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# Distances and Cuts in Planar Graphs

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We prove the following theorem. Let $G = (V, E)$ be a planar bipartite graph, embedded in the euclidean plane. Let $O$ and $I$ be two of its faces. Then there exist pairwise edge-disjoint cuts $C_1, ..., C_t$ so that for each two vertices $v$, $w$ with $v, w \in O$ or $v, w \in I$, the distance from $v$ to $w$ in $G$ is equal to the number of cuts $C_j$ separating $v$ and $w$. This theorem is dual to a theorem of Okamura on plane multicommodity flows, in the same way as a theorem of Karzanov is dual to one of Lomonosov. © 1989 Academic Press, Inc.

## 1. INTRODUCTION

We prove the following theorem:

THEOREM. *Let $G = (V, E)$ be a planar bipartite graph, embedded in the euclidean plane. Let $O$ and $I$ be two of the faces. Then there exist pairwise edge-disjoint cuts $\delta(X_1), ..., \delta(X_t)$ so that for each two vertices $v$, $w$ with $v, w \in O$ or $v, w \in I$, the distance of $v$ to $w$ in $G$ is equal to the number of cuts $\delta(X_j)$ separating $v$ and $w$.*

[Here, for $X \subseteq V$, $\delta(X) := \{e \in E \mid |e \cap X| = 1\}$, while $\delta(X)$ *separates* $v$ and $w$ if $|\{v, w\} \cap X| = 1$.]

Note that for any graph $G$, whatever collection of pairwise edge-disjoint cuts $\delta(X_j)$ we take, for any two vertices $v$, $w$ of $G$, the distance from $v$ to $w$ is always at least as large as the number of these cuts separating $v$ and $w$. The point in the theorem is that we can get equality under the conditions given.

This theorem is "dual" to a theorem of Okamura [9] on plane multicommodity flows, in the same way as the results of Karzanov [4] are dual to those of Lomonosov [6, 7] on multicommodity flows, as we shall explain in Section 2 below. The theorem extends a result of Hurkens,



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Schrijver, and Tardos [3], dual to a theorem of Okamura and Seymour [10]; this result restricts $v$, $w$ to belong to only one fixed face.

The theorem cannot be generalized to the obvious extension with more than two faces, as is shown by the complete bipartite graph $K_{2,3}$. This graph also shows that we cannot allow in the theorem above pairs $v$, $w$ with $v \in O$ and $w \in I$.

## 2. Relation to Multicommodity Flows

In this section we discuss a relation of the theorem above with multicommodity flow problems. Let $G = (V, E)$ be an undirected graph. Let $\{r_1, s_1\}, ..., \{r_k, s_k\}$ be pairs of vertices ($r_i \neq s_i$ for $i = 1, ..., k$). Suppose we wish to decide if

there exist pairwise edge-disjoint paths $P_1, ..., P_k$ so that $P_i$ connects $r_i$ and $s_i$ ($i = 1, ..., k$). (1)

Clearly, the following "cut condition" is a necessary condition:

each cut $\delta(X)$ separates at most $|\delta(X)|$ of the pairs $r_i, s_i$. (2)

Now Lomonosov [6, 7] (extending earlier work by Menger [8], Hu [1], Rothschild and Whinston [12], Papernov [11], Seymour [15]), Okamura [9] (extending earlier work by Okamura and Seymour [10]), and Seymour [16] showed the following three results, each of which uses the following "parity condition":

for each vertex $v$, $|\delta(\{v\})| + |\{i \mid v \in \{r_i, s_i\}\}|$ is even. (3)

Lomonosov's theorem. *If*

*the graph* $H := (\{r_1, s_1, ..., r_k, s_k\}, \{\{r_1, s_1\}, ..., \{r_k, s_k\}\})$ *has at most four vertices, or is isomorphic to* $C_5$ *(the circuit with five vertices), or contains two vertices* $v$, $w$ *so that* $\{v, w\} \cap \{r_i, s_i\} \neq \emptyset$ *for all* $i = 1, ..., k$, (4)

*then the cut condition* (2) *and the parity condition* (3) *together imply* (1).

Okamura's theorem. *If*

*G is planar, so that there are two of its faces, O and I, with for each* $i = 1, ..., k$: $r_i, s_i \in O$ *or* $r_i, s_i \in I$, (5)

*then the cut condition* (2) *and the parity condition* (3) *together imply* (1).

Seymour's theorem. *If*

$$\textit{the graph } (V, E \cup \{\{r_1, s_1\}, ..., \{r_k, s_k\}\}) \textit{ is planar,} \tag{6}$$

*then the cut condition* (2) *and the parity condition* (3) *together imply* (1).

A consequence of these results is that if (4), (5), or (6) holds, and if, moreover, the cut condition (2) holds, then there exist paths $P'_1, P''_1, ..., P'_k, P''_k$ so that both $P'_i$ and $P''_i$ connect $r_i$ and $s_i$ $(i = 1, ..., k)$ and so that each edge of $G$ is in at most two of the paths $P'_1, P''_1, ..., P'_k, P''_k$. (This follows by duplicating each edge of $G$ and each pair $\{r_i, s_i\}$, after which (2) and (3) hold.)

Hence, if (4), (5), or (6) holds, and if $c \in \mathbb{Q}^E_+$ (a "capacity function") and $d \in \mathbb{Q}^k_+$ (a "demand function") so that

$$\begin{aligned} &\text{for each } X \subseteq V, \textstyle\sum (c_e \mid e \in \delta(X)) \\ &\qquad \geqslant \textstyle\sum (d_i \mid i = 1, ..., k; X \text{ separates } r_i \text{ and } s_i), \end{aligned} \tag{7}$$

then there exist paths $P^1_1, ..., P^{t_1}_1, P^1_2, ..., P^{t_2}_2, ..., P^1_k, ..., P^{t_k}_k$ (where each $P^j_i$ connects $r_i$ and $s_i$) and rationals $\lambda^1_1, ..., \lambda^{t_1}_1, \lambda^1_2, ..., \lambda^{t_2}_2, ..., \lambda^1_k, ..., \lambda^{t_k}_k \geqslant 0$ so that

$$\begin{aligned} &\sum_{i=1}^{k} \sum_{\substack{j=1 \\ e \in P^j_i}}^{t_i} \lambda^j_i \leqslant c_e \qquad (e \in E), \\ &\sum_{j=1}^{t_i} \lambda^j_i = d_i \qquad (i = 1, ..., k) \end{aligned} \tag{8}$$

(a "multicommodity flow"). (For (5) this is a result of Papernov [11].) (This result follows from the result in the previous paragraph, by observing that we may take, without loss of generality, $c$ and $d$ to be integral; and hence we can replace each edge $e$ of $G$ by $c_e$ parallel edges, and each pair $\{r_i, s_i\}$ by $d_i$ parallel pairs, after which we apply the previous result.)

In polyhedral terms, this statement is equivalent to: if (4), (5), or (6) holds, then the cone of vectors $(d; c) \in \mathbb{Q}^k \times \mathbb{Q}^E$ defined by the linear inequalities

$$\begin{aligned} &\text{(i)} \quad \textstyle\sum (c_e \mid e \in \delta(X)) \geqslant \sum (d_i \mid i \in \rho(X)) \qquad (X \subseteq V), \\ &\text{(ii)} \quad d_i \geqslant 0 \qquad (i = 1, ..., k), \\ &\text{(iii)} \quad c_e \geqslant 0 \qquad (e \in E) \end{aligned} \tag{9}$$

(where $\rho(X) := \{i = 1, ..., k \mid X$ separates $r_i$ and $s_i\})$, is equal to the cone generated by the following vectors:

$$\begin{aligned} &\text{(i)} \quad (\varepsilon_i; \chi^P) \qquad (i = 1, ..., k; P\ r_i - s_i\text{-path}), \\ &\text{(ii)} \quad (0; \varepsilon_e) \qquad (e \in E). \end{aligned} \tag{10}$$

(Here $\varepsilon_i$ denotes the $i$th unit basis vector in $\mathbb{Q}^k$; $\varepsilon_e$ denotes the $e$th unit basis vector in $\mathbb{Q}^E$; $\chi^P$ is the *incidence vector* of $P$ in $\mathbb{Q}^E$, i.e., $\chi^P(e) = 1$ if $e \in P$ and $= 0$ otherwise.)

By polarity, this last statement is equivalent to: if (4), (5), or (6) holds, then the cone of vectors $(b; l) \in \mathbb{Q}^k \times \mathbb{Q}^E$ defined by the linear inequalities

$$\begin{aligned} &\text{(i)} \quad b_i + \sum_{e \in P} l_e \geqslant 0 \qquad (i = 1, ..., k; P\ r_i - s_i\text{-path}), \\ &\text{(ii)} \quad l_e \geqslant 0 \qquad (e \in E), \end{aligned} \tag{11}$$

is equal to the cone generated by the following vectors:

$$\begin{aligned} &\text{(i)} \quad (-\chi^{\rho(X)}; \chi^{\delta(X)}) \qquad (X \subseteq V), \\ &\text{(ii)} \quad (\varepsilon_i; 0) \qquad (i = 1, ..., k), \\ &\text{(iii)} \quad (0; \varepsilon_e) \qquad (e \in E). \end{aligned} \tag{12}$$

Note that (11)(i) just means that $-b_i$ is a lower bound for the distance from $r_i$ to $s_i$, taking $l$ as a length function. So the statement is equivalent to: if (4), (5), or (6) holds, then for any "length function" $l: E \to \mathbb{Q}_+$, there exist subsets $X_1, ..., X_t$ of $V$ and rationals $\mu_1, ..., \mu_t \geqslant 0$, so that

$$\begin{aligned} &\text{(i)} \quad \sum (\mu_j \mid j = 1, ..., t; i \in \rho(X_j)) \geqslant \text{dist}_l(r_i, s_i) \qquad (i = 1, ..., k), \\ &\text{(ii)} \quad \sum (\mu_j \mid j = 1, ..., t; e \in \delta(X_j)) \leqslant l_e \qquad (e \in E). \end{aligned} \tag{13}$$

[Here, $\text{dist}_l$ denotes the distance, taking $l$ as a length function. Note that equality in (i) can be derived from (ii).]

Now Karzanov [4] showed that if (4) holds, and if $l$ is integral, we can take the $\mu_j$ half-integral. In fact, he showed that if $l$ is integral so that each circuit of $G$ has an even length, we can take the $\mu_j$ integral (thus extending work of Hu [2] and Seymour [13]). Equivalently, if $G$ is bipartite and (4) holds, then there exist pairwise edge-disjoint cuts $\delta(X_1), ..., \delta(X_t)$ so that for each $i = 1, ..., k$, the distance from $r_i$ to $s_i$ is equal to the number of cuts $\delta(X_j)$ separating $r_i$ and $s_i$. (The equivalence follows in one direction by taking $l_e = 1$ for each edge $e$, and in the other direction by replacing each edge $e$ of length $l_e$ by a path consisting of $l_e$ edges.)

The theorem to be proved in this paper is similar, but now with respect to Okamura's condition (5) instead of Lomonosov's condition (4). Note that in a similar way as above, a fractional version of Okamura's theorem can be derived from our theorem. (For more on the duality of path and cut packing, see Karzanov [5].)

Professor A. V. Karzanov communicated to me that a similar theorem with respect to Seymour's condition (6) can be derived from Seymour [14].

## 3. Proof of the Theorem

Suppose that the theorem is not true, and let $G$ be a counterexample with

$$\sum_{F \neq O, I} 2^{e(F)} \qquad \text{as small as possible,} \tag{14}$$

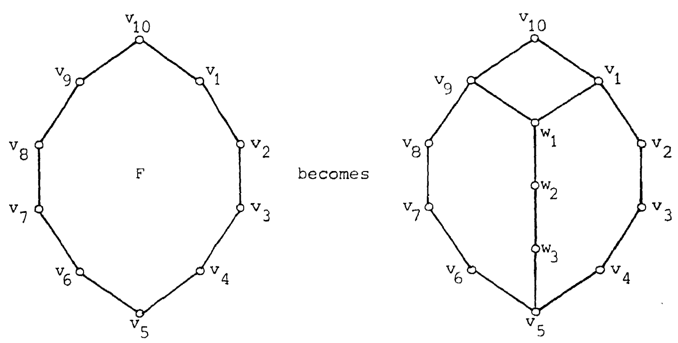
where the sum ranges over all faces $F \neq O, I$, and where $e(F)$ denotes the number of edges surrounding $F$. We may assume that $O$ is the unbounded face.

$G$ has no multiple edges: otherwise, either the circuit $C$ formed by them is a face, in which case we can delete one of the edges, thereby decreasing sum (14), or $C$ contains edges both in its interior and in its exterior, in which case the graph formed by $C$ and its interior or the graph formed by $C$ and its exterior yields a counterexample with smaller sum (14).

We first show:

Claim 1. *Each face $F \neq O, I$ forms a quadrangle (i.e., $e(F) = 4$).*

*Proof of Claim* 1. Let $F$ be some face forming a $k$-gon, with $k \neq 4$. Since $G$ is bipartite and has no parallel edges, $k$ is even and $k \geqslant 6$. We make a counterexample with a smaller sum than (14) as follows. Let $v_1, ..., v_k$ be the vertices surrounding $F$. Add, in the interior of $F$, new vertices $w_1, ..., w_{(1/2)k-2}$ and new edges $\{v_1, w_1\}$, $\{v_{k-1}, w_1\}$, $\{w_i, w_{i+1}\}$ $(i = 1, ..., \frac{1}{2}k - 3)$, and $\{w_{(1/2)k-2}, v_{(1/2)k}\}$. E.g., for $k = 10$,



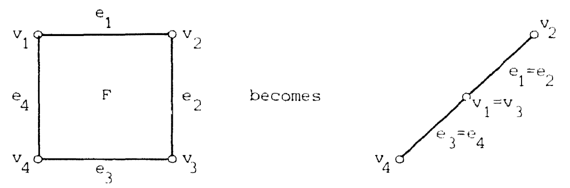

(15)

Note that this modification does not change the distance between any two vertices of the original graph. Therefore, after this modification we have again a counterexample to the theorem, with, however, a smaller sum than (14) (since $2^k > 2^{k-2} + 2^{k-2} + 2^4$), contradicting our assumption. ∎

Next we show:

CLAIM 2. *Let $F$ be a face, with $F \neq O, I$, and let $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, $e_3 = \{v_3, v_4\}$, $e_4 = \{v_4, v_1\}$ be the four edges surrounding $F$. Then there exist vertices $v$, $w$, with $v, w \in O$ or $v, w \in I$, and a shortest path from $v$ to $w$ which uses both $e_1$ and $e_2$.*

*Proof of Claim* 2. Suppose no such $v$, $w$ exist. Identify $v_1$ and $v_3$, $e_1$ and $e_2$, and $e_3$ and $e_4$. So




(16)

After this modification, all distances between vertices $v$, $w$ on $O$ and between vertices $v$, $w$ on $I$, are unchanged. Hence, the new graph is again a counterexample. However, the sum (14) has decreased, contradicting our assumption. ∎

Now we define *dual paths* $Q_1, ..., Q_t$, i.e., paths (including circuits) in the (planar) dual graph of $G$. These dual paths are determined by the following properties: each edge of the graph occurs exactly once in $Q_1, ..., Q_t$; if $F(\neq O, I)$ is surrounded by the edges $e_1, e_2, e_3, e_4$ (in this order), then $e_1, F, e_3$ (or $e_3, F, e_1$) will occur in exactly one of the $Q_j$; the faces $O$ and $I$ only occur as beginning or end faces in $Q_1, ..., Q_t$.

More precisely, $Q_1, ..., Q_t$ are all possible sequence of the form

$$(F_0, e_1, F_1, e_2, ..., F_{k-1}, e_k, F_k) \tag{17}$$

satisfying: (i) for $i = 1, ..., k$: $e_i$ is an edge separating the faces $F_{i-1}$ and $F_i$; (ii) for $i = 1, ..., k-1$: $F_i \notin \{O, I\}$ and $e_i$ and $e_{i+1}$ are opposite edges of $F_i$; (iii) either $F_0 = F_k \notin \{O, I\}$ and $e_1$ and $e_k$ are opposite edges of $F_0$, or $F_0, F_k \in \{O, I\}$; (iv) $k \geq 1$. If $F_0 = F_k \notin \{O, I\}$, we identify all possible sequences obtained from (17) by cyclically shifting it or by reversing it. If $F_0, F_k \in \{O, I\}$, we identify (17) with its reverse. Here edge $e$ is said to *separate* faces $F$ and $F'$ if $F$ and $F'$ are the faces incident to $e$ (possibly $F = F'$). Clearly, in the way described the edges of $G$ are partitioned into dual paths and circuits.

Consider now some fixed $Q_g$, represented by (17). Let for each $i=1, ..., k$, $v_i$ and $w_i$ be vertices so that $e_i=\{v_i, w_i\}$ and so that if we would orient the edges surrounding $F_i$ clockwise, then $e_i$ is oriented from $v_i$ to $w_i$. Then $f_i := \{v_i, v_{i+1}\}$ and $g_i := \{w_i, w_{i+1}\}$ are also edges of $G$ $(i=1, ..., k-1)$. So

$$(v_1, f_1, v_2, f_2, ..., v_{k-1}, f_{k-1}, v_k) \tag{18}$$

is the path along $Q_g$ "on the right side," and

$$(w_1, g_1, w_2, g_2, ..., w_{k-1}, g_{k-1}, w_k) \tag{19}$$

is the path along $Q_g$ "on the left side."

CLAIM 3. *For all* $i, j \in \{1, ..., k\}$: $\text{dist}(v_i, v_j) = \text{dist}(w_i, w_j)$, *where* dist *denotes distance.*

*Proof of Claim* 3. Suppose to the contrary that $\text{dist}(v_i, v_j) \neq \text{dist}(w_i, w_j)$ for some $i, j$. Choose such $i, j$ so that $i<j$ and $j-i$ is as small as possible. By symmetry, we may assume that $\text{dist}(v_i, v_j) < \text{dist}(w_i, w_j)$. As $G$ is bipartite, $j-i \geqslant \text{dist}(w_i, w_j) \geqslant \text{dist}(v_i, v_j)+2 \geqslant 2$.

Let

$$(v_i, \sigma, v_j) \tag{20}$$

be a shortest $v_i - v_j$-path, for some string $\sigma$. Since $\text{dist}(w_i, w_j) \geqslant \text{dist}(v_i, v_j)+2$, it follows that

$$(w_i, e_i, v_i, \sigma, v_j, e_j, w_j) \tag{21}$$

is a shortest $w_i - w_j$-path. Consider the circuit (see Fig. 1)

$$C := (v_i, \sigma, v_j, f_{j-1}, v_{j-1}, ..., f_{i+1}, v_{i+1}, f_i, v_i). \tag{22}$$

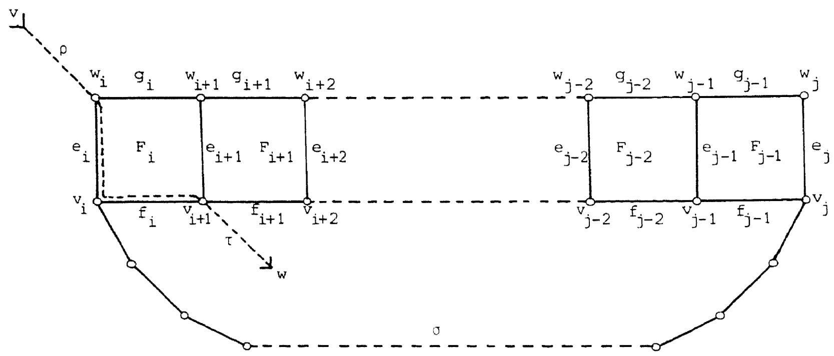


FIGURE 1

$C$ is a simple circuit, i.e., no vertex occurs twice in (22), except for the beginning and end vertex. Indeed, all vertices in (20) are distinct, as it is a shortest path. Moreover, all vertices $v_i, v_{i+1}, ..., v_j$ are distinct, except possibly $v_i = v_j$: if $v_p = v_q$ with $i \leqslant p < q \leqslant j$, then $\text{dist}(v_p, v_q) = 0 < \text{dist}(w_p, w_q)$ (since $G$ has no parallel edges), and hence, by the minimality of $j-i$, $q-p \geqslant j-i$; that is, $p = i$ and $q = j$. Suppose finally, $\sigma = (\sigma', v_q, \sigma'')$ for some strings $\sigma'$, $\sigma''$ and $i+1 \leqslant q \leqslant j-1$. Then $\text{dist}(v_i, v_q) + \text{dist}(v_q, v_j) = \text{dist}(v_i, v_j)$ (as $v_q$ is on the shortest $v_i - v_j$-path (20), and hence, $\text{dist}(v_i, v_q) + \text{dist}(v_q, v_j) = \text{dist}(v_i, v_j) < \text{dist}(w_i, w_j) \leqslant \text{dist}(w_i, w_q) + \text{dist}(w_q, w_j)$. Therefore, $\text{dist}(v_i, v_q) < \text{dist}(w_i, w_q)$ or $\text{dist}(v_q, v_j) < \text{dist}(w_q, w_j)$, contradicting the minimality of $j-i$.

By Claim 2, there exist vertices $v$ and $w$, either both on $O$ or both on $I$, and a shortest $v-w$-path $P$ with

$$P = (v, \rho, w_i, e_i, v_i, f_i, v_{i+1}, \tau, w), \tag{23}$$

where $\rho$ and $\tau$ are strings. Hence, the path

$$P' := (v, \rho, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau, w) \tag{24}$$

is also a shortest $v-w$-path. Since $O, I \notin \{F_i, ..., F_{j-1}\}$ (as $1 \leqslant i < j \leqslant k$), we are in one of the following four cases (as either both $v$ and $w$ are enclosed by $C$ (Cases 1 and 2) or not (Cases 3 and 4)).

*Case* 1. $(v, \rho)$ and $(v_i, \sigma, v_j)$ have a vertex in common, say $u$,

$$\begin{aligned}(v, \rho) &= (\rho', u, \rho''),\\ (v_i, \sigma, v_j) &= (\sigma', u, \sigma''),\end{aligned} \tag{25}$$

for (possibly empty) strings $\rho'$, $\rho''$, $\sigma'$, $\sigma''$. Then

$$(\rho', u, (\sigma')^{-1}, e_i, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau, w) \tag{26}$$

also would be a shortest $v-w$-path, since $(w_i, e_i, \sigma', u)$ is a shortest $w_i - u$-path (as it is part of (21)). But then

$$(\rho', u, (\sigma')^{-1}, f_i, v_{i+1}, \tau, w) \tag{27}$$

would be an even shorter $v-w$-path, which is a contradiction.

*Case* 2. $(v, \rho)$ contains one of the edges $e_{i+1}, ..., e_{j-1}$, say $(v, \rho) = (\rho', v_p, e_p, w_p, \rho'')$ for some $p$ with $i+1 \leqslant p \leqslant j-1$ and certain (possibly empty) strings $\rho'$, $\rho''$. Substitution in $P$ gives:

$$P = (\rho', v_p, e_p, w_p, \rho'', w_i, e_i, v_i, f_i, v_{i+1}, \tau, w). \tag{28}$$

Since $P$ is a shortest $v-w$-path, it follows that $\text{dist}(w_p, w_i) < \text{dist}(v_p, v_i)$, contradicting the minimality of $j-i$.

*Case* 3. $(\tau, w)$ and $(v_i, \sigma, v_j)$ have a vertex in common, say $u$,

$$(\tau, w) = (\tau', u, \tau''),$$
$$(v_i, \sigma, v_j) = (\sigma', u, \sigma''), \tag{29}$$

for (possibly empty) strings $\tau'$, $\tau''$, $\sigma'$, $\sigma''$. So

$$(w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', u) \tag{30}$$

is not longer than

$$(w_i, e_i, \sigma', u) \tag{31}$$

(since (30) is part of the shortest path $P'$). Hence, substituting (31) by (30) in (21),

$$(w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', u, \sigma'', v_j, e_j, w_j) \tag{32}$$

is a shortest $w_i - w_j$-path. In particular, $\text{dist}(v_{i+1}, v_j) < \text{dist}(w_{i+1}, w_j)$, contradicting the minimality of $j-i$.

*Case* 4. $(\tau, w)$ contains one of the edges $e_{i+1}, ..., e_{j-1}$, say $(\tau, w) = (\tau', v_p, e_p, w_p, \tau'')$ for some $p$ with $i+1 \leqslant p \leqslant j-1$ and certain (possibly empty) strings $\tau'$, $\tau''$. Substitution in $P$ gives:

$$P = (v, \rho, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', v_p, e_p, w_p, \tau''). \tag{33}$$

Since $P$ is a shortest $v-w$-path, it follows that $\text{dist}(v_{i+1}, v_p) < \text{dist}(w_{i+1}, w_p)$, contradicting the minimality of $j-i$. ∎

A consequence of Claim 3 is that $Q_g$ will have no self-intersections: if $F_i = F_j$ with $i \neq j$ and $i-j \neq k$, then $v_i = v_{j+1}$, $w_i \neq w_{j+1}$, or $v_{i+1} = v_j$, $w_{i+1} \neq w_j$, as one easily checks. This contradicts Claim 3.

Another consequence of Claim 3 is:

$$\text{no shortest path has more than one edge in common with } Q_g. \tag{34}$$

Next we show:

CLAIM 4. *Each $Q_g$ connects $O$ and $I$.*

*Proof of Claim* 4. Suppose $Q_g$ does not connect $O$ and $I$, for some $g = 1, ..., t$. Then $Q_g$ connects $O$ with $O$, or connects $I$ with $I$, or is a circuit. That is, the edges in $Q_g$ form a cut $\delta(X)$, for some $X \subseteq V$.

I. We first show for each $v, w \in V$ that for each $v-w$-path $P$ there exists a $v-w$-path $P'$ so that

$$\text{length}(P') - \text{int}(P', Q_g) \leqslant \text{length}(P) - \text{int}(P, Q_g), \quad \text{and}$$
$$\text{int}(P', Q_g) \leqslant 1, \tag{35}$$

where int(..., $Q_g$) denotes the number of edges in ... in common with $Q_g$. This is shown by induction on length ($P$). If $\text{int}(P, Q_g) \geqslant 2$, there exist $i, j$ so that $P = (\rho, v_i, e_i, w_i, \sigma, w_j, e_j, v_j, \tau)$ for strings $\rho, \sigma, \tau$, where $\sigma$ does not have any edge in common with $Q_g$ (we use the notation introduced before Claim 3; maybe $v_i, v_j$ and $w_i, w_j$ are interchanged). Since by Claim 3, $\text{dist}(w_i, w_j) = \text{dist}(v_i, v_j)$, there exists a $v-w$-path $\tilde{P}$ with $\text{length}(\tilde{P}) \leqslant \text{length}(P) - 2$ and $\text{int}(\tilde{P}, Q_g) \geqslant \text{int}(P, Q_g) - 2$. Applying the induction hypothesis to $\tilde{P}$ implies the statement above.

II. Now contract all edges ocurring in $Q_g$. This gives a smaller bipartite graph $G'$. For the new distance function dist′ in $G'$ we have

$$\begin{aligned} \text{dist}'(v, w) &= \text{dist}(v, w) - 1, && \text{if } X \text{ separates } v \text{ and } w, \\ \text{dist}'(v, w) &= \text{dist}(v, w), && \text{otherwise.} \end{aligned} \tag{36}$$

To see this, it suffices to show that $\text{dist}'(v, w) \geqslant \text{dist}(v, w) - 1$ for all $v, w$ (by the bipartiteness of $G$ and $G'$). Let $\Pi$ be a shortest $v-w$-path in $G'$. It corresponds to a $v-w$-path $P$ in $G$ with $\text{length}(P) - \text{int}(P, Q_g) = \text{length}(\Pi)$. Hence, by I above, there exists a $v-w$-path $P'$ in $G$ so that $\text{length}(P') - \text{int}(P', Q_g) \leqslant \text{length}(\Pi)$ and $\text{int}(P', Q_g) \leqslant 1$. Hence, $\text{dist}(v, w) \leqslant \text{length}(P') \leqslant \text{length}(\Pi) + 1 = \text{dist}'(v, w) + 1$.

By the minimal property of $G$, in $G'$ there exist pairwise disjoint cuts $\delta(X_1), ..., \delta(X_{t'})$ so that for all pairs of vertices $v, w$ both on $O$ or both on $I$:

$$\text{dist}'(v, w) = |\{i = 1, ..., t' \mid X_i \text{ separates } v \text{ and } w\}|. \tag{37}$$

So by (36), taking $X_{t'+1} := X$, in $G$ we have for all such $v, w$:

$$\text{dist}(v, w) = |\{i = 1, ..., t' + 1 \mid X_i \text{ separates } v \text{ and } w\}|. \tag{38}$$

As $\delta(X_1), ..., \delta(X_{t'+1})$ are pairwise disjoint, $G$ is not a counterexample to he theorem, contradicting our assumption. ∎

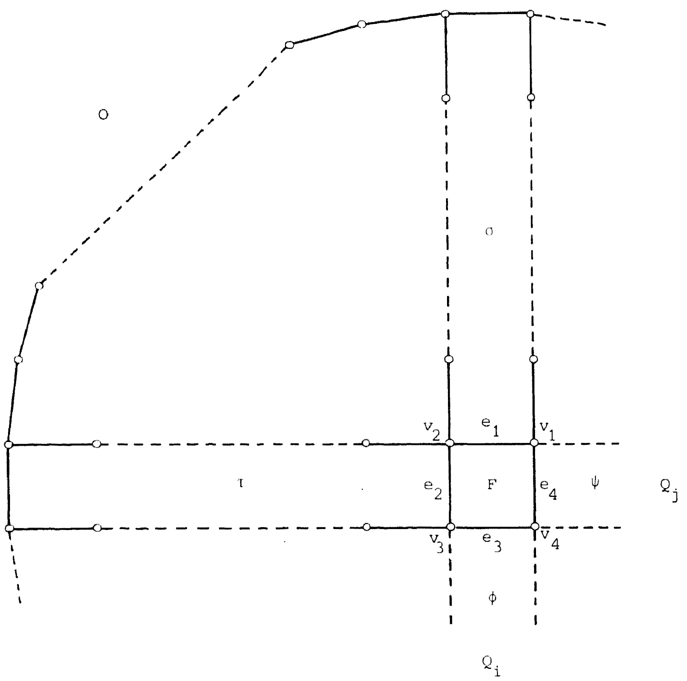
Our final claim will complete the counterexample:

CLAIM 5. *No two distinct $Q_i$ and $Q_j$ have a face $F \neq O, I$ in common.*

*Proof of Claim* 5. Suppose to the contrary

$$\begin{aligned} Q_i &= (O, \sigma, F, \varphi), \\ Q_j &= (O, \tau, F, \psi), \end{aligned} \tag{39}$$

for strings $\sigma$, $\varphi$, $\tau$, $\psi$, and face $F \neq O, I$ $(i \neq j)$. We may assume that $\sigma$ and $\tau$ do not have a face in common (by taking (39) so that $\sigma$ and $\tau$ have minimal length). This gives the following situation




(40)

We may assume that $e_1$ is the last symbol of $\sigma$ and that $e_2$ is the last symbol of $\tau$. By Claim 2, there exist vertices $v$, $w$, both on $O$ or both on $I$, and a shortest $v-w$-path $P$ using $e_2$ and $e_3$:

$$P = (v, \pi, v_2, e_2, v_3, e_3, v_4, \rho, w). \tag{41}$$

As $P$ is a shortest $v-w$-path, with $v, w \in O$ or $v, w \in I$, $P$ has at most one edge in common with each of the $Q_g$ $(g = 1, ..., k)$ (by (34)). Since $P$ crosses both $Q_i$ and $Q_j$ at $F$, while the vertex $v_2$ is contained in the set of vertices enclosed by the dual circuit $(O, \sigma, F, \tau^{-1}, O)$, $P$ should also have its beginning vertex $v$ inside of this circuit. So $v$ is on $O$, and hence also $w$ is on $O$.

Since $P$ has exactly one edge in common with $Q_i$, it follows that $P$ is homotopic (in the space obtained from the euclidean plane by deleting the interiors of $O$ (=unbounded face) and $I$) to the $v-w$-path $P'$ which follows the boundary of $O$ and which contains the first edge of $Q_i$. Similarly, $P$ is homotopic to the $v-w$-path $P''$ which follows the boundary of $O$ and which contains the first edge of $Q_j$. Since $v$ is inside of the circuit $(O, \sigma, F, \tau^{-1}, O)$, while $w$ is outside of it, $P'$ is not homotopic to $P''$, a contradiction. ∎

Claim 5 implies that there are no faces other than $O$ and $I$ (any other face would belong to two different $Q_i$ and $Q_j$). So $G$ is a simple circuit, for which the theorem trivially holds.

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