Journal of Combinatorial Theory, Series B **85**, 255–260 (2002)
doi:10.1006/jctb.2001.2101

# A Short Proof of Guenin's Characterization of Weakly Bipartite Graphs

Alexander Schrijver

*CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands; and Department of Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands*

Received February 19, 2001; published online April 29, 2002

We give a proof of Guenin's theorem characterizing weakly bipartite graphs by not having an odd-$K_5$ minor. The proof curtails the technical and case-checking parts of Guenin's original proof. © 2002 Elsevier Science (USA)

## 1. INTRODUCTION

A *signed graph* is a pair $(G, \Sigma)$, where $G = (V, E)$ is an undirected graph and $\Sigma \subseteq E$. Call a set of edges, or path, or circuit *odd* (*even*, respectively) if it contains an odd (even, respectively) number of edges in $\Sigma$. An *odd circuit cover* is a set of edges intersecting all odd circuits.

Following Grötschel and Pulleyblank [1], a signed graph $(G, \Sigma)$ is called *weakly bipartite* if each vertex of the polyhedron (in $\mathbb{R}^E$) determined by

(1)
$$\text{(i)}\quad x(e) \geqslant 0 \qquad \text{for each edge } e,$$
$$\text{(ii)}\quad \sum_{e \in C} x(e) \geqslant 1 \qquad \text{for each odd circuit } C,$$

is integer, that is, the incidence vector of an odd circuit cover. Weakly bipartite graphs are of importance since a maximum-capacity cut in such graphs can be found in polynomial time (as one can optimize over (1) in polynomial-time, with the ellipsoid method).

For any $U \subseteq V$, the signed graphs $(G, \Sigma)$ and $(G, \Sigma \triangle \delta(U))$ have the same collection of odd circuits. ($\triangle$ denotes symmetric difference; $\delta(U)$ is the edge cut determined by $U$.) Hence being weakly bipartite is invariant under such an operation. We call two such signed graphs *equivalent*.

It is not difficult to see that for each inclusionwise minimal odd circuit cover $B$, the set $B \triangle \Sigma$ is a cut. Hence $|C \cap B|$ is odd for any odd circuit $C$ and any inclusionwise minimal odd circuit cover $B$.



0095-8956/02 $35.00

Guenin [2, 3] gave a characterization of weakly bipartite graphs in terms of forbidden minors, thus proving a special case of a conjecture of Seymour [6]. To describe the characterization, let $(G=(V,E),\Sigma)$ be a signed graph, and let $e \in E$. *Deleting* $e$ means deleting $e$ from $E$ and $\Sigma$. *Contracting* $e$ means first, if $e \in \Sigma$, resetting $\Sigma := \Sigma \triangle \delta(v)$ (where $v$ is some end of $e$), and next contracting $e$ in $G$. This operation is dependent on the choice of $v$, but the result is unique up to equivalence. A signed graph $(G', \Sigma')$ is called a *minor* of a signed graph $(G, \Sigma)$ if $(G', \Sigma')$ arises from $(G, \Sigma)$ by a series of deletions of vertices and edges, contractions of edges, and substitution by an equivalent signed graph. Being weakly bipartite is maintained under taking minors.

The signed graph $\tilde{K}_5 := (K_5, EK_5)$ is *not* weakly bipartite, since $x(e) := \frac{1}{3}$ ($e \in EK_5$) satisfies (1) but is not a convex combination of odd circuit covers (as each odd circuit cover has size at least $4 > \frac{10}{3}$). So any signed graph having $\tilde{K}_5$ as a minor is not weakly bipartite. Guenin [2, 3] proved that also the converse holds:

THEOREM. *A signed graph is weakly bipartite if and only if it has no* $\tilde{K}_5$ *minor.*

We give a proof of Guenin's theorem shorter than that of Guenin. In fact, our proof follows the framework of his proof, but saves considerably on the technical parts of the proof, by applying a lemma proved in the following section.

## 2. A LEMMA

An *odd-*$K_4$ is an undirected graph obtained from $K_4$ by replacing edges by paths such that each triangle of $K_4$ becomes a circuit with an odd number of edges.

LEMMA. *Let* $G = (V, E)$ *be a graph, let* 0 *be a vertex of* $G$, *and let* 1, 2, *and* 3 *be three of its neighbours. Let* $S_1$, $S_2$, *and* $S_3$ *be pairwise disjoint stable sets in* $G$, *with* $i \in S_i$ *for* $i = 1, 2, 3$. *Suppose that for all distinct* $i, j$, *the graph induced by* $S_i \cup S_j$ *contains a path connecting* $i$ *and* $j$. *Then* $G$ *has an odd-*$K_4$ *subgraph containing the edges 01, 02, and 03.*

*Proof.* Consider a counterexample with $|V|+|E|$ minimal. So $V = S_1 \cup S_2 \cup S_3 \cup \{0\}$ and $E$ consists of the edges 01, 02, and 03, and of the edges contained in the paths as described. Hence for distinct $i, j$, there is a unique path $P_{i,j}$ from $i$ to $j$ contained in $S_i \cup S_j$. Also

(2) for distinct $i, j$, $\quad S_i \cup S_j = VP_{i,j}$.

For if $v \in (S_i \cup S_j) \setminus VP_{i,j}$, we can contract the (two) edges incident with $v$ to obtain a smaller counterexample, a contradiction.

Condition (2) implies $|S_1| = |S_2| = |S_3|$. If $|S_1| = 1$, we have an odd-$K_4$ as required, so we can assume that each $|S_i| \geqslant 2$. So each path $P_{i,j}$ has length at least 3. Let $2'$ be the second vertex along $P_{1,2}$, $3'$ the second vertex along $P_{2,3}$, and $1'$ the second vertex along $P_{3,1}$. Contract the edges incident with 0. The new vertex $0'$ is adjacent to $1'$, $2'$, and $3'$. For $i = 1, 2, 3$, let $S'_i := S_i \setminus \{i\}$. So $S'_i$ contains $i'$, and is a stable set in the contracted graph $G'$. Moreover,

(3) for distinct $i, j$, $\quad S'_i \cup S'_j$ contains an $i' - j'$ path.

To prove this, we can assume $i = 1$, $j = 2$. By (2), $1'$ is on $P_{1,2}$. Since also $2'$ is on $P_{1,2}$, this implies that $S_1 \cup S_2$ contains an $1' - 2'$ path avoiding 1 and 2. Hence we have (3).

As $G'$ is smaller than $G$, $G'$ has an odd-$K_4$ subgraph containing $0'1'$, $0'2'$, and $0'3'$. By decontracting, this gives an odd-$K_4$ subgraph in $G$ as required. ∎

## 3. LEHMAN'S THEOREM

Let $(G, \Sigma)$ be a minimally non-weakly bipartite signed graph (minimal under taking minors). We show that $(G, \Sigma)$ contains a $\tilde{K}_5$ minor, which is Guenin's theorem. As in [2], the basis of the proof is a powerful result of Lehman [4] (cf. Padberg [5], Seymour [7]).

Let $n := |E|$, let $r$ be the minimum size of an odd circuit, and let $s$ be the minimum size of an odd circuit cover. Let $M$ ($N$, respectively) be the matrix whose rows are the incidence vectors of the minimum-size odd circuits (minimum-size odd circuit covers, respectively). Now Lehman proved that both $M$ and $N$ have precisely $n$ rows, that $rs > n$, and that the rows of $M$ can be reordered so that

$$MN^T = J + (rs - n)\, I = N^T M. \tag{4}$$

This implies that we can index the minimum-size odd circuits as $C_1, \ldots, C_n$ and the minimum-size odd circuit covers as $B_1, \ldots, B_n$ in such a way that for all $i, j = 1, \ldots, n$,

$$|C_i \cap B_j| = 1 \text{ if } i \neq j, \qquad \text{and} \qquad |C_i \cap B_j| = q \text{ if } i = j, \tag{5}$$

where $q := rs - n + 1$. Since $q = |C_1 \cap B_1|$ is odd and $\geqslant 2$ (as $rs > n$), we have $q \geqslant 3$.

The fact that $N^T M = J + (rs-n) I$ is equivalent to:

(6)
(i) for each $e \in E$ there are precisely $q$ indices $i$ with $e \in C_i \cap B_i$,
(ii) for all distinct $e, f \in E$ there is precisely one index $i$ with $e \in B_i$ and $f \in C_i$.

An important observation (of Guenin [2]) is that for all distinct $i, j = 1, \ldots, n$:

(7) the only odd circuits contained in $C_i \cup C_j$ are $C_i$ and $C_j$; the only odd circuit covers contained in $B_i \cup B_j$ are $B_i$ and $B_j$.

For let $C$ be an odd circuit contained in $C_i \cup C_j$. Then $C_i \triangle C_j \triangle C$ contains an odd circuit, $C'$ say. This implies that $C \cup C' \subseteq C_i \cup C_j$ and $C \cap C' \subseteq C_i \cap C_j$ (for if $e \in C \cap C'$ then $e \notin C_i \triangle C_j$). Hence $|C| + |C'| \leqslant |C_i| + |C_j|$. So also $C$ and $C'$ are minimum-size odd circuits and $C \cup C' = C_i \cup C_j$. As $|C_i \cap B_i| \geqslant 3$ we have $|C \cap B_i| \geqslant 2$ or $|C' \cap B_i| \geqslant 2$. Therefore $C$ or $C'$ is equal to $C_i$, and the other equal to $C_j$. The proof for odd circuit covers is the same.

## 4. CONSTRUCTION OF A $\tilde{K}_5$ MINOR

Fix an edge $e \in E$, with ends $v_1$ and $v_2$, say. By (6)(i) we can assume that $e$ is contained in $C_i \cap B_i$ for $i = 1, \ldots, q$. Then, by (6),

(8) any two sets among $C_1 \setminus \{e\}, \ldots, C_q \setminus \{e\}, B_1 \setminus \{e\}, \ldots, B_q \setminus \{e\}$ are disjoint, except that $|(C_i \setminus \{e\}) \cap (B_i \setminus \{e\})| = q-1$ for $i = 1, \ldots, q$.

To see this, choose distinct $i, j = 1, \ldots, q$. Then $C_i \cap B_j = \{e\}$, as $|C_i \cap B_j| = 1$. Moreover, $C_i \cap C_j = \{e\}$, for suppose $f \in C_i \cap C_j$ with $e \neq f$. Then $f \in C_i \cap C_j$ and $e \in B_i \cap B_j$, contradicting (6)(ii). One similarly shows that $B_i \cap B_j = \{e\}$. This proves (8).

As in Guenin [2] one has

(9) for distinct $i, j = 1, \ldots, q$, $C_i$ and $C_j$ have no vertex $\neq v_1, v_2$ in common.

Otherwise $(C_i \cup C_j) \setminus \{e\}$ contains a path $P$ from $v_1$ to $v_2$ different from $C_i \setminus \{e\}$ and $C_j \setminus \{e\}$. By (7), $(C_i \cup C_j) \setminus \{e\}$ contains no odd circuit. Hence $P$ and $C \setminus \{e\}$ have the same parity, and so $P \cup \{e\}$ is an odd circuit in $C_i \cup C_j$, contradicting (7). This proves (9).

Since $B_i \triangle \Sigma$ is a cut for each $i = 1, 2, 3$, there exist $U_1, U_2, U_3 \subseteq V$ such that

$$(10) \qquad \delta(U_i) = B_j \triangle B_k = (B_j \cup B_k) \setminus \{e\}$$

for all distinct $i, j, k \in \{1, 2, 3\}$. As $e \notin B_j \triangle B_k$, we can assume $v_1, v_2 \notin U_i$. Also

(11) $\qquad U_i$ induces a connected subgraph of $G$.

If not, there is a $K \subseteq U_i$ such that $\delta(K)$ is a nonempty proper subset of $\delta(U_i)$. Then $B_j \triangle \delta(K)$ is an odd circuit cover contained in $B_j \cup B_k$, distinct from $B_j$ and $B_k$, contradicting (7).

By (10), $\delta(U_1 \triangle U_2 \triangle U_3) = \delta(U_1) \triangle \delta(U_2) \triangle \delta(U_3) = \varnothing$, and hence $U_1 \triangle U_2 \triangle U_3 = \varnothing$ (as $G$ is connected and $v_1, v_2 \notin U_1 \triangle U_2 \triangle U_3$). So there exist pairwise disjoint sets $V_1, V_2, V_3$ of vertices such that $U_i = V_j \cup V_k$ for all distinct $i, j, k \in \{1, 2, 3\}$. Define $V_0 := V \setminus (V_1 \cup V_2 \cup V_3)$.

Conditions (8) and (10) imply that $\delta(U_j) \cap \delta(U_k) = B_i \setminus \{e\}$ for distinct $i, j, k$. Hence $B_i \setminus \{e\}$ is the set of edges connecting either $V_i$ and $V_0$, or $V_j$ and $V_k$. So any edge not in $(B_1 \cup B_2 \cup B_3) \setminus \{e\}$ is spanned by one of the sets $V_0, V_1, V_2, V_3$.

Let $\{i, j, k\} = \{1, 2, 3\}$. Since $C_i$ does not contain any edge in $(B_j \cup B_k) \setminus \{e\} = \delta(U_i)$, the set $VC_i$ is disjoint from $U_i = V_j \cup V_k$. As $|C_i \cap B_i| \geqslant 3$ we know that $VC_i$ intersects $V_i$.

We can reset $\Sigma$ to an equivalent signing

$$(12) \qquad \Sigma := B_1 \triangle B_2 \triangle B_3 \triangle \delta(V_0).$$

So $\Sigma$ consists of $e$ and all edges connecting distinct sets among $V_1, V_2, V_3$. For each $i = 1, 2, 3$ and $k = 1, 2$, let $e_{i,k}$ be the first edge along the path $C_i \setminus \{e\}$ that belongs to $B_i$, when starting from vertex $v_k$. So both $e_{i,1}$ and $e_{i,2}$ connect $V_0$ and $V_i$.

Let $(H, \Sigma)$ be the minor of $(G, \Sigma)$ obtained by deleting all edges except those in $C_1 \cup C_2 \cup C_3$ and those spanned by $V_1 \cup V_2 \cup V_3$, and contracting all remaining edges that are not in $\Sigma \cup \{e_{i,k} \mid i = 1, 2, 3; k = 1, 2\}$.

$H$ can be described as follows. $H$ contains the edge $e$, connecting the vertices $\bar{v}_1$ and $\bar{v}_2$ to which $v_1$ and $v_2$ are contracted (we have $\bar{v}_1 \neq \bar{v}_2$ by (9)). For each $i = 1, 2, 3$, the part of the path $C_i \setminus \{e\}$ that is inbetween $e_{i,1}$ and $e_{i,2}$ belongs to one contracted vertex of $H$, call it $i$. This vertex $i$ is adjacent to $\bar{v}_1$ and $\bar{v}_2$ by the edges $e_{i,1}$ and $e_{i,2}$. For each $i = 1, 2, 3$, $V_i$ has been contracted to $i$ and a number of other vertices, together forming the stable set $S_i$ (say) in $H$. Any further edge of $H$ connects $S_i$ and $S_j$ for some distinct $i, j \in \{1, 2, 3\}$.

By (11), the subgraph of $H$ induced by $S_i \cup S_j$ is connected (for all distinct $i, j = 1, 2, 3$). So by the lemma, the graph $H - \bar{v}_2$ has an odd-$K_4$ subgraph containing the edges $\bar{v}_1 1$, $\bar{v}_1 2$, and $\bar{v}_1 3$. As $\bar{v}_2$ is adjacent to $\bar{v}_1$, 1, 2, and 3, it follows that $(H, \Sigma)$ has a $\tilde{K}_5$ minor.

## REFERENCES


1. M. Grötschel and W. R. Pulleyblank, Weakly bipartite graphs and the max-cut problem, *Oper. Res. Lett.* **1** (1981), 23–27.
2. B. Guenin, A characterization of weakly bipartite graphs, *in* "Integer Programming and Combinatorial Optimization, Proceedings, 6th IPCO Conference, Houston, Texas, 1998" (R. E. Bixby *et al.*, Eds.), Lecture Notes in Computer Science, Vol. 1412, pp. 9–22, Springer-Verlag, Berlin, 1998.
3. B. Guenin, A characterization of weakly bipartite graphs, *J. Combin. Theory Ser. B* **83** (2001), 112–168.
4. A. Lehman, The width-length inequality and degenerate projective planes, *in* "Polyhedral Combinatorics, Proceedings, DIMACS Workshop, Morristown, New Jersey, 1989" (W. Cook and P. D. Seymour, Eds.), pp. 101–105, Amer. Math. Soc., Providence, 1990.
5. M. Padberg, Lehman's forbidden minor characterization of ideal 0-1 matrices, *Discrete Math.* **111** (1993), 409–420.
6. P. D. Seymour, The matroids with the max-flow min-cut property, *J. Combin. Theory Ser. B* **23** (1977), 189–222.
7. P. D. Seymour, On Lehman's width-length characterization, *in* "Polyhedral Combinatorics, Proceedings, DIMACS Workshop, Morristown, New Jersey, 1989" (W. Cook and P. D. Seymour, Eds.), pp. 107–117, Amer. Math. Soc., Providence, 1990.