The Radon Transform: First Steps

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In this note we discuss some aspects of the Radon transform mentioned in the review of Helgason's book in this Newsletter.

In short, the Radon transform of a function $f(x,y)$ of two variables is the set of line integrals, with obvious generalizations to higher dimensions. It plays a fundamental role in a large class of applications which fall under the heading of tomography. In a narrow sense, tomography is the problem of reconstructing the interior of an object by passing radiation through it and recording the resulting intensity over a range of directions. It is the problem of finding f from the above mentioned line integrals and is related to the inversion of the Radon transform.

Before discussing a simple example of how to compute the Radon transform, we will tell more about the background of the applications. In mathematical physics there is a notorious class of difficult problems: the ill-posed problems. The notion of a *well*-posed problem is due to Hadamard: a solution must exist, be unique, and depend continuously on the data. In ill-posed problems the last condition may be violated, and then important difficulties may arise, especially when the data are not complete or not accurate. Tomography falls in this class of ill-posed problems.

Probably the most widely known applications of tomography are in medicine. Computer assisted tomography (CAT-scan) uses X-rays directed from a range of directions to reconstruct the density in a thin slice of the body (the Greek word *τομος* means *section*). Recent advances in medical tomography include nuclear magnetic resonance (NMR), where strong magnetic fields are used to make hydrogen atoms resonate. One advantage over the CAT-scan is that the use of potentially harmful X-rays can be avoided.

An important feature of the medical applications of tomography is its 'nondestructive' character. Also in industry there is a considerable need to investigate the integrity and remaining reliable lifetime of components and structures by using nondestructive evaluation. Once again the components are subjected to penetrating radiation with the aim of deducing information about their internal states.

The search for oil depends heavily upon the analysis of seismic data. This is another example of the reconstruction of internal features of a body from monitored reflections of radiation or energy flows.

The Radon transform is an interesting example of a mathematical problem that was considered and solved long before its applicability was seen. In fact, this problem, as well as its three-dimensional version, was solved by J. Radon in 1917 and later rediscovered in various settings such as probability theory (recovering a probability distribution from its marginal distributions) and astronomy (determining the velocity distribution of stars from the distribution of radial velocities in various directions). Of course, much work was needed to adapt the Radon inversion formula to the incomplete information available in practice. The computational solution of ill-posed problems of the form arising in the general area of tomography is a very active research topic in computational mathematics. Although the last decade has yielded very useful algorithms, much work remains to be done; for instance in 3-D problems. At CWI research on reconstruction problems started quite recently. There are promissing contacts with industry (on NMR and seismic problems) and with researchers from medical disciplines.

To describe the rôle of line integrals in tomography we start with the equation

$$I = I_0 e^{-\mu x},$$

for the beam density of a narrow beam of X-ray photons through some homogeneous material, where I_0 is the input intensity (number of photons per second per unit cross-sectional area) and I is the observed intensity after the beam passes the distance x through the material. The linear *attenuation coefficient* μ depends, among other things, on the density of the material. This formula has to be changed for material that is inhomogeneous, where μ depends on a space parameter. In two variables the analogue of the above equation becomes

$$I = I_0 \exp[-\int_L \mu(x,y) ds],$$

where the line integral is along the beam path L, which is parametrized by s (see figure 1).

By moving the source and detector it is possible to obtain a set of line integrals. Taking logarithms, this constitutes a sampling of the Radon transform. Then an appropriate inversion or reconstruction algorithm is applied to recover an approximation to the attenuation coefficient distribution over a transverse section of some portion of the human body.

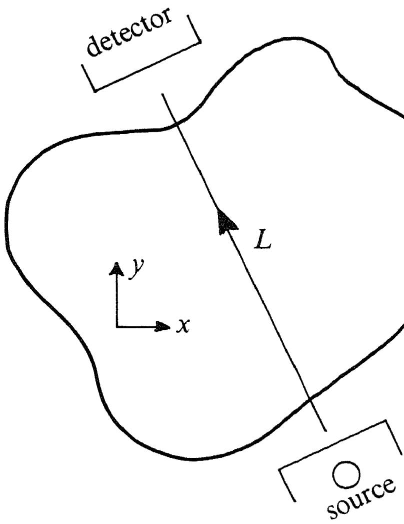
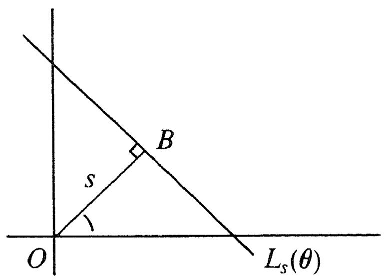


FIGURE 1. The beam passes through the region characterized by $\mu(x,y)$ along the line L

A typical coordinate system for setting up the Radon transform is the following. A line L_s in $\mathbb{R}^2$ with distance s from the origin $O = (0,0)$ in x,y-plane is further characterized by an angle θ (see figure 2)



$B = (s\cos\theta,\ s\sin\theta)$ is a fixed point on L

FIGURE 2

So each line has two parameters s, θ which look like polar coordinates, but in fact they are not: if $s = 0$ different values of θ yield different lines $L_0(\theta)$.

Let $A = (x,y) = (r\cos\phi, r\sin\phi)$ (polar coordinates) denote a variable point on $L_s(\theta)$. Then, if the distance from A to B equals t, we have

$$\begin{cases} x = s\cos\theta - t\sin\theta, \\ y = s\sin\theta + t\cos\theta. \end{cases} \tag{1}$$

To define the Radon transform we assume that $f : \mathbb{R}_2 \rightarrow \mathbb{R}$ is continuous and integrable, and we write

$$Rf(s,\theta) = \int_{-\infty}^{\infty} f(x,y)dt, \quad (x,y) \in L_s(\theta), \tag{2}$$

where t appears in the above notation for $(x,y) \in L_s(\theta)$. By allowing negative values of s, we can restrict the θ-domain to $[0,\pi]$, since

$$Rf(s,\theta) = Rf(-s,\pi+\theta).$$

In practical problems f is compactly supported, that is $f = 0$ if r is large enough, say $r \geqslant 1$.

For higher dimensions a vector notation is very useful. For $\mathbb{R}_2$ we start with the unit vector ω with polar angle θ. So the point B in figure 2 can be written as the vector $B = s\omega$, with scalar s. $L_0(\theta)$ runs through the origin and is parallel with $L_s(\theta)$. Let $y \in L_0(\theta)$. Then $\omega \cdot y = 0$. So $L_s(\theta)$ is characterized by the set of end points of vectors x that can be written as

$$x = y + s\omega, \quad \text{with } \omega \cdot y = 0,$$

or as the set of end points of vectors $x \in \mathbb{R}_2$ satisfying $x \cdot \omega = s$. Hence, the Radon transform can be written as

$$Rf(s,\theta) = \int_{y \cdot \omega = 0} f(y + s\omega) dy = \int_{x \cdot \omega = s} f(x) dx,$$

where, for convenience, we now suppose that the argument of f is a vector. The above definition is for $x, y, \omega \in \mathbb{R}_2$. However, by integrating over (hyper) planes, the same notation can be used for $\mathbb{R}_n$.

Especially fruitful is the introduction of the δ-function notation. Recall that this generalized function has the property

$$\int_{-\infty}^{\infty} f(x)\delta(x) dx = f(0)$$

for smooth functions f. So the line integral over the X-axis can be expressed as

$$Rf(0,0) = \int_{-\infty}^{\infty} f(x,0) dx = \int_{-\infty}^{\infty} \{ \int_{-\infty}^{\infty} f(x,y)\delta(y) dy \} dy.$$

In general we can write

$$Rf(s,\theta) = \int_{-\infty}^{\infty} \int f(x,y)\delta(s - x\cos\theta - y\sin\theta) dx dy$$

$$= \int\int f(x)\delta(s - x \cdot \omega) dx, \quad x \in \mathbb{R}_2,$$

and for n-dimensions we can use the same notation

$$Rf(s,\theta) = \int\int f(x)\delta(s - x \cdot \omega) dx,$$

$s \in \mathbb{R}$, $x, \omega \in \mathbb{R}_n$; θ is now a $(n-1)$-dimensional vector containing the polar angles for defining a $(n-1)$-dimensional hyperplane in $\mathbb{R}_n$, and ω is the corresponding unit vector.

EXAMPLE. Let $f(x,y) = \exp(-x^2-y^2) = \exp(-r^2)$. We use

$$\int\int_{-\infty}^{\infty} f(x,y)\delta(s-x\cos\theta-y\sin\theta)dxdy.$$

The transformation (rotation)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

yields directions of integration parallel and perpendicular to $L_s(\theta)$. From $x^2+y^2 = u^2+v^2$ (the mapping is an isometry), we obtain

$$Rf(s,\theta) = \int\int_{-\infty}^{\infty} \exp(-u^2-v^2)\delta(s-u)dudv = \sqrt{\pi}e^{-s^2}.$$

On the other hand, using (1), $r^2 = x^2+y^2 = s^2+t^2$, and definition (2) we easily obtain the same result. It follows that the Radon transform of the Gaussian distribution yields again a Gaussian.

The theory of the Radon transform can be put in the framework of the Fourier transforms. Since for the latter inversion formulas are readily available, the inversion of Radon transformations is, in principle, established. Radon was not aware of this link with Fourier transformation, and he put the inversion in the following form: Let $F_Q(q)$ be the mean of $Rf(s,\theta)$ over all $L_s(\theta)$ on a distance q from a point $Q = (x,y) = (r\cos\phi, r\sin\phi)$, i.e.,

$$F_Q(q) = \frac{1}{2\pi}\int_0^{2\pi} Rf[q+r\cos(\phi-\theta),\theta]d\theta,$$

then

$$f(x,y) = -\frac{1}{\pi}\int_0^{\infty} q^{-1}dF_Q(q) \tag{3}$$

(in the notation of a Stieltjes integral).

The conditions on f are: continuous and compactly supported. For a recent elementary proof see NIEVERGELT [4].

For the pure mathematician this may give an end to the matter. For the applied mathematician there are two important difficulties:

- the inversion of Radon transforms is an ill-posed problem (inaccurate data may produce instabilities)
- the number of line integrals (i.e., data) is limited; also the directions (θ-values) may be restricted to a narrow range.

The numerical analyst usually applies algebraic inversion techniques for integral equations, instead of using the analytical inversion theorem. The latter, however, plays a fundamental rôle in diverse areas of Radon transformations and tomography, especially when it is written in terms of Fourier transforms.

For a very nice introductory monograph the reader is referred to DEANS [1]. The book of NATTERER [3] goes further in the direction of mathematical foundation of this topic. The IEEE-Special Issue [5] gives an interesting introduction to both the mathematical and the applied aspects of tomography. In the references below excellent bibliographics are included. Recent contributions in which reconstruction is considered as a statistical problem by modelling both noise and the to be reconstructed object f as stochastic processes are described in VARDI ET AL. [6] and GEMAN and MCCLURE [2].

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