha_{\tilde{w}} = \tilde{\rho}_{\tilde{w}}. \]

To see this, it suffices to show that

\[ \text{there exists a } z \in \mathbb{Z}^+ \text{ and a stable set } S \text{ such that } \tilde{w} := w + \sum_{e \in \delta(u)} z_e \chi^e \text{ is } \]

nonnegative and such that $\tilde{w}(S) = \tilde{\rho}_w$ and $S$ intersects each edge incident with $u$.

\[ \text{(3.6)} \]

This suffices, since if $z'$ arises from $z$ by replacing the negative entries by 0, and

\[ w' := w + \sum_{e \in \delta(u)} z'_e \chi^e, \]

\[ \text{(3.7)} \]
then \( w'(S) = \hat{w}(S) - \sum(z_e|z_e < 0) \) and \( \hat{\rho}_w \leq \hat{\rho}_{\hat{w}} - \sum(z_e|z_e < 0) \), as \( w' = \hat{w} - \sum(z_e|z_e < 0) \). This implies (3.5).

To prove (3.6), first suppose that \( N(u) \) is a stable set. Let \( G' \) be the graph obtained from \( G \) by contracting the edges in \( \delta(u) \). Then \( G' \) contains no bad \( K_4 \). Let \( t \) be the new vertex. Let \( w' : VG' \rightarrow \mathbb{Z}_+ \) be defined by \( w'(t) := w(N(u)) - w(u) \) and \( w'(v) := w(v) \) if \( v \neq t \). Since \( G' \) is smaller than \( G \), we know \( \alpha_{w'}(G') = \hat{\rho}_w(G') \).

Consider a \( w' \)-cover \( \mathcal{F}' \) in \( G' \) of cost \( \hat{\rho}_w(G') \). Let \( \lambda \) be the number of circuits in \( \mathcal{F}' \) that are not circuits in \( G \). So they traverse \( t \) and can be made to circuits in \( G \) by adding two edges incident with \( u \). It gives, for some \( \hat{w} \), a \( \hat{w} \)-cover \( \mathcal{F} \) in \( G \) of cost \( \hat{\rho}_w(G') + \lambda \) such that \( \hat{w} \) coincides with \( w \) on \( V \setminus (N(u) \cup \{u\}) \) and such that \( \hat{w}(u) = \lambda \) and \( \hat{w}(N(u)) = w'(t) + \lambda \). Hence the cost is \( \hat{\rho}_w(G') + \hat{w}(u) \) and \( \hat{w}(N(u)) - \hat{w}(u) = w(N(u)) - w(u) \). This last implies that \( \hat{w} = \hat{w} + \sum_{e \in \delta(u)} z_e \chi^e \) for some \( z \in \mathbb{Z}^{\delta(u)} \).

Now let \( S' \) be a stable set in \( G' \) with \( w'(S') = \alpha_{w'}(G') \). If \( t \in S' \), define \( S := (S' \setminus \{t\}) \cup N(u) \), and if \( t \notin S' \), define \( S := S' \cup \{u\} \). So \( S \) is a stable set in \( G \). Then \( w(S) = w'(S') + w(u) \) and \( S \) intersects each edge incident with \( u \). So
\[
(3.8) \quad \hat{w}(S) = w'(S') + \hat{w}(u) = \hat{\rho}_w(G') + \hat{w}(u) \geq \hat{\rho}_{\hat{w}}(G).
\]
This gives (3.6) in case \( N(u) \) is a stable set.

If \( N(u) \) is not a stable set, let \( G' := G - u - N(u) \) and \( w' := w|VG' \). By the minimality of \( G \), \( \alpha_{w'}(G') = \hat{\rho}_w(G') \). Let \( \mathcal{F}' \) be a \( w' \)-cover in \( G' \) of cost \( \hat{\rho}_w(G') \). By adding to \( \mathcal{F}' \) a number of times a triangle incident with \( u \) we obtain a \( \hat{w} \)-cover \( \mathcal{F} \) in \( G \) for some \( \hat{w} : V \rightarrow \mathbb{Z}_+ \), where \( \hat{w} \) coincides with \( w \) on \( V \setminus (\{u\} \cup N(u)) \), where \( \hat{w}(N(u)) - \hat{w}(u) = w(N(u)) - w(u) \), and where \( \mathcal{F} \) has cost \( \hat{\rho}_w(G') + \hat{w}(u) \).

Now let \( S' \) be a stable set in \( G' \) with \( w'(S') = \alpha_{w'}(G') \). Define \( S := S' \cup \{u\} \). So \( S \) is a stable set in \( G \). Then \( w(S) = w'(S') + w(u) \) and \( S \) intersects each edge incident with \( u \). Moreover, \( \hat{w}(S) = w'(S') + \hat{w}(u) = \hat{\rho}_w(G') + \hat{w}(u) \geq \hat{\rho}_{\hat{w}}(G) \). So we have (3.6), and hence (3.5).

Choose \( z \) in (3.5) with \( z|\delta(u) \) as small as possible. Choose \( f \in \delta(u) \) with \( z_f \geq 1 \). We can assume that \( z_f = 1 \) and \( z_e = 0 \) for all other edges \( e \), as we can reset \( w := \hat{w} - \chi^f \). (This resetting does not change the value of \( w(V \setminus \{u\}) - w(u) \).) Then (3.2) follows from the minimality of \( w(V \setminus \{u\}) - w(u) \).

We finally show (3.1). By the definition of \( \hat{\rho}_{w+\chi^f} = \alpha_{w+\chi^f} \). Also we have \( \alpha_{w+\chi^f} \leq \alpha_w + 1 \), since any stable set \( S \) satisfies \( (w + \chi^f)(S) \leq w(S) + 1 \). As \( \hat{\rho}_w \leq \hat{\rho}_{w+\chi^f} \), this implies (3.1).

As of now we assume that \( w \) and \( f \) satisfy (3.1) and (3.2). Let \( f \) connect vertices \( u \) and \( u' \). Since by the minimality of \( G \), \( G \) has no isolated vertices, there exists a minimum-cost \( w + \chi^f \)-cover consisting only of edges and odd circuits, say, \( e_1, \ldots, e_t, C_1, \ldots, C_k \). We choose \( f \) and \( e_1, \ldots, e_t, C_1, \ldots, C_k \) such that
\[
(3.9) \quad |VC_1| + \cdots + |VC_k|
\]
is as small as possible. Then
\[
(3.10) \quad \text{at least two of the } C_i \text{ traverse } f.
\]
To see this, let \( G' := G - f \). If \( \alpha_w(G') = \alpha_w(G) \), then by induction \( G' \) has a \( w \)-cover of cost \( \alpha_w \). As this is a \( w \)-cover in \( G \) as well, this would imply \( \alpha_w = \hat{\rho}_w \), a contradiction.

So \( \alpha_w(G') > \alpha_w(G) \). That is, there exists a stable set \( S \) in \( G' \) with \( w(S) > \alpha_w \). Necessarily, \( S \) contains both \( u \) and \( u' \). Then, for any circuit \( C \) traversing \( f \),
\[
(3.11) \quad |VC \cap S| \leq \left\lfloor \frac{1}{2} |VC| \right\rfloor + 1.
\]
Also, $f$ is not among $e_1, \ldots, e_t$, since otherwise $\rho_w \leq \rho_{w+x'} - 1$, contradicting (3.1).

Setting $l$ to the number of $C_i$ traversing $f$, we obtain

$$\tilde{\rho}_{w+x'} \leq \alpha_w + 1 \leq w(S) = (w + \chi f)(S) - 2 \leq -2 + \sum_{j=1}^t |e_j \cap S| + \sum_{i=1}^k |VC_i \cap S|$$

(3.12) \quad \leq -2 + t + \sum_{i=1}^k \frac{1}{2} |VC_i| + l = \tilde{\rho}_{w+x'} + l - 2.

So $l \geq 2$, which is (3.10).

By (3.10) we can assume that $C_1$ and $C_2$ traverse $f$. It is convenient to assume that $EC_1 \setminus \{f\}$ and $EC_2 \setminus \{f\}$ are disjoint; this can be achieved by adding parallel edges. So $EC_1 \cap EC_2 = \{f\}$.

Then,

if $C$ is an odd circuit with $EC \subseteq EC_1 \cup EC_2$, then $f \in EC$ and $EC_1 \triangle EC_2 \Delta EC$

(3.13) again is an odd circuit.

To see this, define $C'_i := C$. As $EC_1 \triangle EC_2 \Delta EC$ is an odd cycle (a cycle is an edge-disjoint union of circuits), it can be decomposed into circuits $C'_2, \ldots, C'_p$, with $C'_2, \ldots, C'_q$ odd and $C'_{q+1}, \ldots, C'_p$ even ($q \geq 2$). Choose for each $i = q + 1, \ldots, p$ a perfect matching $M_i$ in $C'_i$. Let $e'_1, \ldots, e'_r$ be the edges in the matchings $M_i$ and in $\{f\} \setminus EC$. Then

$$\chi^{VC_1} + \chi^{VC_2} = \sum_{i=1}^q \chi^{VC'_i} + \sum_{j=1}^r \chi^{e'_j}$$

and

$$\frac{1}{2} |VC_1| + \frac{1}{2} |VC_2| = \frac{1}{2} |EC_1| + \frac{1}{2} |EC_2| - 1 = r - 1 + \frac{1}{2} \sum_{i=1}^q |EC'_i|$$

(3.15) \quad \geq r + \sum_{i=1}^q \frac{1}{2} |VC'_i|.

So replacing $C_1, C_2$ by $C'_1, \ldots, C'_q$ and adding $e'_1, \ldots, e'_r$ to $e_1, \ldots, e_t$ again gives a $w + \chi f$-cover of cost at most $\tilde{\rho}_{w+x'}$.

If $f \notin EC$, then $f$ is among $e'_1, \ldots, e'_r$. Hence deleting $f$ gives a $w$-cover of cost at most $\tilde{\rho}_{w+x'} - 1 \leq \alpha_w$, contradicting (3.1). So $f \in EC$. As this is true for any odd circuit in $EC_1 \cup EC_2$ we know that $f \in EC'_i$ for $i = 1, \ldots, q$ and that $q = 2$.

If $p \geq 3$ or $r \geq 1$, then $|EC'_1| + |EC'_2| < |EC_1| + |EC_2|$, contradicting the minimality of (3.9). This proves (3.13).

First, it implies

$$a circuit in EC_1 \cup EC_2 is odd if and only if it contains f.$$  

(3.16)

A second consequence is as follows. Let $P_i$ be the $u - u'$ path $C_i \setminus \{f\}$. Orient the edges occurring in the path $P_i := C_i \setminus \{f\}$ in the direction from $u$ to $u'$ for $i = 1, 2$.

Then

$$the orientation is acyclic.$$

(3.17)
For suppose there exists a directed circuit \( C \). Then \((EC_1 \cup EC_2) \setminus EC\) contains a directed \( u - u' \) path, and hence an odd circuit \( C' \). Hence, by (3.13), \( EC_1 \cup EC_2 \cup EC' \) is an odd circuit, however, containing the even circuit \( EC \), a contradiction.

Let \( A \) and \( B \) be the color classes of the bipartite graph \((VP_1 \cup VP_2, EP_1 \cup EP_2)\) such that \( u, u' \in A \). So

\[
(3.18) \quad A := \{v \in VP_1 \cup VP_2 \mid \text{there exists an even-length directed } u - v \text{ path}\},
\]

\[
B := \{v \in VP_1 \cup VP_2 \mid \text{there exists an odd-length directed } u - v \text{ path}\}.
\]

Define

\[
(3.19) \quad X := VP_1 \cap VP_2
\]

\[
\text{and } U := \left\{v \in V \mid w(v) = \sum_{j=1}^{t} |e_j \cap \{v\}| + \sum_{j=1}^{k} |VC_j \cap \{v\}| \right\}.
\]

We next show the following technical, but straightforward to prove, claim.

Claim 2. Let \( z \in A \), let \( Q \) be an even length directed \( u - z \) path, and let \( S \) be a stable set in \( G \). Then

\[
(3.20) \quad (w - \chi^Q)(S) \geq \alpha_w - \left\lfloor \frac{1}{2} |VQ| \right\rfloor + 1
\]

if and only if

\[
(3.21) \quad \begin{align*}
(i) & \quad |e_j \cap S| = 1 \text{ for each } j = 1, \ldots, t, \\
(ii) & \quad |VC_j \cap S| = \left\lfloor \frac{1}{2} |VC_j| \right\rfloor \text{ for } j = 3, \ldots, k, \\
(iii) & \quad S \subseteq U, \\
(iv) & \quad S \text{ contains } B \setminus VQ \text{ and is disjoint from } A \setminus VQ, \\
(v) & \quad S \text{ contains } B \cap X \text{ and is disjoint from } A \cap X.
\end{align*}
\]

Proof of Claim 2. We can assume that \( EQ \subseteq EC_1 \). Set \( W := VC_1 \setminus VQ \). So \( |W| \) is even. Consider the following sequence of (in)equalities:

\[
(3.22) \quad \begin{align*}
(w - \chi^Q)(S) &= w(S) - |VQ \cap S| \leq (w + \chi^J)(S) - |VQ \cap S| \\
&\leq \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=1}^{k} |VC_j \cap S| - |VQ \cap S| = \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=2}^{k} |VC_j \cap S| + |W \cap S| \\
&\leq t + \sum_{j=2}^{k} \left\lfloor \frac{1}{2} |VC_j| \right\rfloor + |W \cap S| = \rho_{w+\chi^J} - \left\lfloor \frac{1}{2} |VC_1| \right\rfloor + |W \cap S| \\
&\leq \rho_{w+\chi^J} - \frac{1}{2} |VC_1| + \frac{1}{2} |W| = \alpha_w + 1 - \left\lfloor \frac{1}{2} |VQ| \right\rfloor.
\end{align*}
\]

Hence (3.20) holds if and only if equality holds throughout in (3.22), which is equivalent to (3.21).

End of Proof of Claim 2.

By (3.17), we can order the vertices in \( X \) as \( v_0 = u, v_1, \ldots, v_s = u' \) such that both \( P_1 \) and \( P_2 \) traverse them in this order. For \( j = 0, \ldots, s \), let \( P_j \) be the collection of directed \( u - x \) paths, where \( x = v_j \) if \( v_j \in A \) and \( x \) is an in-neighbor of \( v_j \) if \( v_j \in B \). So \( x \in A \).

Let \( j \) be the largest index for which there exists a path \( Q \in P_j \) with

\[
(3.23) \quad \alpha_{w - \chi^Q} \leq \alpha_w - \left\lfloor \frac{1}{2} |VQ| \right\rfloor.
\]
Such a $j$ exists, since (3.23) holds for the trivial directed $u - u$ path, as $\alpha_{u - u} \leq \alpha_u$. Also, $j < s$, since otherwise $VQ = VC$ for some odd circuit $C$, and hence with (3.2) we have

$$\tilde{\rho}_w \leq \tilde{\rho}_{w - \chi V^c} + \left| \frac{1}{2} |VC| \right| = \alpha_{w - \chi V^c} + \left| \frac{1}{2} |VC| \right| \leq \alpha_w,$$

contradicting (3.1).

Let $Q_1$ and $Q_2$ be the two paths in $P_{j+1}$ that extend $Q$. By the maximality of $j$, we know

$$\alpha_{w - \chi V Q_i} \geq \alpha_w - \left| \frac{1}{2} |VQ_i| \right| + 1.$$

Hence there exist stable sets $S_1$ and $S_2$ with

$$\alpha_{w - \chi V Q_i}(S_i) \geq \alpha_w - \left| \frac{1}{2} |VQ_i| \right| + 1$$

for $i = 1, 2$. So, for $i = 1, 2$, (3.21) holds for $Q_i, S_i$. By (3.21)(iv), $S_1$ and $S_2$ coincide on $VP_1 \cup VP_2$ except on $VQ_1 \cup VQ_2$. In other words,

$$\alpha_{w - \chi V Q_i}(S_1 \triangle S_2) \cap (VP_1 \cup VP_2) \subseteq VQ_1 \cup VQ_2.$$

By (3.21)(v), $S_1$ and $S_2$, moreover, coincide on $X$.

Let $H$ be the subgraph of $G$ induced by $S_1 \triangle S_2$. So $H$ is a bipartite graph, with color classes $S_1 \setminus S_2$ and $S_2 \setminus S_1$. Define

$$Y_i := VQ_i \setminus VQ$$

for $i = 1, 2$. Then

$$Y_i \cap Y_j = \emptyset,$$

for $i, j = 1, 2$. Then

$$H$$

contains a path connecting $Y_1$ and $Y_2$.

For suppose not. Let $K$ be the union of the components of $H$ that intersect $Y_1$. So $K$ is disjoint from $Y_2$. Define $S := S_1 \triangle K$. Then $S \cap Y_1 = S_2 \cap Y_1$ and $S \cap Y_2 = S_1 \cap Y_2$. This implies that $Q, S$ satisfy (3.23). Hence (3.20) holds, contradicting (3.23). This proves (3.29).

Let $C$ be the (even) circuit formed by the two directed $v_j - v_{j+1}$ paths. So $Y_1$ and $Y_2$ are subsets of $V C$. Let $R$ be a shortest path in $H$ that connects $Y_1$ and $Y_2$; say it connects $y_1 \in Y_1$ and $y_2 \in Y_2$.

Since $y_1, y_2 \in S_1 \triangle S_2$, we know by (3.21)(v) that $y_1, y_2 \not\in X$. By (3.21)(iv), if $y_1 \in S_1 \setminus S_2$, then $y_1 \in A$ and if $y_1 \in S_2 \setminus S_1$, then $y_1 \in B$. Similarly, if $y_2 \in S_2 \setminus S_1$, then $y_2 \in A$ and if $y_2 \in S_1 \setminus S_2$, then $y_2 \in B$.

So if $R$ is even, then $y_1$ and $y_2$ belong to different sets $A, B$, and if $R$ is odd, then $y_1$ and $y_2$ belong to the same set among $A, B$. Hence $R$ forms with part of $C$ an odd circuit.

By (3.27) and as $(S_1 \triangle S_2) \cap X = \emptyset$, there exist a directed $u - v_j$ path $N'$ and a directed $v_{j+1} - u'$ path $N''$ that are (vertex-)disjoint from $S_1 \triangle S_2$. $N'$, $N''$, and $f$ make a $v_j + v_{j+1}$ path $N$. Then $N, R,$ and $C$ make an odd $K_4$, with 3-valent vertices $v_j, v_{j+1}, y_1, y_2$.

By assumption, it is not a bad $K_4$; that is, it satisfies (1.4). Suppose first that $R$ has even length. Then by (1.4) $N$ also has even length. Hence $v_j$ and $v_{j+1}$ belong to different sets $A, B$. Then by (1.4) and the symmetry of $y_1$ and $y_2$, we may assume that $y_1$ is adjacent to $v_j$ and that $y_2$ is adjacent to $v_{j+1}$. Hence, as $y_1, y_2 \in S_1 \cup S_2$, $v_j$ and $v_{j+1}$ do not belong to $S_1 \cap S_2$, and so $v_j, v_{j+1} \not\in B$ (by (3.21)(v)), a contradiction.
So $R$ has length 1. Hence $N$ has length 1 as well, and $v_j, v_{j+1}, y_1, y_2$ lie in the same color class of the bipartition $A, B$ of $C$. So we know that

$$v_j = u, \quad v_{j+1} = u', \quad y_1, y_2 \in A, \quad \text{and } R \text{ has length 1.}$$

Let $D$ be the set of edges of $G$ connecting two vertices in $A$. So $f \in D$ and $y_1 y_2 \in D$. Hence $|D| \geq 2$. We consider the edges in $D$ as chords of the circuit $C$ with $EC = EP_1 \cup EP_2$.

Now any edge $d$ in $D$ can play the same role as $f$, since if $C_1'$ and $C_2'$ denote the two odd circuits in $EC \cup \{d\}$, then

$$C_1', C_2', C_3, \ldots, C_k, e_1, \ldots, e_t \text{ form a } w + \chi^d \text{-cover of cost } \tilde{\rho}_{w + \chi^d} = \tilde{\rho}_{w + \chi^f}. $$

Indeed, as $\chi^{C_1'} + \chi^{C_2'} = \chi^d + \chi^{C_1} + \chi^{C_2} - \chi^f$, the collection $C_1', C_2', C_3, \ldots, C_k, e_1, \ldots, e_t$ is a $w + \chi^d \text{-cover of cost } \tilde{\rho}_{w + \chi^d}$ with $|VC_1'| + |VC_2'| + |VC_3| + \cdots + |VC_k|$ at most $(3.9)$. Hence $(3.31)$ follows from the choice of $f$.

So each $d \in D$ has all the properties derived for $f$ so far, and it would lead to the same circuit $C$ and to the same bipartition $A, B$ of $C$.

This is used to prove that

$$(3.32) \quad \text{any edge in } D \text{ crosses any chord of } C.$$ 

Indeed, we need only to prove this for $f$. However, by the minimality of $(3.9)$ all circuits among $C_1, \ldots, C_k$ are chordless, so each chord of $C$ crosses $f$.

Let $n := |D|$, and let $s_1, s_2, \ldots, s_{2n}$ be the ends of the edges in $D$, in cyclic order. Let $f_1, \ldots, f_{2n}$ be the edges in $D$ incident with $s_1, \ldots, s_{2n}$, respectively. So $f_{n+j} = f_j$ for all $j$ (taking indices mod $2n$). For $j = 1, \ldots, 2n$, let $R_j$ be the $s_{j-1} - s_j$ path along $C$ that does not contain any other of the vertices $s_i$.

By Lemma 2.1, we can assume that $2, \ldots, n \not\in J$, where $J$ is as defined in $(2.2)$. Let $Q_1$ be the path of the form $Q = R_{j+1} R_{j+2} \cdots R_n$ with $0 \leq j \leq n$ such that

$$(3.33) \quad \alpha_{w - \chi Q} \geq \alpha_w - \left[ \frac{1}{2} |VQ| \right] + 1$$

and such that $j$ is maximal. This path exists, since for $j = 0$ we have $(3.33)$, as otherwise $(3.24)$ would again yield a contradiction.

Trivially, $j < n$, since the empty path does not satisfy $(3.33)$. Let $Q_2 := R_{j+2} R_{j+3} \cdots R_{j+n}$. Since $Q_2$ also satisfies $(3.33)$ (as, again, $(3.24)$ would yield a contradiction otherwise), there exist stable sets $S_1$ and $S_2$ with

$$(3.34) \quad (w - \chi^{Q_i})(S_i) \geq \alpha_w - \left[ \frac{1}{2} |VQ_i| \right] + 1$$

for $i = 1, 2$. So, for $i = 1, 2$, $(3.21)$ holds for $Q_1, S_i$ where we can take for $f$ any edge not incident with an internal vertex of $Q_i$. By $(3.21)(iv)$,

$$(3.35) \quad (S_1 \triangle S_2) \cap VC \subseteq VQ_1 \cup VQ_2.$$ 

We (re)define $H$ as the subgraph of $G$ induced by $S_1 \triangle S_2$. Define

$$(3.36) \quad Y_1 := VR_{j+1} \text{ and } Y_2 := VR_{n+1} \cup VR_{n+2} \cup \cdots \cup VR_{n+j+1}.$$ 

Then

$$(3.37) \quad H \text{ contains a path connecting } Y_1 \text{ and } Y_2.$$
For suppose not. Let $K$ be the union of the components of $H$ that intersect $Y_1$. So $K$ is disjoint from $Y_2$. Define $S := S_1 \triangle K$. Then $S \cap Y_1 = S_2 \cap Y_1$ and $S \cap Y_2 = S_1 \cap Y_2$. This implies that $Q := R_{j+2} R_{j+3} \cdots R_n$ and $S$ satisfy (3.21), taking $f := f_n$. Hence (3.20) holds for $Q$, contradicting the maximality of $j$. This proves (3.37).

Let $R$ be a shortest path in $H$ that connects $Y_1$ and $Y_2$; say it connects $y_1 \in Y_1$ and $y_2 \in Y_2$. By (3.35), any internal vertex of $R$ that is on $C$ is an internal vertex of $R_{j+2} R_{j+3} \cdots R_n$. If $y_1 \in S_1 \setminus S_2$, as $y_1$ is not an internal vertex of $Q_2$, we know $y_1 \in A$. Similarly, if $y_1 \in S_2 \setminus S_1$, then $y_1 \in B$. Similarly, if $y_2 \in S_2 \setminus S_1$, then $y_2 \in A$, and if $y_2 \in S_1 \setminus S_2$, then $y_2 \in B$. So $R$ together with the $y_1 - y_2$ part of $R_{j+1} R_{j+2} \cdots R_{n+j+1}$ forms an odd cycle. Hence it contains an odd circuit, and so $R$ contains an odd bow. By (2.1), this bow connects two vertices in some $R_{j+2}, \ldots, R_n$. This contradicts the fact that $j + 2, \ldots, n \notin J$. □

Figure 1.1 gives a strongly t-perfect graph that contains a had $K_4$. So the implication in Theorem 3.1 cannot be reversed. However one has the following corollary.

**Corollary 3.2.** For any graph $G$, the following are equivalent:

(3.38) 
(i) $G$ contains no bad $K_4$;
(ii) each subgraph of $G$ is t-perfect;
(iii) each subgraph of $G$ is strongly t-perfect.

**Proof.** The implication (i)$\Rightarrow$(iii) follows from Theorem 3.1, while the implication (iii)$\Rightarrow$(ii) follows by the observations made in section 1.

The implication (ii)$\Rightarrow$(i) was proved by Barahona and Mahjoub [1]. It suffices to show that a bad $K_4$ is not t-perfect. Choose a smallest counterexample $G$. As $G$ is t-perfect, $G \neq K_4$. If (1.4) does not hold, then $G$ has a vertex $v$ such that contracting the edges in $\delta(v)$ gives an odd $K_4$-subdivision $G'$ that again does not satisfy (1.4). As $G'$ again is a t-perfect odd $K_4$ (as one easily checks), this contradicts the minimality of $G$. □

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