Europ. J. Combinatorics (2000) **21**, 487–502
doi:10.1006/eujc.1999.0355
Available online at http://www.idealibrary.com on IDEAL®

Adjacency, Inseparability, and Base Orderability in Matroids

J. KEIJSPER, R. PENDAVINGH AND A. SCHRIJVER

Two elements in an oriented matroid are inseparable if they have either the same sign in every signed circuit containing them both or opposite signs in every signed circuit containing them both. Two elements of a matroid are adjacent if there is no $\mathcal{M}(K_4)$-minor using them both, and in which they correspond to a matching of K_4.

We prove that two elements e, f of an oriented matroid are inseparable if and only if e, f are inseparable in every $\mathcal{M}(K_4)$ or U_4^2-minor containing them. This provides a link between inseparability in oriented matroids (introduced by Bland and Las Vergnas) and adjacency in binary matroids (introduced by Seymour).

We define the concepts of base orderable and strongly base orderable subsets of a matroid, generalizing the definitions of base orderable and strongly base orderable matroids. Strongly base orderable subsets can be used to obtain packing and covering results, generalizing results of Davies and McDiarmid, as was shown in a previous paper.

In this paper, we prove that any pairwise inseparable subset of an oriented matroid is base orderable. For binary matroids we derive the following characterization: a subset is strongly base orderable if and only if it is pairwise adjacent.

1. INTRODUCTION

In this paper, we discuss two notions of 'closeness' of elements in a matroid, show how they are related, and give further relations to (strong) base orderability in matroids.

The first notion is inseparability in oriented matroids, introduced by Bland and Las Vergnas [2]. Two elements of an oriented matroid are *inseparable* if they have either the same sign in every signed circuit containing them both or opposite sign in every signed circuit containing them both.

The other notion is adjacency in matroids. This concept was introduced by Seymour [14], to generalize the adjacency relation in a graph (edges of a graph are adjacent if they have a common vertex). Two elements of a matroid are *adjacent* if the matroid has no $\mathcal{M}(K_4)$-minor using these elements as 'opposite' edges of K_4.

In Section 3, we prove that two elements of an oriented matroid are inseparable if and only if they are adjacent in the underlying matroid, and they are inseparable in every (oriented) U_4^2-minor that contains them. Hence, for regular (binary orientable) matroids, the notions of inseparability and adjacency are equivalent.

In the later sections of the paper, we use this result to study two properties that a subset F of the ground set of a matroid can have. (We abbreviate $B \cup \{x\}$ by $B + x$ and $B \setminus \{x\}$ by $B - x$.)

(1) F is base orderable if for every pair of bases B, B', there is an injection $\pi: B \cap F \to B'$, such that $B - x + \pi(x)$ and $B' - \pi(x) + x$ are bases, for every $x \in B \cap F$. Such an injection π is called a *bo-injection*.
(2) F is strongly base orderable if for every pair of bases B, B', there is an injection $\pi : B \cap F \to B'$, such that $(B \setminus X) \cup \pi(X)$ and $(B' \setminus \pi(X)) \cup X$ are bases for all $X \subseteq B \cap F$ with $|\pi(X) \setminus F| \leq 1$. Such an injection π is called an *sbo-injection*.

These definitions extend the definitions of base orderable and strongly base orderable matroids (cf. [16]): a matroid M is (strongly) base orderable if $E(M)$ is (strongly) base orderable. A strongly base orderable subset is trivially base orderable as well.

0195–6698/00/040487 + 16 $35.00/0

It is not difficult to see that any strongly base orderable subset is pairwise adjacent (a subset F of the ground set of a matroid is *pairwise adjacent* if every pair of elements from F is adjacent). The main result of Section 7 is that for binary matroids the converse also holds. This generalizes the fact that a binary matroid is strongly base orderable if and only if it does not have an $\mathcal{M}(K_4)$-minor. It also extends the result from [7] that a set of edges with a common vertex in a graph is strongly base orderable in the cycle matroid of the graph.

To prove that any pairwise adjacent subset in a binary matroid is strongly base orderable, we first show that we can essentially restrict ourselves to regular matroids (Section 6), and then we use the fact that regular matroids are orientable. This seems to be a detour, but for an oriented matroid it appears to be surprisingly easy to prove that any pairwise inseparable subset is base orderable (Section 5). From this proof we obtain a bo-injection π, which in the binary case can be proved to be an sbo-injection.

Familiarity with matroid theory is assumed (for background information, see [11] or [16]). The necessary preliminaries on oriented matroids are summarized in Section 2.

2. Preliminaries

In this section, we give a short summary of oriented matroid theory, which is sufficient for our purposes. For more information, the reader is referred to [1].

Let E be a finite set. A *signed subset* of E is a pair $X = (X^+, X^-)$, where X^+ (the set of *positive elements* of X) and X^- (the set of *negative elements* of X) are disjoint subsets of E. The set $\underline{X} := X^+ \cup X^-$ is called the set *underlying* the signed set X. The notation $-X$ is used for the signed set (X^-, X^+) 'opposite' to X. Signed sets are identified with signed incidence vectors. That is, we will think of a signed subset of E as an element of $\{+, -, 0\}^E$. In this notation, we have for any signed set X, and $e \in E$ that $X_e = +$ if $e \in X^+$, $X_e = -$ if $e \in X^-$ and $X_e = 0$ if $e \notin \underline{X}$.

An *oriented matroid* M is a pair $(E, \mathcal{C})$, where E is a finite set called the *ground set* of M, and $\mathcal{C}$, the set of *signed circuits* of M, is a collection of signed subsets of E satisfying the following (signed circuit) axioms.

(C0) $\emptyset \notin \mathcal{C}$.
(C1) $X \in \mathcal{C} \Rightarrow -X \in \mathcal{C}$ (*symmetry*).
(C2) $X, Y \in \mathcal{C}, \underline{X} \subseteq \underline{Y} \Rightarrow X = Y$ or $X = -Y$ (*incomparability*).
(C3) $X, Y \in \mathcal{C}, X \neq -Y, e \in X^+ \cap Y^- \Rightarrow \exists Z \in \mathcal{C}$:
$Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$ (*weak elimination*).

Note that the collection $\underline{\mathcal{C}} = \{\underline{X} \mid X \in \mathcal{C}\}$ satisfies the the circuit axioms for a matroid. The matroid $\underline{M} = (E, \underline{\mathcal{C}})$ is called the *underlying matroid* of M. A matroid M is *orientable* if there is an oriented matroid with underlying matroid M. A binary matroid is orientable if and only if it is regular [2].

Reorientation of an element in an oriented matroid means flipping its sign in every signed circuit containing it. If an oriented matroid can be obtained from another by reorientation of some set of elements, the two oriented matroids are said to be in the same *reorientation class* of the matroid underlying both.

Any oriented matroid M with set of signed circuits $\mathcal{C}$ has a unique signature $\mathcal{C}^*$ of the cocircuits of $\underline{M}$ (that is, a signing of all the cocircuits of $\underline{M}$) such that

$$(\perp) \qquad \text{for all } C \in \mathcal{C} \text{ and } D \in \mathcal{C}^*\text{: } C \perp D.$$

Here, $C \perp D$ means

$$\underline{C} \cap \underline{D} \neq \emptyset \Rightarrow \exists e, f \in \underline{C} \cap \underline{D} : C_e D_e = -C_f D_f.$$

The set $\mathcal{C}^*$ then satisfies the signed circuit axioms, and hence if E is the ground set of M, then $M^* := (E, \mathcal{C}^*)$ is an oriented matroid, the *dual* of M (so by definition, the matroid underlying M^* is the dual of $\underline{M}$). The signed circuits of M^* are the *signed cocircuits* of M. We have $M^{**} = M$. The fundamental property ($\perp$) of a pair of dual-oriented matroids is called *orthogonality*. If M is a binary oriented matroid (regular matroid), then it satisfies an even stronger orthogonality property: for any circuit C and cocircuit D of M,

$$\underline{C} \cap \underline{D} \neq \emptyset \Rightarrow |\{e \in E(M) : \ C_e D_e = -\}| = |\{e \in E(M) : \ C_e D_e = +\}|.$$

Minors of oriented matroids are defined similarly to minors of ordinary matroids. If $M = (E, \mathcal{C})$ is an oriented matroid, and $X \subseteq E$, then $M \backslash X$ is the oriented matroid on $E \setminus X$ with set of signed circuits $\{C \in \mathcal{C} \mid \underline{C} \subseteq E \setminus X\}$; the contraction M/X on the same ground set is defined by the collection of signed circuits consisting of the signed sets with mimimal nonempty support among

$$\{D \mid \underline{D} \subseteq E \setminus X \text{ and } \exists C \in \mathcal{C} : C_e = D_e \, \forall e \in E \setminus X\}.$$

The *bases* of an oriented matroid are the bases of the underlying matroid. If M is an oriented matroid on E, B is a basis of M, and $e \in E \setminus B$, then the *fundamental circuit of e with respect to B* is the unique signed circuit C of M with $\underline{C}$ contained in $B + e$ such that $C_e = +$. It is denoted by $C(B, e)$. Similarly, if $f \in B$, then $C^*(B, f)$, the *fundamental cocircuit of f with respect to B* is the unique signed cocircuit D of M with $\underline{D} \subseteq (E \setminus B) + f$ and $D_f = +$. Now, for $e \in E \setminus B$ and $f \in B$ we have

$$f \in C(B, e) \Leftrightarrow e \in C^*(B, f) \Leftrightarrow C(B, e)_f C^*(B, f)_e = -$$

(for the last equivalence, use ($\perp$) and the fact that $C(B, e) \cap C^*(B, f) = \{e, f\}$).

Hereafter, given an oriented matroid M, $E(M)$ will denote the ground set of M, and $\mathcal{C}(M)$, $\mathcal{C}^*(M)$, $\mathcal{B}(M)$ will denote the collections of signed circuits, signed cocircuits, and bases of M, respectively. The same notation will be used for the ground set, circuits, cocircuits and bases of an ordinary (unoriented) matroid.

3. Separability in Oriented Matroids

Given an oriented matroid M, two elements $e, f \in E(M)$ are said to be *separated* by a pair of circuits $C, C' \in \mathcal{C}(M)$ if $e, f \in \underline{C} \cap \underline{C'}$ and $C_e C_f = -C'_e C'_f$. If circuits separating e, f exist, e and f are *separable*. Otherwise, they are *inseparable*. The inseparable pair e, f is *covariant* if $C_e = C_f$ for all circuits containing both e and f, and *contravariant* if $C_e = -C_f$ for all circuits containing both e and f.

For any circuit C with $e, f \in \underline{C}$ we can find a cocircuit D with $\underline{C} \cap \underline{D} = \{e, f\}$, and since $C \perp D$, we must have $C_e C_f = -D_e D_f$ for such a cocircuit. Hence, from a pair of circuits C, C' separating e, f we can construct a pair of cocircuits D, D' with $-D_e D_f = C_e C_f = -C'_e C'_f = D'_e D'_f$. This shows that e, f are separable in M if and only if they are separable in M^*, and that e, f are covariant in M if and only if they are contravariant in M^*.

Clearly, reorienting any element of M does not affect separability (note that it does affect covariance and contravariance). If an orientable matroid has only one reorientation class, then separability of elements is a property of the underlying unoriented matroid. Regular matroids

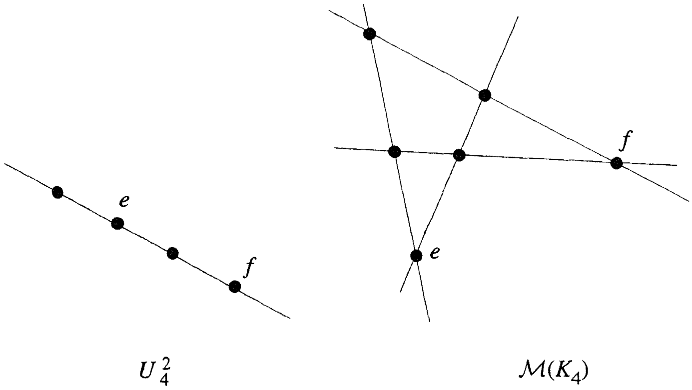


FIGURE 1. Orientations of U_4^2 and $\mathcal{M}(K_4)$ in which e, f are separable.

have exactly one reorientation class [2]. Since graphic matroids are regular, separability is a property of the graph from which the graphic matroid is constructed. For example, two edges of K_4 are separable exactly when they have no vertex in common.

For matroids having more than one reorientation class, such as U_4^2, the situation is different. The unoriented matroid U_4^2 does not distinguish between any pair of elements from its ground set E. A reorientation class of U_4^2 determines, and is determined by, an embedding of E in the projective line. In an orientation of U_4^2, a pair of elements is separable exactly when they are not consecutive on the projective line.

Figure 1 shows affine pictures of $\mathcal{M}(K_4)$ and U_4^2, with a separable pair indicated in both matroids. A signed circuit in the affine picture of an oriented matroid is a minimal signed set of points such that the convex hull of the positive elements intersects the convex hull of the negative elements.

Let M and M' be oriented matroids, and let $F \subseteq E(M)$, $F' \subseteq E(M')$. We say that (M', F') is a *minor* of (M, F) if M' is a minor of M, and $F' \subseteq F$. By the definitions of separability and minor, if e and f are separable in M' and $(M', \{e, f\})$ is a minor of $(M, \{e, f\})$, then e and f are separable in M.

We now characterize separability.

THEOREM 1. *Let M be an oriented matroid. The following are equivalent for $e, f \in E(M)$:*

(i) e and f are separable in M.
(ii) $(M, \{e, f\})$ has a minor $(N, \{e, f\})$ such that e and f are separable in N, and $\underline{N}$ is isomorphic to U_4^2 or $\mathcal{M}(K_4)$.

PROOF. It suffices to show that $(i) \Rightarrow (ii)$, since the reverse implication is clear from the remark above. So assume that $(M, \{e, f\})$ is minor-minimal with the property that e, f are separable in M. Let C, C' be circuits with $C_e = C_f$ and $C'_e = -C'_f$. Then

$$\underline{C} \cap \underline{C'} = \{e, f\}, \underline{C} \cup \underline{C'} = E(M), \text{ and } \underline{M} \text{ is simple and cosimple,}$$

since elements common to C and C', other than e, f, can be contracted and elements not in C or C' can be deleted. In particular, this eliminates loops and coloops. From each parallel class, we need only one element, hence each parallel class has only one element (note that the pair

e, f is not parallel or coparallel). As separability is closed under duality, this also eliminates coparallel elements. It follows that:

$$\text{if } C'' \text{ is a circuit containing } e \text{ and } f, \text{ then } \underline{C''} = \underline{C} \text{ or } \underline{C''} = \underline{C'}.$$

If $C''_e = C''_f$ then C'', C' separates e, f, so $\underline{C''} = E(M)\setminus\underline{C'}+e+f = \underline{C}$. A similar argument shows that $C''_e = -C''_f$ implies $\underline{C''} = \underline{C'}$.

There exists a cocircuit D with $\underline{D} \cap \underline{C} = \{e, f\}$, hence $D_e = -D_f$ and $\underline{D} \subseteq \underline{C'}$. Similarly we have a cocircuit D' with $D'_e = D'_f$ and $\underline{D'} \subseteq \underline{C}$. If one of these inclusions were proper, the dual M^* would have circuits separating e, f but with $\underline{D} \cup \underline{D'} \neq E(M)$, contradicting the minimality of M. Thus $\underline{D} = \underline{C'}$ and $\underline{D'} = \underline{C}$. We claim that:

$$\underline{C} - a \text{ is a basis of } M \text{ for all } a \in C.$$

As C is a circuit, $\underline{C} - a$ is independent for all $a \in C$. Since $\underline{C} - e$ properly contains the hyperplane $E(M)\setminus\underline{D}$, it is a spanning set. Hence $\underline{C}-e$ is a basis, and therefore the independent sets $\underline{C} - a$ of the same cardinality are all bases.

Since by the same argument, $\underline{C'} - a$ is a basis for every $a \in \underline{C'}$, we have $|\underline{C}| = |\underline{C'}| = r(M) + 1$ and hence $E(M) = 2|\underline{C}| - 2 = 2r(M)$.

Define $A_x := \underline{C}(\underline{C} - f, x) - x$ and $B_x := \underline{C}(\underline{C} - e, x) - x$ for all $x \notin \underline{C}$. Thus $A_x \subseteq \underline{C} - f$ and $B_x \subseteq \underline{C} - e$. We claim that:

$$\text{if } A_x \cap B_x \neq \emptyset \text{ for some } x \notin \underline{C}, \text{ then } \underline{M} \text{ is isomorphic to } U_4^2.$$

Indeed, choose $y \in A_x \cap B_x$. Then $\underline{C}+x-y$ contains a circuit C'' as it has more elements than the basis $\underline{C}-e$. As $\underline{C}-e+x-y$ and $\underline{C}-f+x-y$ are bases, $e, f \in \underline{C''}$. Since $y \in \underline{C}\setminus\underline{C''}$, we have $\underline{C''} \neq \underline{C}$. Thus $\underline{C''} = \underline{C'}$, and hence $\underline{C''} = \{e, f, x\}$. Then $r(M) + 1 = |C| = |C''| = 3$, and $|E(M)| = 4$. Since M has no parallel elements, $\underline{M} = U_4^2$. This proves the claim.

So we may assume that $A_x \cap B_x = \emptyset$ for all $x \notin \underline{C}$. Note that $A_x \cup B_x = \underline{C}$, since $(A_x + x) \cup (B_x + x) - x$ must contain a circuit, and this circuit is contained in $\underline{C}$. Thus A_x and B_x partition $\underline{C}$.

We say that two elements $x, y \notin \underline{C}$ *cross* if both $A_x \not\subseteq A_y$ and $A_y \not\subseteq A_x$. We claim that there is a crossing pair $x, y \notin \underline{C}$.

If not, we may choose $x \notin \underline{C}$ such that $A_x \subseteq A_y$ for all $y \notin \underline{C}$. Then $B_y \subseteq B_x$ for all $y \notin \underline{C}$, so B_x spans each $y \notin \underline{C}$, and therefore it spans $E \setminus A_x$. Since the circuit $A_x + x$ has at least three elements, there is an element $a \in A_x$ other than e. The fundamental cocircuit $\underline{C^*}(\underline{C} - a, e)$ is disjoint from B_x and hence from the span of B_x, but it intersects $\underline{C} - a$ only in e, and hence it intersects $A_x - a$ only in e. It follows that a, e are coparallel, a contradiction.

Now, $\underline{M}$ is isomorphic to $\mathcal{M}(K_4)$.

Let $x, y \notin \underline{C}$ be crossing and take $x' \in A_x\setminus A_y$ and $y' \in A_y\setminus A_x$. Then $\underline{C}-e+x-x'+y-y'$ spans x', since x' is contained in the circuit $B_y + y$, which is contained in $\underline{C} - e + x + y - y'$, and similarly it spans y'. So $\underline{C} - e + x - x' + y - y'$ spans the basis $\underline{C} - e$, and because it has the cardinality of a basis, it is a basis. Similarly, $\underline{C} - f + x - x' + y - y'$ is a basis. Hence, $\underline{C} + x - x' + y - y'$ contains a circuit C'' through e and f. Since $x', y' \in \underline{C} \setminus \underline{C''}$ we have $\underline{C''} \neq \underline{C}$. It follows that $\underline{C'} = \underline{C''} = \{e, f, x, y\}$, so $r(M) + 1 = |C| = |C'| = 4$, and $\underline{C} = \{e, f, x', y'\}$. Also, $A_x + x$, $A_y + y$, $B_x + x$ and $B_y + y$ are circuits. This ensures that $\underline{M}$ is isomorphic to $\mathcal{M}(K_4)$.

One may see this as follows. We know that the triples exx', eyy', fxy' and fyx' are circuits (using shorthand notation for sets). We claim that the other 16 triples are all bases: four are proper subsets of the circuit $efxy$, another four are proper subsets of the circuit $efx'y'$. The remaining eight triples can be shown to be spanning.

We have found the cycle matroid of a K_4 with matchings ef, xy, and $x'y'$. □

The following result of [5] is an immediate corollary, since a matroid is series-parallel if and only if it has no $\mathcal{M}(K_4)$ or U_4^2-minor.

COROLLARY 1. *If M is an oriented matroid, then every pair of elements of M is inseparable if and only if $\underline{M}$ is series-parallel.*

For binary oriented matroids, Theorem 1 implies that two elements are inseparable if and only if they do not correspond to opposite edges of K_4 in any $\mathcal{M}(K_4)$-minor containing them both. By a definition of Seymour [14], two distinct elements e, f of a matroid M are *adjacent* exactly if they have this mentioned property, that is, e, f are adjacent if M has no $\mathcal{M}(K_4)$-minor containing e, f in which $\{e, f\}$ corresponds to a matching of K_4. We will call a set $F \subseteq E(M)$ *pairwise adjacent* in the matroid M if every two elements from F are adjacent in M.

COROLLARY 2. *If M is a regular oriented matroid and $e, f \in E(M)$, then e and f are inseparable in M if and only if e and f are adjacent in $\underline{M}$.*

Thus in a regular matroid M, the adjacent pairs are exactly the inseparable pairs in the unique orientation of M (unique up to reorienting elements). A straightforward consequence of this observation is the following characterization of adjacency in graphic matroids due to Seymour [14].

COROLLARY 3. *Let e, f be disjoint edges of a graph G. Let the ends of e be s_1, s_2, and let the ends of f be t_1, t_2. Then the following are equivalent:*

(a) e, f are not adjacent in $\mathcal{M}(G)$,
(b) there are four paths $P_{11}, P_{12}, P_{21}, P_{22}$ of G, such that P_{11}, P_{22} have no common vertices, P_{12}, P_{21} have no common vertices, and P_{ij} joins s_i to t_j $(i, j = 1, 2)$.

4. PAIRWISE INSEPARABLE SETS

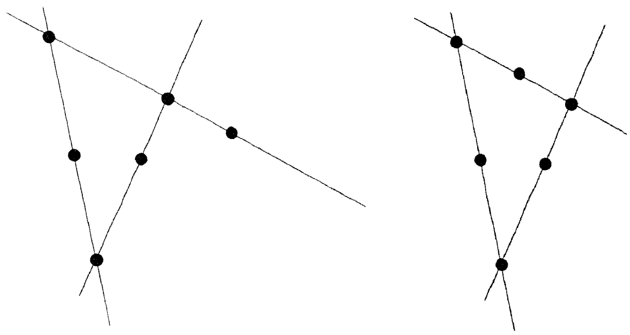
For an oriented matroid M, the *inseparability graph* of M is the graph with vertex set $E(M)$ in which $e, f \in E(M)$ are connected by an edge if and only if they are inseparable in M. Inseparability graphs were studied in [4, 5, 12]. In this section, we study cliques in the inseparability graph, in other words, we study *pairwise inseparable sets* in oriented matroids. This will help us understand pairwise adjacent sets in binary matroids, since, by a result of Seymour [13], in a 3-connected binary nonregular matroid, the only pairwise adjacent subsets are singletons, and since, by Theorem 1, in a regular matroid M, a subset F of $E(M)$ is pairwise adjacent if and only if F is pairwise inseparable in some orientation of M.

We investigate pairwise inseparable subsets of size 3. Oxley [10] showed that the following holds for a triple e, f, g in a 3-connected matroid M with rank and corank at least 3.

(1) If M is nonbinary, and the triple e, f, g is not used by a U_4^2-minor of M, there is a $\mathcal{W}^3$-minor of M using the triple as its rim or as its spokes ($\mathcal{W}^3$ denotes the 3-whirl).
(2) If M is binary, then M has an $\mathcal{M}(K_4)$-minor using e, f, g.

We apply this result to a pairwise inseparable triple e, f, g in an orientation of M.

THEOREM 2. *If $F = \{e, f, g\}$ is a pairwise inseparable triple in a 3-connected oriented matroid M with rank and corank at least 3, then (M, F) has a minor (N, F) such that $\underline{N}$ is isomorphic to a 3-wheel or 3-whirl, using e, f, g as the rim or the spokes.*



$\mathcal{W}_A^3$ $\mathcal{W}_B^3$

FIGURE 2. Two orientations of the whirl $\mathcal{W}^3$.

PROOF. F is not used by a U_4^2-minor, as any triple in an orientation of U_4^2 contains a separable pair. So if $\underline{M}$ is nonbinary, the theorem follows from Oxley's result (1). If $\underline{M}$ is binary, then by Oxley's result (2), F is contained in an $\mathcal{M}(K_4)$-minor. Moreover, a pairwise inseparable triple in $\mathcal{M}(K_4)$, alias the 3-wheel, is a triangle or a vertex cut. □

Figure 2 shows two affine diagrams of oriented matroids in distinct reorientation classes of the whirl $\mathcal{W}^3$. The rim and the set of spokes of $\mathcal{W}^3$ are pairwise inseparable triples in the orientation $\mathcal{W}_A^3$, but not in the orientation $\mathcal{W}_B^3$ (on the contrary, they are pairwise *separable* triples in $\mathcal{W}_B^3$).

The following fundamental property of pairwise inseparable triples will prove to be very useful.

LEMMA 1. *Let M be an oriented matroid, and let F be a pairwise inseparable triple in M. Then F is not contained in the intersection of a circuit and a cocircuit.*

PROOF. For an inseparable pair $\{e, f\} \subseteq E(M)$, set $\varepsilon_{ef} = +$ if e, f are covariant, and $\varepsilon_{ef} = -$ if e, f are contravariant. For a pairwise inseparable triple $F = \{e, f, g\}$, we define $\delta_F := \varepsilon_{ef}\varepsilon_{eg}\varepsilon_{fg}$. Clearly, δ_F is invariant under reorientation. If a circuit contains F, then $\delta_F = +$, and if a cocircuit contains F, then $\delta_F = -$. This shows that a pairwise inseparable triple is not both contained in a circuit and a cocircuit. □

From results in [15] and [14], one can deduce that for 3-connected regular matroids, the converse also holds.

COROLLARY 4. *Let M be a 3-connected regular matroid, and let $F \subseteq E(M)$ be a triple of elements from M. Then F is pairwise adjacent if and only if F is not contained in the intersection of a circuit and a cocircuit.*

Corollary 4 is not true if we replace the regular matroid by a general oriented matroid and 'pairwise adjacent' by 'pairwise inseparable'. Indeed, as we pointed out above, in the orientation $\mathcal{W}_B^3$ (see Figure 2) of the 3-whirl, the rim is not pairwise inseparable. However the rim of a 3-whirl is not contained in a cocircuit.

Lemma 1 also shows that a pairwise inseparable nonseparating cocircuit of an oriented matroid forms a vertex in the underlying matroid, as defined by Kelmans [8] (he defines a matroid *vertex* as a nonseparating cocircuit intersecting every circuit in at most two elements).

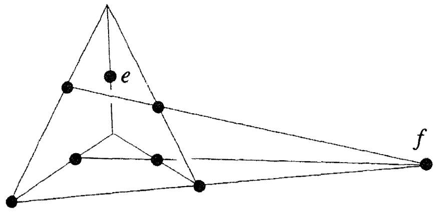


FIGURE 3. An orientation of the matroid J. The set $\{e, f\}$ is not base orderable.

5. BASE ORDERABLE SETS

Let M be a matroid, and F a nonempty subset of $E(M)$. Then F is *base orderable* in M if:

for every pair of bases B, B', there is an injection $\pi : B \cap F \to B'$ such that $B - x + \pi(x)$ and $B' - \pi(x) + x$ are bases, for every $x \in B \cap F$.

Such an injection we will call a *bo-injection*. This definition generalizes the definition of a base orderable matroid. Indeed, M is a base orderable matroid if and only if $E(M)$ is base orderable in M. Since bases of an oriented matroid are by definition bases of the underlying matroid, we can also view the above as a definition of base orderable sets in oriented matroids.

It is not difficult to see that a base orderable set in matroid M is also base orderable in M^*. By a result of Brualdi [3], each singleton is base orderable. Moreover, if M is a matroid, and $F \subseteq E(M)$ is base orderable in M, then F' is base orderable in M', for any minor (M', F') of (M, F). Because a matching $\{e, f\}$ in the graph K_4 is not base orderable in $\mathcal{M}(K_4)$, this means that:

any base orderable set in a matroid is pairwise adjacent.

The converse does not hold: the orientable matroid J, shown in Figure 3 has no $\mathcal{M}(K_4)$-minor, but $\{e, f\}$ is not base orderable in J. (The matroid J is one of the forbidden minors for base orderability in ternary matroids, found by Oxley [9].) However, if we demand that M is orientable, and F is pairwise inseparable in some orientation of M, then F is necessarily base orderable. (It is not difficult to check in Figure 3 that e and f are separable in this orientation of J.)

THEOREM 3. *If M is an oriented matroid, and F is pairwise inseparable in M, then F is base orderable in M.*

PROOF. Let F be pairwise inseparable in M, and let B, B' be bases of M. We must define a bo-injection $\pi : B \cap F \to B'$. For all $x \in B \cap F$, we need to choose $\pi(x) \in C^*(B, x)$ to ensure that $B - x + \pi(x)$ is a basis, and $\pi(x) \in C(B', x)$ to ensure that $B' - \pi(x) + x$ is a basis.

Let $\Pi(x)$ be the signed subset of $E(M)$ defined by:

$$\Pi(x)_y := C^*(B, x)_y C(B', x)_y \qquad \text{for all } y.$$

We have $x \in \Pi(x)^+$, and since $C^*(B, x) \perp C(B', x)$, it follows that $\Pi(x)^- \neq \emptyset$.

Choose $\pi(x) \in \Pi(x)^-$ arbitrarily, for all $x \in B \cap F$. It suffices to show that:

π is an injection.

Suppose to the contrary that $\pi(e) = \pi(f) =: g$, for distinct $e, f \in B \cap F$. By definition of Π, we have $g \in C^*(B, x)$ and $g \in C(B', x)$, with $C^*(B, x)_g = -C(B', x)_g$ for $x = e, f$. This is equivalent to:

$$x \in C(B, g) \text{ and } x \in C^*(B', g), \text{ with } C(B, g)_x = -C^*(B', g)_x, \text{ for } x = e, f.$$

It follows that $C(B, g)_e C(B, g)_f = C^*(B', g)_e C^*(B', g)_f$, contradicting the fact that e and f are inseparable. □

For regular matroids, Theorem 3 and Corollary 2 imply the following equivalence.

COROLLARY 5. *Let M be a regular matroid, and $F \subseteq E(M)$. Then F is base orderable if and only if F is pairwise adjacent.*

6. STRONGLY BASE ORDERABLE SETS

Let M be a matroid, and F a nonempty subset of $E(M)$. Then F is *strongly base orderable* in M if:

> for every pair of bases B, B', there is an injection $\pi : B \cap F \to B'$ such that $(B \setminus X) \cup \pi(X)$ and $(B' \setminus \pi(X)) \cup X$ are bases for all $X \subseteq B \cap F$ with $|\pi(X) \setminus F| \leq 1$.

Such an injection will be called an *sbo-injection* for B and B' in M. This definition generalizes the concept of a strongly base orderable matroid. Strongly base orderable sets are preserved under taking subsets, duals and minors, like base orderable sets. Moreover, any strongly base orderable set is a base orderable set (and hence pairwise adjacent). Any singleton-subset of $E(M)$ is a strongly base orderable set for M, again by the result of Brualdi [3].

To motivate our interest in strongly base orderable sets, we mention a previous result. The following Davies and McDiarmid [6] type theorem, on packing 'common spanning sets' of two matroids intersecting in a common strongly base orderable set was proved in [7].

THEOREM 4. *Let $\mathcal{M}$ be a matroid on E_1 and let $\mathcal{N}$ be a matroid on E_2. Let $E := E_1 \cup E_2$ and $F := E_1 \cap E_2$. Suppose that F is strongly base orderable in $\mathcal{M}$ as well as in $\mathcal{N}$, and suppose that both $\mathcal{M}$ and $\mathcal{N}$ have k pairwise disjoint bases. Then E contains pairwise disjoint sets $S_1, \ldots, S_k$, such that $S_i \cap E_1$ is a spanning set of $\mathcal{M}$, and $S_i \cap E_2$ is a spanning set of $\mathcal{N}$, $i = 1, \ldots, k$.*

It was also shown in [7] that a substar in a graph G is strongly base orderable in $\mathcal{M}(G)$ (a *substar* in a graph is a set of edges having a vertex in common). An efficient algorithm for packing connectors in a graph was derived from these results. (An *S–T connector* in a graph G, of which the vertex-set $V(G)$ is partitioned into S and T, is a subset F of the edge-set $E(G)$ such that every component of (V, F) intersects both S and T.)

In the next section, we will prove that a subset of a binary matroid is pairwise adjacent if and only if it is strongly base orderable (this generalizes the result that substars in a graph are strongly base orderable). To be able to restrict ourselves to 3-connected matroids there, we study in this section how strongly base orderable sets behave under taking direct sums and 2-sums.

If M_1 and M_2 are matroids with $E(M_1) \cap E(M_2) = \emptyset$, then the *direct sum* $M_1 \oplus M_2$ of M_1 and M_2 is the matroid with ground set $E(M_1) \cup E(M_2)$ and collection of bases

$$\{B_1 \cup B_2 \mid B_i \in \mathcal{B}(M_i), i = 1, 2\}.$$

LEMMA 2. *Let M be the direct sum of the matroids M_1 and M_2, and let F_1, F_2 be strongly base orderable in M_1, M_2, respectively. Then $F := F_1 \cup F_2$ is strongly base orderable in M.*

PROOF. Left to the reader. □

If M_1 and M_2 are matroids with $E(M_1) \cap E(M_2) = \{z\}$, where z is neither a loop nor a coloop of M_i, $i = 1, 2$, then the *2-sum* $M_1 \oplus_2 M_2$ of M_1 and M_2, with *basepoint* z, is the matroid with ground set $(E(M_1) \cup E(M_2)) - z$, and collection of bases

$$\{B_1 \cup B_2 - z \mid B_1 \in \mathcal{B}(M_1),\ B_2 \in \mathcal{B}(M_2),\ \text{and } z \in B_1 \Delta B_2\}.$$

Here, $X \Delta Y$ denotes the symmetric difference $(Y \setminus X) \cup (X \setminus Y)$.

LEMMA 3. *Let M be a 2-sum of the matroids M_1 and M_2, with basepoint z.*

(i) If F_i is strongly base orderable in M_i, then F_i is strongly base orderable in M ($i = 1, 2$).
(ii) If $F_i + z$ is strongly base orderable in M_i for $i = 1, 2$, then $F := F_1 \cup F_2$ is strongly base orderable in M.

PROOF. (i) This is left to the reader.
(ii) Let B and B' be bases of M. We need to show the existence of an sbo-injection $\pi : B \cap F \to B'$.

Let us write $B = B_1 \cup B_2 - z$ and $B' = B'_1 \cup B'_2 - z$ as in the definition of the 2-sum, where B_i, B'_i are bases of M_i. As $F_i + z$ is strongly base orderable in M_i, there exists an sbo-injection $\pi_i : B_i \cap (F_i + z) \to B'_i$.

By symmetry, we may assume that $z \notin B_1$. Consider the map $\pi : B \cap F \to B'$ defined by:

$$\begin{aligned}
&\pi(e) = \pi_2(e) \text{ if } e \in F_2,\\
&\pi(e) = \pi_1(e) \text{ if } e \in F_1, \pi_1(e) \neq z, \text{ and}\\
&\pi(e) = \pi_2(z) \text{ if } e \in F_1, \pi_1(e) = z.
\end{aligned}$$

We will show that π is an sbo-injection.

Let $A \subseteq B \cap F$ be such that $|\pi(A) \setminus F| \leq 1$. Define $A_1 := B_1 \cap A$. Furthermore, let $A_2 := B_2 \cap A$ if $z \notin \pi_1(A_1)$ and $A_2 := (B_2 \cap A) + z$ otherwise. Then $A = A_1 \cup A_2 - z$ and $\pi(A) = \pi_1(A_1) \cup \pi_2(A_2) - z$. Hence, we have:

$$(B \setminus A) \cup \pi(A) = [(B_1 \setminus A_1) \cup \pi_1(A_1)] \cup [(B_2 \setminus A_2) \cup \pi_2(A_2)] - z,$$

and

$$(B' \setminus \pi(A)) \cup A = [(B'_1 \setminus \pi_1(A_1)) \cup A_1] \cup [(B'_2 \setminus \pi_2(A_2)) \cup A_2] - z.$$

Since $|\pi_1(A_1) \setminus (F_1 + z)| + |\pi_2(A_2) \setminus (F_2 + z)| = |\pi(A) \setminus F| \leq 1$, it follows that $(B_1 \setminus A_1) \cup \pi_1(A_1)$ and $(B'_1 \setminus \pi(A_1)) \cup A_1$ are bases of M_1 and $(B_2 \setminus A_2) \cup \pi_2(A_2)$ and $(B'_2 \setminus \pi_2(A_2)) \cup A_2$ are bases of M_2.

To see that $(B \setminus A) \cup \pi(A)$ is a basis of M, it suffices to show that z is a member of exactly one of $(B_1 \setminus A_1) \cup \pi_1(A_1)$ and $(B_2 \setminus A_2) \cup \pi_2(A_2)$.

If $z \in (B_1 \setminus A_1) \cup \pi_1(A_1)$, then $z \in \pi_1(A_1)$ and hence $z \in B'_1$, and $z \in A_2$ by definition. Since $B' = B'_1 \cup B'_2 - z$ is a basis, we have that $z \notin B'_2$ and hence $z \notin \pi_2(A_2)$. It follows that $z \notin (B_2 \setminus A_2) \cup \pi_2(A_2)$.

If $z \notin (B_1 \setminus A_1) \cup \pi_1(A_1)$, then $z \notin \pi_1(A_1)$ and hence $z \notin A_2$. Since $B = B_1 \cup B_2 - z$ is a basis, and we assumed that $z \notin B_1$, we have that $z \in B_2$. It follows that $z \in (B_2 \setminus A_2) \cup \pi_2(A_2)$.

The proof that $(B' \setminus \pi(A)) \cup A$ is a basis is similar. □

7. Binary Matroids

In this section, we prove the following extension of Corollary 5.

THEOREM 5. *For a binary matroid M, with $F \subseteq E(M)$, the following are equivalent:*

(a) F is strongly base orderable,
(b) F is base orderable,
(c) F is pairwise adjacent.

This theorem generalizes the fact that a binary matroid is strongly base orderable, if and only if it is base orderable, if and only if it has no $\mathcal{M}(K_4)$-minor (see [16]).

For oriented matroids, one might expect a similar result, with 'pairwise adjacent' replaced by 'pairwise inseparable'. However, a strongly base orderable subset of an oriented matroid need not be pairwise inseparable: U_4^2 is a strongly base orderable matroid, but it contains a separable pair. We could not prove, or disprove, the other implication (that a pairwise inseparable subset of an oriented matroid is strongly base orderable). Nevertheless, to obtain a partial result in this direction, which suffices to finish the argument for the binary case, we reconsider the proof of Theorem 3.

According to the proof of Theorem 3, for any oriented matroid M, with $F \subseteq E(M)$ pairwise inseparable, and for any two bases B, B' of M we can define a bo-injection $\pi : B \cap F \to B'$ by choosing $\pi(x)$ (for $x \in B \cap F$) arbitrarily in

$$\Pi(x)^- := \{y \in E(M) \mid C^*(B, x)_y C(B', x)_y = -\}.$$

We now impose one further restriction on π: we choose $\pi(x) \notin F$ if possible. Thus, hereafter, given an oriented matroid M, a pairwise inseparable subset F of its ground set, and two bases B, B' of M, a bo-injection *associated with* B, B' in (M, F) is a function $\pi : B \cap F \to B'$ satisfying

$$\pi(x) \in \Pi(x)^-$$

and

$$\pi(x) \in F \Rightarrow \Pi(x)^- \subseteq F,$$

for any $x \in B \cap F$ (such a function exists by the proof of Theorem 3). Then the contents of the following lemma is that if such a bo-injection is not an sbo-injection for B, B', then there is a small certificate for that, i.e., a small subset of $B \cap F$ which cannot be exchanged.

LEMMA 4. *Let M be an oriented matroid, with $F \subseteq E(M)$ pairwise inseparable. Then either F is strongly base orderable, or there are bases B, B' of M, and a subset X of $B \cap F$, with $|X| = 2$ and $|\pi(X) \setminus F| \leq 1$ such that one of $(B \setminus X) \cup \pi(X)$, $(B' \setminus \pi(X)) \cup X$ is not a basis (where π is a bo-injection associated with B, B' in (M, F)).*

Let (M, F) be a minor-minimal counterexample. Thus, F is not strongly base orderable in M, and there are bases B, B' of M showing this. Let π be a bo-injection associated with B, B' in (M, F). Let $X \subseteq B \cap F$ be a smallest set with $|\pi(X) \setminus F| \leq 1$ and such that at least one of $(B \setminus X) \cup \pi(X)$ and $(B' \setminus \pi(X)) \cup X$ is not a basis. Then $|X| > 2$, since we are dealing with a counterexample to the lemma. By minimality of X, $X \subseteq B \setminus B'$, since π is the identity on $B \cap B' \cap F$. By contracting elements in $B \cap B'$ and deleting elements outside $B \cup B'$, we obtain a minor M' of M with bases $B \setminus B'$, $B' \setminus B$, for which the restriction of π to $(B \setminus B') \cap F$ is an associated bo-injection, and X is a 'bad' set. Hence, $B \cap B' = \emptyset$ and $B \cup B' = E(M)$, by minimality of (M, F).

We may assume that $(B \setminus X) \cup \pi(X)$ is not a basis. (Otherwise, we consider the dual matroid, where $E(M) \setminus (B' - \pi(X) + X) = B - X + \pi(X)$ is not a basis. Note that Π is the same for the dual matroid, since $C^*_M(B, x) = C_{M^*}(E(M) \setminus B, x) = C_{M^*}(B', x)$, and similarly, $C_M(B', x) = C^*_{M^*}(B, x)$.)

Consider $N := M/(B - X) \setminus (B' - \pi(X))$. Then, X is a basis of N, and $\pi(X)$ is a circuit of N (by the minimality of X).

Since $E(N) = X \cup \pi(X)$, we have $|E(N) \setminus F| \leq 1$, so N has a pairwise inseparable set containing all but one element of $E(N)$. This means that N has no U^2_4- or $\mathcal{M}(K_4)$-minor, since neither one of these matroids contains a pairwise inseparable set of size 1 less than its ground set. Hence N is series-parallel. So there is a graph H such that $N = \mathcal{M}(H)$, with $|X| + 1$ vertices (as $r(N) = |X|$) and containing a circuit $C = \pi(X)$ with $|X|$ edges, and $|X|$ vertices. Let v be the unique vertex of H not in C. No element $x \in X$ is spanned by C, since $x + \pi(X - x)$ is a basis of N. Hence, all $x \in X$ are adjacent to v in H. Moreover, no pair of elements from X is parallel since X is a basis. Therefore, the elements of X have distinct endpoints on C. Because $|X| > 2$, it follows that H is a wheel, contradicting the fact that N is series-parallel.

PROOF OF THEOREM 5. Since the implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are trivial, we are only concerned with proving that (c) $\Rightarrow$ (a). Let (M, F) be a minor-minimal counterexample. So M is a binary matroid, $F \subseteq E(M)$ is pairwise adjacent but not strongly base orderable in M, whereas for any proper minor (N, F') of (M, F) we have that F' is a strongly base orderable set for N.

If M is not 3-connected, it can be written as a direct sum or as a 2-sum of two of its proper minors M_1, M_2. Denoting $F \cap E(M_i)$ by F_i, it is not difficult to derive from the fact that pairwise adjacent sets are closed under taking minors and subsets the following statements (see also [14]).

(a) If $M = M_1 \oplus M_2$, then F_i is pairwise adjacent in M_i, $i = 1, 2$.
(b) If $M = M_1 \oplus_2 M_2$ with basepoint z (so $E(M_1) \cap E(M_2) = \{z\}$), and $F_2 = \emptyset$, then $F = F_1$ is pairwise adjacent in M_1.
(c) If $M = M_1 \oplus_2 M_2$ with basepoint z, and $F_i \neq \emptyset$ $(i = 1, 2)$, then $F_i + z$ is pairwise adjacent in M_i, $i = 1, 2$.

Because M_i is a proper minor of M, in each of these cases it follows from the minimality of (M, F) that F_i (or $F_i + z$) is strongly base orderable in M_i, $i = 1, 2$. Then by Lemma 2 or Lemma 3, F is strongly base orderable in M, a contradiction.

Hence, M is 3-connected. Moreover, since any singleton-subset of $E(M)$ is strongly base orderable in M, we have $|F| \geq 2$. By a theorem of Seymour [13], no pair of elements in a 3-connected nonregular binary matroid is adjacent. So M is regular (orientable).

Now (viewing M as an oriented matroid), by Lemma 4, there are two bases B, B' of M, satisfying $E(M) = B \cup B'$ and $B \cap B' = \emptyset$, subsets $X = \{a, b\}$ of B and $\pi(X) = \{c, d\}$ of B', such that $F \supseteq \{a, b, c\}$, and

$$\begin{aligned}
&\pi(a) = c \in \Pi(a)^-, \text{ so } c \in C^*(B, a) \cap C(B', a),\\
&\pi(b) = d \in \Pi(b)^-, \text{ so } d \in C^*(B, b) \cap C(B', b),\\
&B - a - b + c + d \text{ is not a basis, so } a \in C(B, d) \text{ and } b \in C(B, c)\\
&(\text{i.e., } d \in C^*(B, b) \text{ and } c \in C^*(B, b)).
\end{aligned}$$

Here, $\pi : B \cap F \to B'$is the bo-injection associated with B, B' in (M, F). By definition, because $\pi(a) = c \in F$, we have $\Pi(a)^- \subseteq F$.

The strong orthogonality property for a regular matroid implies

$$|\Pi(a)^-| = |\Pi(a)^+|.$$

Because $a \in \Pi(a)^+$ and $c \in \Pi(a)^-$, we may conclude that there exists an $e \in B' - c$ with $e \in \Pi(a) = C^*(B, a) \cap C(B', a)$ if and only if there exists an $e \in B' - c$ such that $e \in \Pi(a)^-$.

However, if there exists an $e \in B' - c$ such that $e \in \Pi(a)^- \subseteq F$, then a, c, e are three different elements of F in the intersection of the circuit $C(B', a)$ and the cocircuit $C^*(B, a)$. This contradicts Lemma 1.

To derive this contradiction, we have argued that it suffices to find an $e \in B' - c$ with $e \in \Pi(a) = C^*(B, a) \cap C(B', a)$. This is done in the next lemma.

In the remaining lemma, we use the binary representation matrix of M, defined as follows. For any two disjoint bases B, B' of a binary matroid M, let $P_{B,B'}$ be the $B \times B'$ 0,1 matrix with $P_{B,B'}(b, b') = 1$ if and only if $B - b + b'$ is a basis.

Let B and B' be disjoint bases in a binary matroid M. Let $P := P_{B,B'}$. Thus

$$P(b, b') = 1 \Leftrightarrow b \in C(B, b') \Leftrightarrow b' \in C^*(B', b).$$

Let $Q := P^T_{B',B}$. Thus

$$Q(b, b') = 1 \Leftrightarrow b' \in C(B', b) \Leftrightarrow b \in C^*(B, b').$$

Then $\det P \neq 0$ and $Q = (P^{-1})^T$. Moreover, if $P(X, Y)$ denotes the submatrix of P indexed by $X \subseteq B$ and $Y \subseteq B'$, then for $|X| = |Y|$,

$$(B \setminus X) \cup Y \text{ is a basis of } M \Leftrightarrow \det P(X, Y) \neq 0.$$

Also, $Q(b, b') = \det P(B - b, B' - b')$.

Henceforth, all calculations are in $GF(2)$.

LEMMA 5. *Let M be a binary matroid, with disjoint bases B and B'. Let $P := P_{B,B'}$, and $Q := P^T_{B',B}$. Let $a, b \in B$ and $c, d \in B'$ with a, b, c, d distinct and $\{a, b, c\}$ pairwise adjacent. Suppose $P(a, c) = P(a, d) = P(b, c) = P(b, d) = 1$, and $Q(a, c) = Q(b, d) = 1$. Then $P(a, e) = Q(a, e) = 1$ for some $e \in B' - c$.*

PROOF. Suppose not. Choose a counterexample with $|B|$ minimal. We first prove a series of claims (i)–(vii).

(i) For each $f \in B - a - b$ one has $P(f, c) = 1$ or $P(f, d) = 1$. For each $g \in B' - c - d$ one has $P(a, g) = 1$ or $P(b, g) = 1$.

Assume that one of the statements is false. If the first statement is false, we choose $g \in B' - c - d$ with $P(f, g) = 1$ and $P(a, g)$ minimal (that is, $= 0$ if possible). If the second statement is false, we choose $f \in B - a - b$ with $P(f, g) = 1$.

Let $M' := M/g$. Then $B - f$ and $B' - g$ are disjoint bases of M'. Let $P' := P_{B-f,B'-g}$ and $Q' := P^T_{B'-g,B-f}$.

Then for each $x \in B - f$ and $y \in B' - g$ one has $P'(x, y) = P(x, y) + P(x, g)P(f, y)$. Indeed, $P'(x, y) = 1 \Leftrightarrow B - f - x + y$ is a basis of $M' \Leftrightarrow B - f - x + y + g$ is a basis of $M \Leftrightarrow P(x, y)P(f, g) + P(x, g)P(f, y) = 1$ (the determinant).

Therefore, for all $x \in \{a, b\}$ and $y \in \{c, d\}$ one has $P'(x, y) = 1$, since $P(x, g)P(f, y) = 0$ (as either statement is false). Moreover, for each $y \in B' - g$: if $P(a, y) = 0$, then $P'(a, y) = 0$, since otherwise $P(a, y) = 0$ and $P'(a, y) = 1$, implying $P(a, g) = P(f, y) = 1$, contradicting our choice of g with $P(a, g)$ minimal.

Additionally, for each $x \in B - f$ and $y \in B' - g$ one has $Q'(x, y) = Q(x, y)$, since $Q'(x, y) = 1 \Leftrightarrow B'-g-y+x$ is a basis of $M' \Leftrightarrow B'-y+x$ is a basis of $M \Leftrightarrow Q(x, y) = 1$.

Thus, we would obtain a smaller counterexample, contradicting our assumption. This proves (i).

(ii) There are no $f, f' \in B - a - b$ and $g, g' \in B' - c - d$ with $P(f, g) = P(f', g') = 1$, $P(f, g') = P(f', g) = 0$, $P(a, g) = P(a, g') = 0$, $P(b, g) = P(b, g')$, $P(f, c) = P(f', c)$ and $P(f, d) = P(f', d)$.

Suppose such f, f', g, g' exist. Let $M' := M/\{g, g'\}$. Then $B - f - f'$ and $B' - g - g'$ are disjoint bases of M'. Let $P' := P_{B-f-f',B'-g-g'}$ and $Q' := P^T_{B'-g-g',B-f-f'}$.

Then for each $x \in B - f - f'$ and $y \in B' - g - g'$ one has $P'(x, y) = P(x, y) + P(x, g)P(f, y) + P(x, g')P(f', y)$ and $Q'(x, y) = Q(x, y)$. In particular, $P'(x, y) = 1$ for all $x \in \{a, b\}$ and $y \in \{c, d\}$, and $P'(a, y) = P(a, y)$ for all $y \in B' - g - g'$. This is however a smaller counterexample, a contradiction, showing (ii).

(iii) There are no $f \in B - a - b$ and $g \in B' - c - d$ with $P(f, d) = P(a, g) = 0$, and $P(f, g) = 1$.

Suppose such f, g exist. Then by (i), $P(b, g) = 1$ and $P(f, c) = 1$, and so P has the following submatrix.

	c	d	g
a	1	1	0
b	1	1	1
f	1	0	1

Hence a, b, f, c, d, g span an $M(K_4)$ with b and c opposite, a contradiction. This gives (iii).

(iv) There are no $f, f' \in B - a - b$, $g, g' \in B' - c - d$ such that $P(a, g) = P(a, g') = 0$, $P(f, g) = P(f', g') = 1$ and $P(f, g') = P(f', g) = 0$.

Suppose such f, f', g, g' exist. By (i), $P(b, g) = P(b, g') = 1$. By (iii), $P(f, d) = P(f', d) = 1$. By (ii), $P(f, c) \neq P(f', c)$. Hence (by symmetry of f and f'), we can assume that $P(f, c) = 1$ and $P(f', c) = 0$. Then the submatrix of P spanned by a, b, f, f' and c, d, g, g' is as follows.

	c	d	g	g'
a	1	1	0	0
b	1	1	1	1
f	1	1	1	0
f'	0	1	0	1

This implies that a and b are not adjacent, a contradiction. This proves (iv).

There exists at least one $y \in B' - c - d$ with $P(a, y) = 0$, since otherwise we have the contradiction

$$0 = \sum_{y \in B'} Q(a, y)P(b, y) = P(b, c) = 1,$$

as $P(a, y) = 1$ and $y \neq c$ implies $Q(a, y) = 0$ (because we have a counterexample to the lemma).

Now choose $g \in B' - c - d$ such that $P(a, g) = 0$, and such that the number of $x \in B$ with $P(x, g) = 1$ is minimal. Choose $f \in B - a - b$ with $P(f, g) = 1$ (this exists, since $Q(b, d) = 1$). Note that $P(b, g) = 1$ by (i), and $P(f, d) = 1$ by (iii).

(v) For each $y \in B'$, if $P(b, y) = 0$, then $P(f, y) = 0$.

Suppose $P(b, y) = 0$ and $P(f, y) = 1$. Then by (i), $P(a, y) = 1$. Hence P contains the following submatrix.

	d	g	y
a	1	0	1
b	1	1	0
f	1	1	1

Thus a, b, d, f, g, y span $M(K_4)$ with a, b opposite, a contradiction. This shows (v).

(vi) For each $y \in B'$, if $P(a, y) = 0$, then $P(f, y) = 1$.

Suppose $P(a, y) = P(f, y) = 0$. Then by (i), $P(b, y) = 1$. By the minimality of the choice of g, there exists an x with $P(x, g) = 0$ and $P(x, y) = 1$. This contradicts (iv), proving (vi).

(vii) $P(f, c) = 1$.

Consider

$$\sum_{y \in B'} Q(a, y) P(b, y) = 0 = \sum_{y \in B'} Q(a, y) P(f, y).$$

Since we have a counterexample to the lemma, for each $y \in B'$: $Q(a, y) = 1$ implies $y = c$ or $P(a, y) = 0$; hence, by (vi) and (i), $Q(a, y) = 1$ implies $y = c$ or $P(b, y) = P(f, y) = 1$. Thus $P(b, c) = P(f, c) = 1$. This proves (vii).

As P is nonsingular, P does not have equal rows. Hence $P(f, y) \neq P(b, y)$ for some $y \in B'$. By (v), $P(f, y) = 0$ and $P(b, y) = 1$, and by (vi), $P(a, y) = 1$. Hence P contains the following submatrix.

	c	g	y
a	1	0	1
b	1	1	1
f	1	1	0

Thus a, b, c, f, g, y span an $M(K_4)$ with b, c opposite, a contradiction. This proves Lemma 5, and Theorem 5. □

REFERENCES

1. A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids, Encyclopedia of Mathematics and its Applications*, **46**, Cambridge University Press, Cambridge, 1993.
2. R. G. Bland and M. Las Vergnas, Orientability of matroids, *J. Combin. Theory, Ser. B.*, **24** (1978), 94–123.
3. R. A. Brualdi, Comments on bases in dependence structures, *Bull. Austr. Math. Soc.*, **1** (1969), 161–167.
4. R. Cordovil and P. Duchet, On sign-invariance graphs of uniform oriented matroids, *Discrete Math.*, **79** (1990), 251–257.
5. I. P. F. da Silva, Note on inseparability graphs of matroids having exactly one class of orientations, *Discrete Math.*, **171** (1997), 77–87.
6. J. Davies and C. McDiarmid, Disjoint common transversals and exchange structures, *J. London Math. Soc.*, **14** (1976), 55–62.
7. J. Keijsper, An algorithm for packing connectors, *J. Comb. Theory, Ser. B.*, **74** (1998), 397–404.

8. A. K. Kelmans, Concept of a vertex in a matroid and 3-connected graphs, *J. Graph Theory*, **4** (1980), 13–19.
9. J. G. Oxley, A characterization of the ternary matroids with no $M(K_4)$-minor, *J. Comb. Theory, Ser. B*, **42** (1987), 212–249.
10. J. G. Oxley, On nonbinary 3-connected matroids, *Trans. Am. Math. Soc.*, **300** (1987), 663–679.
11. J. G. Oxley, *Matroid Theory*, Oxford Science Publications, Clarendon Press, Oxford University Press, New York, 1992.
12. J.-P. Roudneff, Inseparability graphs of oriented matroids, *Combinatorica*, **9** (1989), 75–84.
13. P. D. Seymour, Minors of 3-connected matroids, *Europ. J. Combinatorics*, **6** (1985), 375–382.
14. P. D. Seymour, Adjacency in binary matroids, *Europ. J. Combinatorics*, **7** (1986), 171–176.
15. P. D. Seymour, Triples in matroid circuits, *Europ. J. Combinatorics*, **7** (1986), 177–185.
16. D. J. A. Welsh, *Matroid Theory, L. M. S. Monographs,* **8**, Academic Press (Harcourt Brace Jovanovich Publishers), London, 1976.

Received 13 October 1998 and accepted in revised form 22 September 1999

J. KEIJSPER AND R. PENDAVINGH
University of Amsterdam,
Department of Mathematics,
Plantage Muidergracht 24,
Amsterdam,
The Netherlands

A. SCHRIJVER
University of Amsterdam,
Department of Mathematics,
Plantage Muidergracht 24,
Amsterdam,
The Netherlands

AND
CWI,
Kruislaan 413,
Amsterdam,
The Netherlands