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Short Proofs on Multicommodity Flows and Cuts

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We give a short proof of a theorem of Karzanov on the packing of cuts, and derive a theorem of Lomonosov on the existence of integer multicommodity flows (implying theorems of Hu, Rothschild and Whinston, Dinits, Papernov, and Seymour).

1. KARZANOV'S THEOREM

Consider for any graph $H = (W, F)$ the following property:

H does not have one of the following two graphs as subgraph: (1)

(a) (b)

It is not difficult to check that (1) is equivalent to (assuming H has no isolated vertices): H is

either (i) the complete graph K_4;
or (ii) the circuit C_5; (2)
or (iii) the union of two stars (i.e., there are two vertices covering all edges of H).

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We first give a proof of the following theorem of Karzanov [6] (proved by Seymour [13] for the case $|F| = 2$). By $d(u, w)$ we mean the distance of u and w in graph G, and by $\delta(X)$ the cut determined by $X \subseteq V$. We say that $\delta(X)$ *separates* $\{u, w\}$ if $u \neq w$ and $|X \cap \{u, w\}| = 1$.

KARZANOV'S THEOREM. *Let $G = (V, E)$ be a connected bipartite graph, and let $H = (W, F)$ be a graph satisfying* (1), *with $W \subseteq V$. Then G has pairwise edge-disjoint cuts $\delta(X_1), ..., \delta(X_t)$ so that for each $\{r, s\} \in F$,*

$$d(r, s) = \text{number of cuts } \delta(X_j) \text{ separating } \{r, s\}. \tag{3}$$

Proof. Let $G = (V, E)$ be a counterexample with $|E|$ as small as possible. We first note the following:

For each $X \subseteq V$ with $\delta(X) \neq \emptyset$ there exist $\{r, s\} \in F$ and a path P connecting r and s so that $|P \backslash \delta(X)| \leqslant d(r, s) - 2$ (4)

(taking a path as an edge set). This can be seen by considering the bipartite graph G' obtained from G by contracting all edges in some $\delta(X)$. If r, s, P as in (4) do not exist for this $\delta(X)$, we know that for all $\{r, s\} \in F$:

$$\begin{aligned} d'(r', s') &= d(r, s) - 1 \qquad \text{if } \delta(X) \text{ separates } \{r, s\}; \\ d'(r', s') &= d(r, s), \qquad \text{if } \delta(X) \text{ does not separate } \{r, s\}. \end{aligned} \tag{5}$$

Here r' and s' are the images of r and s in G', and d' denotes the distance function in G'. As G' has fewer edges than G, in G' there exist pairwise edge-disjoint cuts $\delta(X'_1), ..., \delta(X'_t)$ so that for each $\{r, s\} \in F$ one has: $d'(r', s') =$ number of cuts $\delta(X'_j)$ separating $\{r', s'\}$. In the original graph this gives cuts $\delta(X_1), ..., \delta(X_t)$, which together with $\delta(X)$ have the required property.

From (4) we derive the following:

Claim. For all $u, w \in V$ with $u \neq w$ there exists $\{r, s\} \in F$ so that $\{r, s\} \cap \{u, w\} = \emptyset$ and

$$\begin{aligned} d(r, s) + d(u, w) &\geqslant d(r, w) + d(u, s), \\ d(r, s) + d(u, w) &\geqslant d(r, u) + d(w, s). \end{aligned} \tag{6}$$

Proof. Define

$$X := \{v \in V \mid d(u, v) + d(v, w) = d(u, w)\}. \tag{7}$$

So X is the set of vertices which are on at least one shortest $u - w$-path.

First suppose $X = V$. By (4) applied to $\{u\}$, there exist $\{r, s\} \in F$ and an $r - s$-path P so that $|P \backslash \delta(\{u\})| \leqslant d(r, s) - 2$. So P is a shortest $r - s$-path intersecting $\delta(\{u\})$ twice. This directly implies that $u \notin \{r, s\}$. To see that $w \notin \{r, s\}$, suppose $s = w$, say. Then $|P \backslash \delta(\{u\})| = |P| - 2 \geqslant d(u, r) + d(u, w) - 2 = 2d(u, r) + d(r, w) - 2 > d(r, s) - 2$, a contradiction.

So we know $\{r, s\} \cap \{u, w\} = \emptyset$. Moreover,

$$d(r, s) + d(u, w) = d(r, s) + d(u, r) + d(r, w) \geqslant d(r, w) + d(u, s). \tag{8}$$

One similarly shows the second inequality in (6).

Next suppose $X \neq V$. As above, let G' be the graph obtained from G by contracting the edges in $\delta(X)$, and let d' denote the distance function in G'. By (4), there exists $\{r, s\} \in F$ so that $d'(r, s) \leqslant d(r, s) - 2$. Then

$$\begin{aligned} d'(u, s) &\geqslant d(u, s) - 1, & d'(r, w) &\geqslant d(r, w) - 1, \\ d'(w, s) &\geqslant d(w, s) - 1, & d'(r, u) &\geqslant d(r, u) - 1. \end{aligned} \tag{9}$$

To see the first inequality, let P be a $u-s$-path in G with $|P \backslash \delta(X)| = d'(u, s)$. Choose P so that $|P \cap \delta(X)|$ is as small as possible. Suppose $|P \cap \delta(X)| \geqslant 2$. Then we can split P as $P'P''$ so that $|P' \cap \delta(X)| = 2$. Let P' go from u to $v \in X$. Since P' is not fully contained in X, we know $|P'| \geqslant d(u, v) + 2$. Let $\tilde{P}'$ be a shortest $u-v$-path in G. Then $|\tilde{P}'| = d(u, v) \leqslant |P'| - 2$, and $\tilde{P}'$ is fully contained in X. This implies for $\tilde{P} := \tilde{P}'P''$ that $|\tilde{P} \backslash \delta(X)| \leqslant |P \backslash \delta(X)|$ and $|\tilde{P} \cap \delta(X)| = |P \cap \delta(X)| - 2$, contradicting the minimality of $|P \cap \delta(X)|$. So $|P \cap \delta(X)| \leqslant 1$, implying $d'(u, s) = |P \backslash \delta(X)| \geqslant |P| - 1 \geqslant d(u, s) - 1$. The other inequalities in (9) are proved similarly.

Since $d'(r, s) \leqslant d(r, s) - 2$, (9) implies $\{r, s\} \cap \{u, w\} = \emptyset$. Moreover, there exists a shortest $r-s$-path in G' traversing a vertex v in X. Hence,

$$\begin{aligned} d(r, s) + d(u, w) &\geqslant d'(r, s) + d(u, w) + 2 \\ &= d'(r, v) + d'(v, s) + d(u, v) + d(v, w) + 2 \\ &\geqslant d'(r, w) + d'(u, s) + 2 \geqslant d(r, w) + d(u, s). \end{aligned} \tag{10}$$

The second inequality in (6) is shown similarly. This proves the Claim.

The claim implies that for each pair $\{u, w\}$ of vertices of G there exists an $\{r, s\} \in F$ disjoint from $\{u, w\}$. So H is not a union of two stars, and hence $H = K_4$ or $H = C_5$ (assuming H has no isolated vertices).

If $H = K_4$, let $W = \{r_1, r_2, r_3, r_4\}$. Then by the Claim,

$$\begin{aligned} d(r_1, r_2) + d(r_3, r_4) \geqslant d(r_1, r_3) + d(r_2, r_4) &\geqslant d(r_1, r_4) + d(r_2, r_3) \\ &\geqslant d(r_1, r_2) + d(r_3, r_4). \end{aligned} \tag{11}$$

Hence we have equality throughout, that is,

$$d(t, u) + d(v, w) = d(t, v) + d(u, w) \qquad \text{for all distinct } t, u, v, w \in W. \tag{12}$$

This implies that there exists a function $\phi\colon W \to \mathbb{R}_+$ so that $d(u, v) = \phi(u) + \phi(v)$ for each two distinct $u, v \in W$. (Take for $v \in W$: $\phi(v) := \frac{1}{2}(d(u, v) + d(v, w) - d(u, w))$ for arbitrary $u, w \in W$ with $v \neq u \neq w \neq v$. The fact that this is independent of the choice of u, w follows from (12).)

Since all vertices in W are distinct, $\phi(v) > 0$ for at least one $v \in W$. By (4), there exist $\{r, s\} \in F$ and a path P connecting r and s so that $|P \backslash \delta(\{v\})| \leqslant d(r, s) - 2$. So P passes v, and $|P| = d(r, s) = \phi(r) + \phi(s)$. However,

$$|P| \geqslant d(r, v) + d(v, s) = \phi(r) + 2\phi(v) + \phi(s) > \phi(r) + \phi(s), \tag{13}$$

a contradiction.

If $H = C_5$, let $W = \{r_1, r_2, r_3, r_4, r_5\}$ and $F = \{\{r_i, r_{i+1}\} \mid i = 1, ..., 5\}$, taking indices mod 5. Applying the Claim to $u := r_i$, $w := r_{i+2}$ we obtain $\{r, s\} := \{r_{i+3}, r_{i+4}\}$ (as it is the unique pair in F disjoint from $\{u, w\}$), and

$$\begin{aligned} d(r_i, r_{i+2}) + d(r_{i+3}, r_{i+4}) &\geqslant d(r_i, r_{i+3}) + d(r_{i+2}, r_{i+4}) \qquad (i = 1, ..., 5), \\ d(r_i, r_{i+2}) + d(r_{i+3}, r_{i+4}) &\geqslant d(r_i, r_{i+4}) + d(r_{i+2}, r_{i+3}) \qquad (i = 1, ..., 5). \end{aligned} \tag{14}$$

Adding up these ten inequalities, we obtain the same sum at both sides of the inequality sign. So we have equality in each of (14). This is equivalent to (12), and we obtain a contradiction in the same way as above. ∎

Note. The following two examples show that condition (1) in fact is necessary in Karzanov's theorem (where single lines are edges of G, and double lines are edges of H),

(15)

2. Implications of Karzanov's Theorem

As was noted by Karzanov [4] and Seymour [14], cut packing results (like Karzanov's theorem above) can be interpreted in terms of polyhedral cones, and thus by polarity of cones are related to multicommodity flows.

Consider the linear space $\mathbb{R}^F \times \mathbb{R}^E$. Let K be the convex cone in $\mathbb{R}^F \times \mathbb{R}^E$ generated by all vectors,

$$\begin{array}{lll} \text{(i)} & (\varepsilon^f; \chi^P) & \text{for } f=\{a,b\}\in F,\ P\subseteq E \text{ forming a path from } a \text{ to } b; \\ \text{(ii)} & (\underline{0}; \varepsilon^e) & \text{for } e\in E. \end{array} \tag{16}$$

Here ε^f denotes the fth unit basis in $\mathbb{R}^F$ (so $\varepsilon^f(f')=1$ if $f'=f$, and $=0$ otherwise). Similarly, ε^e denotes the eth unit basis vector in $\mathbb{R}^E$. Moreover, χ^P denotes the incidence vector of P in $\mathbb{R}^E$ (so $\chi^P(e)=1$ if $e\in P$, and $=0$ otherwise).

Let L be the convex cone generated by all vectors:

$$\begin{array}{lll} \text{(i)} & (-\chi^{\rho(X)}; \chi^{\delta(X)}) & \text{for } X\subseteq V; \\ \text{(ii)} & (\varepsilon^f; \underline{0}) & \text{for } f\in F; \\ \text{(iii)} & (\underline{0}; \varepsilon^e) & \text{for } e\in E. \end{array} \tag{17}$$

Here $\rho(X) := \{\{a,b\}\in F \mid \delta(X) \text{ separates } \{a,b\}\}$.

We here take for the *polar* of a cone C in $\mathbb{R}^n$ the cone

$$C^* := \{x\in\mathbb{R}^n \mid x^Ty\geqslant 0 \text{ for all } y\in C\}. \tag{18}$$

The following consequence of Karzanov's theorem is contained in the work of Papernov [10].

COROLLARY 1. *If* $H=(W,F)$ *satisfies* (1) *then* $K^*=L$.

Proof. The inclusion $L\subseteq K^*$ follows from the fact that each vector in (16) has nonnegative inner product with each vector in (17), which is trivial. For example,

$$\begin{aligned}(\varepsilon^f;\chi^P)(-\chi^{\rho(X)};\chi^{\delta(X)}) &= -1+|P\cap\delta(X)|\geqslant 0 \quad \text{if } \delta(X) \text{ separates } f, \\ &= |P\cap\delta(X)|\geqslant 0 \quad \text{if } \delta(X) \text{ does not separate } f.\end{aligned} \tag{19}$$

To see the inclusion $K^*\subseteq L$, take $(x;y)\in\mathbb{R}^F\times\mathbb{R}^E$ having nonnegative inner product with all vectors in K. In order to show $(x;y)\in L$, we may assume that x and y are integral, and in fact consist of even integers. Since $(x;y)$ has nonnegative inner product with all vectors in (16)(ii), we know that $y\geqslant 0$. By inserting new vertices, we can replace each edge e of G by a path of length $y(e)$ (contracting e if $y(e)=0$). This makes the new graph G', which is bipartite, as each $y(e)$ is even. Since $(x;y)$ has nonnegative inner product with all vectors in (16)(i), we know that for each $f=\{a,b\}\in F$,

$$-x(f)\leqslant \operatorname{dist}_{G'}(a,b). \tag{20}$$

By Karzanov's theorem, G' has pairwise disjoint cuts $\delta(X_1'), ..., \delta(X_t')$ so that each pair $\{a, b\} \in F$ is separated by $\text{dist}_{G'}(a, b)$ of these cuts. So in G we obtain cuts $\delta(X_1), ..., \delta(X_t)$ so that each edge e of G is contained in at most $y(e)$ of these cuts, and so that each pair $f = \{a, b\} \in F$ is separated by at least $-x(f)$ of these cuts, Hence

$$(x; y) \geqslant \sum_{i=1}^{t} (-\chi^{\rho(X_i)}, \chi^{\delta(X_i)}), \tag{21}$$

proving that $(x; y)$ is a nonnegative linear combination of the vectors in (17). So $(x; y)$ belongs to L. ▮

By duality we have:

COROLLARY 2. *If $H = (W, F)$ satisfies* (1) *then $L^* = K$.*

Proof. $L^* = (K^*)^* = K$. ▮

In other words, each vector which has nonnegative inner product with all vectors in (17) is a nonnegative linear combination of the vectors in (16). Using a method described by Karzanov [6], we derive the following result, proved first by Rothschild and Whinston [11] (cf. [9, 15]) for $|F| = 2$, by Dinits (cf. [1]) for F being a union of two stars, by Lomonosov [7, 8] and Seymour [16] if $|\bigcup F| = 4$, and by Lomonosov [7, 8] for F forming a five-circuit. In this corollary, we allow E and F to have multiple edges (in the results discussed above, multiple edges are irrelevant).

COROLLARY 3. *Let F satisfy* (1), *and let the graph $(V, E \cup F)$ be eulerian (counting multiplicities). Then there exist pairwise edge-disjoint paths P_f (for $f \in F$) so that P_f connects a and b if $f = \{a, b\}$, if and only if,*

$$|\delta(X)| \geqslant |\rho(X)| \qquad \textit{for all} \quad X \subseteq V. \tag{22}$$

Proof. Since (22) is trivially a necessary condition, we show sufficiency. Suppose the corollary is not true, and let G, H form a counterexample with $|E|$ as small as possible. Then

$$\text{no pair } f \text{ in } F \text{ is parallel to an edge } e \text{ in } E, \tag{23}$$

as otherwise deleting e and f would give a smaller counterexample.

Condition (22) being satisfied means that the all-one vector $(\underline{1}; \underline{1})$ in $\mathbb{R}^F \times \mathbb{R}^E$ has nonnegative inner product with all vectors in (17), i.e., is in L^*. So by Corollary 2, $(\underline{1}; \underline{1}) \in K$. Hence for all $f = \{a, b\} \in F$ there exists paths $P_{f1}, ..., P_{ft_f}$ connecting a and b, and rationals $\lambda_{f1}, ..., \lambda_{ft_f} > 0$ so that

$$\begin{aligned} &\text{(i)} \quad \sum_{i=1}^{t_f} \lambda_{fi} = 1 && \text{for all} \quad f \in F; \\ &\text{(ii)} \quad \sum_{f \in F} \sum_{i=1}^{t_f} \lambda_{fi} \chi^{P_{fi}}(e) \leqslant 1 && \text{for all} \quad e \in E. \end{aligned} \tag{24}$$

Consider some $f' = \{a, b\} \in F$. Let $e' = \{a, v'\}$, $e'' = \{v', v''\}$ be the first two edges of $P_{f'1}$. Let G' be the graph obtained from G by replacing e' and e'' by a new edge $e''' := \{a, v''\}$.

If G' again satisfies (22), then in G' there would exist paths as required (as G' has fewer edges than G), and hence also in G there exist paths as required (as we can replace an occurrence of e''' by e' and e'').

So G' does not satisfy (22). Let $X \subseteq V$ be so that $|\rho(X)| > |\delta'(X)|$, where $\delta'(X)$ denotes the cut in G' with sides X and $V \backslash X$. Since $|\delta(X)| \geqslant |\rho(X)|$ and $|\delta'(X)| \geqslant |\delta(X)| - 2$, it follows that

$$|\delta(X)| = |\rho(X)| = |\delta'(X)| + 2 \tag{25}$$

and that $e', e'' \in \delta(X)$. This implies

$$\begin{aligned} |\rho(X)| &= \sum_{f \in \rho(X)} 1 = \sum_{f \in \rho(X)} \sum_{i=1}^{t_f} \lambda_{fi} \\ &< \sum_{f \in F} \sum_{i=1}^{t_f} \lambda_{fi} |P_{fi} \cap \delta(X)| = \sum_{e \in \delta(X)} \left(\sum_{f \in F} \sum_{i=1}^{t_f} \lambda_{fi} \chi^{P_{fi}}(e) \right) \\ &\leqslant \sum_{e \in \delta(X)} 1 = |\delta(X)|. \end{aligned} \tag{26}$$

(The strict inequality follows from the facts that $|P_{fi} \cap \delta(X)| \geqslant 1$ if $f \in \rho(X)$, and that $|P_{f'1} \cap \delta(X)| \geqslant 2$.) However, this contradicts the fact that $|\delta(X)| = |\rho(X)|$. ∎

As is well known, this corollary implies a half-integral multicommodity flow theorem of Lomonosov [7] (extending the max-flow min-cut theorem of Ford and Fulkerson [2] and Hu's two-commodity flow theorem [3]; cf. also [10]).

COROLLARY 4. *Let F satisfy* (1), *let $c: E \to \mathbb{Z}_+$ be a "capacity" function, and let $d: F \to \mathbb{Z}_+$ be a "demand" function. Then each pair $f = \{a, b\} \in F$ can be connected by a half-integral flow φ_f in G of value $d(f)$, in such a way that the total flow passing any edge e of G is at most $c(e)$, if and only if each cut has capacity not smaller than its demand.*

Proof. Replace each edge e of G by $2c(e)$ parallel edges, and each pair f in F by $2d(f)$ "parallel" pairs. After that, apply Corollary 3 to the extended structure. ∎

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