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ON FRACTIONAL MULTICOMMODITY FLOWS AND DISTANCE FUNCTIONS

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We give some results on the existence of fractional and integral solutions to multicommodity flow problems, and on the related problem of decomposing distance functions into cuts. One of the results is: Let $G=(V, E)$ be a planar bipartite graph. Then there exist subsets $W_1, \ldots, W_t$ of V so that for each pair v', v'' of vertices on the boundary of G, the distance of v' and v'' in G is equal to the number of $j=1, \ldots, t$ with $|\{v', v''\} \cap W_j| = 1$ and so that the cuts $\delta(W_j)$ are pairwise disjoint.

1. Introduction

In this paper we show some results on fractional and integral multicommodity flows, and on the packing of cuts in planar graphs. Among the results shown is the following:

Let $G=(V, E)$ be a planar bipartite graph. Then there exist subsets $W_1, \ldots, W_t$ of V so that for each pair v', v'' of vertices on the outer face of G, the distance of v' and v'' in G is equal to the number of $j=1, \ldots, t$ with $|\{v', v''\} \cap W_j| = 1$ and so that the cuts $\delta(W_j)$ are pairwise disjoint

(see Theorem 1 below). Before discussing the results, we first give as a motivation an introduction to multicommodity flows and cut packing, and their 'polarity'.

It is an NP-complete problem to decide if in a given undirected graph $G=(V, E)$, with given pairs of vertices (*ports*) $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$,

there exist k pairwise edge-disjoint paths $P_1, \ldots, P_k$, where P_i connects r_i and s_i $(i=1, \ldots, k)$ (1)

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(Even et al. [1]). There are however some special cases where good characterizations and polynomial-time algorithms have been found. The larger part of these good characterizations consist of the assertion that the following, obviously necessary, *cut condition* is also sufficient:

$$\text{for each } W \subseteq V : |\delta(W)| \geqslant |\sigma(W)| . \tag{2}$$

Here $\delta(W) := \{e \in E \mid |e \cap W| = 1\}$ and $\sigma(W) := \{i \mid |\{r_i, s_i\} \cap W| = 1\}$. It is easy to see that, if G is connected, we may restrict W in (2) to subsets W for which both W and $V \backslash W$ induce a connected subgraph of G.

Many of these results are restricted to the case where the following *parity condition* holds:

$$\text{for each vertex } v \text{ of } G : |\delta(\{v\})| + |\sigma(\{v\})| \text{ is even.} \tag{3}$$

In one stream of research the given ports are restricted to certain configurations. This stream has begun with the work of Menger [9], Hu [3] and Papernov [12], and has culminated in the work of Lomonosov [7, 8] and Seymour [16]. Lomonosov showed that for any given set of pairs $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ the following two statements are equivalent:

for each graph $G = (V, E)$ with $V \supseteq \{r_1, s_1, \ldots, r_k, s_k\}$, the cut condition (2) and the parity condition (3) imply (1). (4)

the graph $H := (\{r_1, s_1, \ldots, r_k, s_k\}, \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ has at most 4 vertices, or is a 5-circuit (possibly with multiple edges), or contains two vertices v', v'' so that $\{r_i, s_i\} \cap \{v', v''\} \neq \emptyset$ for $i = 1, \ldots, k$. (5)

Condition (5) is equivalent to the graph H not having either of the two graphs depicted in Fig. 1 as a subgraph.

Fig. 1.

Lomonosov's theorem implies that if $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ satisfies (5) and $G = (V, E)$ is a graph with $V \supseteq \{r_1, s_1, \ldots, r_k, s_k\}$, then for any 'capacity' function $c \in \mathbb{Z}_+^E$ and any 'demand' function $d \in \mathbb{Z}_+^k$, the following are equivalent:

there exist paths $P_1^1, \ldots, P_1^{t_1}, P_2^1, \ldots, P_2^{t_2}, \ldots, P_k^{t_k}$ (where each P_i^j connects r_i and s_i, for $i = 1, \ldots, k$, $j = 1, \ldots, t_i$) and rational numbers $\lambda_1^1, \ldots, \lambda_1^{t_1}, \lambda_2^1, \ldots, \lambda_2^{t_2}, \ldots, \lambda_k^{t_k} \geqslant 0$ so that:

$$\text{(i)}\quad \sum_{j=1}^{t_i} \lambda_j^t = d_i \quad (i=1,\ldots,k),$$

$$\text{(ii)}\quad \sum_{\substack{i=1\\ e\in P_i^j}}^{k} \sum_{j=1}^{t_i} \lambda_i^j \leqslant c_e \quad (e\in E). \tag{6}$$

$$\text{for each } W\subseteq V: c(\delta(W)) \geqslant d(\sigma(W)). \tag{7}$$

(Here $c(F) := \sum_{e\in F} c_e$ for $F\subseteq E$ and $d(J) := \sum_{j\in J} d_j$ for $J\subseteq \{1,\ldots,k\}$.) It is not difficult to see that (6) always implies (7). Conversely, Lomonosov's result implies that if (5) and (7) are satisfied, then we can take each λ_i^j equal to $\frac{1}{2}$ in (6) (by replacing each edge e of G by $2c_e$ parallel edges, and each port $\{r_i, s_i\}$ by $2d_i$ parallel ports).

The assertion:

$$\forall c\in \mathbb{Z}_+^E\ \forall d\in \mathbb{Z}_+^k: (6) \Leftrightarrow (7) \tag{8}$$

is equivalent to the following: Let ε_i denote the ith unit basis vector in $\mathbb{R}^k$, χ^P denote the incidence vector of P in $\mathbb{R}^E$, and ε_e denote the eth unit basis vector in $\mathbb{R}^E$. Then the cone $C\subseteq \mathbb{R}^k\times\mathbb{R}^E$ generated by the vectors:

$$\begin{aligned} &(\varepsilon_i; \chi^P) && (i=1,\ldots,k;\ P\ r_i-s_i\text{-path}),\\ &(0;\varepsilon_e) && (e\in E) \end{aligned} \tag{9}$$

is determined by the following system of linear inequalities in the vector variable $(d;c)\in\mathbb{R}^k\times\mathbb{R}^E$:

$$\begin{aligned} &d_i\geqslant 0 && (i=1,\ldots,k),\\ &c_e\geqslant 0 && (e\in E),\\ &c(\delta(W))-d(\sigma(W))\geqslant 0 && (W\subseteq V). \end{aligned} \tag{10}$$

By polarity (interchanging the roles of generators and constraints), this is equivalent to the assertion that the cone generated by the vectors:

$$\begin{aligned} &(-\chi^{\sigma(W)}; \chi^{\delta(W)}) && (W\subseteq V),\\ &(\varepsilon_i; 0) && (i=1,\ldots,k),\\ &(0;\varepsilon_e) && (e\in E), \end{aligned} \tag{11}$$

(again, for $J\subseteq\{1,\ldots,k\}$, χ^J denotes the incidence vector of J in $\mathbb{R}^k$, while for $J\subseteq E$, χ^J denotes the incidence vector of J in $\mathbb{R}^E$) is determined by the following system of linear inequalities in the vector variable $(m;l)\in\mathbb{R}^k\times\mathbb{R}^E$:

$$\begin{aligned} &m_i+\sum_{e\in P} l_e\geqslant 0 && (i=1,\ldots,k;\ P\ r_i-s_i\text{-path}),\\ &l_e\geqslant 0 && (e\in E). \end{aligned} \tag{12}$$

Hence (8) is equivalent to:

for any 'length' function $l: E\to\mathbb{Z}_+$ there exist $W_1,\ldots,W_t\subseteq V$ and

$\mu_1, \ldots, \mu_t \geq 0$ so that:

(i) for each $i = 1, \ldots, k$: the minimum length of any $r_i - s_i$-path is at most $\sum (\mu_j \mid j = 1, \ldots, t;\ i \in \sigma(W_j))$;

(ii) for each $e \in E : l_e \geq \sum (\mu_j \mid j = 1, \ldots, t;\ e \in \delta(W_j))$. (13)

(This can be seen by taking $m_i := -$(minimum length of any $r_i - s_i$-path) in (12).)

Karzanov [4] showed that if (5) holds, then we can take all μ_i equal to $\frac{1}{2}$ in (13). In fact, he showed that (5) is equivalent to:

if $G = (V, E)$ is bipartite and $V \supseteq \{r_1, s_1, \ldots, r_k, s_k\}$, then there exist $W_1, \ldots, W_t \subseteq V$ so that:

(i) for each $i = 1, \ldots, k$: the minimum number of edges in any $r_i - s_i$-path is at most $|\{j = 1, \ldots, t \mid i \in \sigma(W_j)\}|$;

(ii) the cuts $\delta(W_1), \ldots, \delta(W_t)$ are pairwise disjoint. (14)

(13) now follows by replacing each edge e by a path of length $2l_e$. Bipartiteness in (14) is 'dual' to the parity condition (3).

A second stream of research restricts G to planar graphs. First, Okamura and Seymour [11] showed that the cut condition (2) and the parity condition (3) imply (1) if:

G is planar, and all $r_1, s_1, \ldots, r_k, s_k$ are vertices on the boundary of G. (15)

Okamura [10] extended this result by relaxing (15) to:

G is planar, and there exist faces I and O (where we can assume O to be the outer face, without loss of generality), so that for each $i = 1, \ldots, k : r_i, s_i \in I$ or $r_i, s_i \in O$. (16)

Seymour [17] showed that (2) and (3) imply (1) if:

the graph $(V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ is planar. (17)

In Oberwolfach the following extension of the Okamura-Seymour theorem, due to Van Hoesel and Schrijver [2], conjectured by Kurt Mehlhorn, was presented:

Let $G = (V, E)$ be a planar graph. Let O and I be the outer and some other fixed face. Let $C_1, \ldots, C_k$ be curves in $\mathbb{R}^2 \backslash (I \cup O)$, with end points being vertices on $I \cup O$, so that for each vertex v of G the degree of v in G has the same parity as the number of curves C_i beginning or ending in v (counting a curve beginning and ending in v for two). Then there exist pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G so that P_i is homotopic to C_i in the space $\mathbb{R}^2 \backslash (I \cup O)$ for $i = 1, \ldots, k$, if and only if for each path Q in the dual graph of G from I or O to I or O, the number of edges in Q is not smaller than the number of times Q necessarily intersects the curves C_i. (18)

With this last number we mean $\sum_{i=1}^{k}(\min\{|D \cap Q| \mid D$ homotopic to $C_i\})$. Mehlhorn's conjecture was motivated by work on grid graphs (cf. [6]), related to the problem of the automatic design of integrated circuits. It is not difficult to see that (18) implies the Okamura-Seymour theorem.

In this contribution to the Proceedings, we discuss some problems, observations and results related to the above, which were inspired by discussions we had in Oberwolfach.

2. Distance functions in planar graphs

In the same manner as (13) (under condition (5)) follows from Lomonosov's theorem, by considering cones one can derive the following from the Okamura-Seymour theorem: Let $G=(V, E)$ be a planar graph, and let $l: E \rightarrow \mathbb{Z}_+$ be a 'length' function. Then there exist subsets $W_1, \ldots, W_t$ of V and $\mu_1, \ldots, \mu_t \geq 0$ so that:

(i) for each pair v', v'' of vertices on the boundary of G the minimum length of any v'-v''-path is at most $\sum(\mu_j \mid j=1, \ldots, t;\ |\{v', v''\} \cap W_j| = 1)$;

(ii) for each $e \in E: l(e) \geq \sum(\mu_j \mid j=1, \ldots, t;\ e \in \delta(W_j))$. (19)

In fact, we can take each μ_j equal to $\frac{1}{2}$, as follows from the following theorem:

Theorem 1. *Let $G=(V, E)$ be a planar bipartite graph. Then there exist subsets $W_1, \ldots, W_t$ of V so that for each pair v', v'' of vertices on the boundary of G, the minimum number of edges in any v'-v''-path is equal to the number of $j=1, \ldots, t$ with $|\{v', v''\} \cap W_j| = 1$ and so that the cuts $\delta(W_j)$ are pairwise disjoint.*

We show how this theorem can be derived from the Okamura-Seymour theorem. First, let $C=(V, E)$ be a circuit with k vertices and k edges:

$$\begin{aligned} V &= \{v_1, \ldots, v_k\}, \\ E &= \{e_1 = \{v_0, v_1\}, \ldots, e_k = \{v_{k-1}, v_k\}\}, \end{aligned} \tag{20}$$

where $v_0 = v_k$. Let $\binom{V}{2}$ and $\binom{E}{2}$ denote the set of undirected pairs of elements from V and E, respectively. Let M be the $\binom{V}{2} \times \binom{E}{2}$ matrix given by:

$$\begin{aligned} M_{\{v_i, v_j\}, \{e_g, e_h\}} &= 1 \quad \text{if } \{v_i, v_j\} \text{ and } \{e_g, e_h\} \text{ "cross"}; \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{21}$$

where $\{v_i, v_j\}$ and $\{e_g, e_h\}$ are said to cross if v_i and v_j belong to different components of the graph $C \backslash \{e_g, e_h\}$. We show that the matrix M is nonsingular, with $\binom{E}{2} \times \binom{V}{2}$ inverse N given by:

$$\begin{aligned} N_{\{e_g, e_h\}, \{v_i, v_j\}} &= +\tfrac{1}{2} \quad \text{if } \{v_i, v_j\} = \{v_g, v_h\} \quad \text{or} \quad \{v_i, v_j\} = \{v_{g-1}, v_{h-1}\}, \\ &= -\tfrac{1}{2} \quad \text{if } \{v_i, v_j\} = \{v_g, v_{h-1}\} \quad \text{or} \quad \{v_i, v_j\} = \{v_{g-1}, v_h\}, \\ &= 0 \qquad \text{otherwise}. \end{aligned} \tag{22}$$

Proposition. $N = M^{-1}$.

Proof. Choose $\{e_g, e_h\}$, $\{e_a, e_b\} \in \binom{E}{2}$. Then

$$(NM)_{\{e_g,e_h\},\{e_a,e_b\}} = \tfrac{1}{2}M_{\{v_g,v_h\},\{e_a,e_b\}} + \tfrac{1}{2}M_{\{v_{g-1},v_{h-1}\},\{e_a,e_b\}} - \tfrac{1}{2}M_{\{v_g,v_{h-1}\},\{e_a,e_b\}} - \tfrac{1}{2}M_{\{v_{g-1},v_h\},\{e_a,e_b\}}. \tag{23}$$

If $\{g, h\} = \{a, b\}$ then it is easy to see that this last expression is equal to 1. If $\{g, h\} \neq \{a, b\}$, then without loss of generality $g \notin \{a, b\}$. Then

$$M_{\{v_g,v_h\},\{e_a,e_b\}} = M_{\{v_{g-1},v_h\},\{e_a,e_b\}} \quad \text{and}$$
$$M_{\{v_g,v_{h-1}\},\{e_a,e_b\}} = M_{\{v_{g-1},v_{h-1}\},\{e_a,e_b\}}, \tag{24}$$

which implies that (23) is 0. □

[It can be shown that $|\det M| = 2^{\binom{k-1}{2}}$.]

Proof of Theorem 1. Without loss of generality, G is 2-connected. Let $v_1, \ldots, v_k$ be the vertices on the boundary of G in order, and let $e_1 = \{v_0, v_1\}, \ldots, e_k = \{v_{k-1}, v_k\}$ be the edges on the boundary of G (where $v_0 := v_k$). Let M and N be the matrices as above with respect to the circuit $(W := \{v_1, \ldots, v_k\},\ F := \{e_1, \ldots, e_k\})$. Let $m : \binom{W}{2} \to \mathbb{Z}_+$ be defined by: $m(\{v_i, v_j\}) :=$ minimum number of edges in any $v_i - v_j$-path. Let $d := Nm$. Since G is planar and bipartite, Nm is a nonnegative integer vector. In fact, for each $g = 1, \ldots, k$:

$$\sum_{\substack{h=1\\ h\neq g}}^{k} d_{\{e_g,e_h\}} = m_{\{v_{g-1},v_g\}} = 1, \tag{25}$$

as easily follows from the definition of N (or from $Md = m$). Therefore, for each $g \in \{1, \ldots, k\}$ there is a unique $h \neq g$ such that $d_{\{e_g,e_h\}} = 1$, i.e. the collection $\{\{e_g, e_h\} \mid d_{\{e_g,e_h\}} = 1\}$ partitions $\{e_1, \ldots, e_k\}$.

Now let G^* be the (planar) dual graph of G. Put a new vertex w_g on every edge e_g^* of G^* corresponding to edge e_g of G, and next delete the vertex of G^* corresponding to the unbounded face, together with all edges incident with it. Call the graph thus obtained H.

So the collection $\{\{w_g, w_h\} \mid d_{\{e_g,e_h\}} = 1\}$ partitions $\{w_1, \ldots, w_k\}$. Let these pairs be the ports for H. Since each w_g has degree 1 in H, the parity condition (3) is satisfied. Also the cut condition (2) is satisfied. Indeed, let Z be a subset of the vertex set Y of H so that both Z and $Y \setminus Z$ induce a connected subgraph of H. We may assume that there exist g and h so that $w_{g+1}, w_h \in Z$ and $w_g, w_{h+1} \notin Z$. Then

$$|\delta(Z)| \geq m_{\{v_g,v_h\}} = (Md)_{\{v_g,v_h\}} = |\sigma(Z)|. \tag{26}$$

So the cut condition is satisfied.

Hence, by the Okamura-Seymour theorem, there exist pairwise edge-disjoint paths $Q_1, \ldots, Q_{\frac{1}{2}k}$ in H connecting the ports. In G this gives pairwise edge-disjoint cuts $\sigma(W_1), \ldots, \sigma(W_{\frac{1}{2}k})$ so that for any g, h, if $d_{\{e_g, e_h\}} = 1$, then $e_g, e_h \in \delta(W_j)$ for some j. Hence for all i, j:

$$\begin{aligned} m_{\{v_i, v_j\}} = (Md)_{\{v_i, v_j\}} &= \sum_{\{e_g, e_h\} \in \binom{E}{2}} M_{\{v_i, v_j\}, \{e_g, e_h\}} d_{\{e_g, e_h\}} \\ &= |\{f = 1, \ldots, \tfrac{1}{2}k \mid |\{v_i, v_j\} \cap W_f| = 1\}| . \qquad (27) \quad \square \end{aligned}$$

The above reasoning also implies that for any planar bipartite graph G there is a unique partitioning of the edges on the boundary C into pairs $\pi_1, \ldots, \pi_{\frac{1}{2}k}$ of edges so that for any two vertices v', v'' on the boundary of G, the distance from v' to v'' in G is equal to the number of pairs π_j which cross (i.e. separate) v' and v'' on C.

Another application of the above proposition is the following. Let $C = (V, E)$ be a circuit (satisfying (20)). Call a function $m: \binom{V}{2} \to \mathbb{R}_+$ *realizable as a distance function of a planar graph with boundary C*, or briefly *realizable*, if there exists a planar graph $G = (V', E')$, with $V' \supseteq V$, $E' \supseteq E$ and with boundary C, and a length function $l: E \to \mathbb{R}_+$ so that for all $v', v'' \in V$, $m(\{v', v''\})$ is equal to the minimum length of any $v' - v''$-path in G.

Theorem 2. *A function $m: \binom{V}{2} \to \mathbb{R}_+$ is realizable, if and only if for all $i, j = 1, \ldots, k$ we have $m(\{v_i, v_j\}) + m(\{v_{i-1}, v_{j-1}\}) \geq m(\{v_i, v_{j-1}\}) + m(\{v_{i-1}, v_j\})$ (taking $m(\{v_i\}) := m(\{v_j\}) := 0$).*

Proof. Necessity being trivial, we show sufficiency. We construct a graph G as follows. Let $w_1, \ldots, w_k = w_0$ be points on the unit circle (in the cyclic order given). Add all line-segments $\overline{w_g w_h}$ $(g, h = 1, \ldots, k; g \neq h)$. Let W be the set of points which are on two or more of these line-segments. Clearly, the figure now forms a planar graph H, with vertex set W. Let H^* be the dual graph. Put a new point v_i on the edge of H^* corresponding to edge $\overline{w_i w_{i+1}}$ of H $(i = 0, \ldots, k-1)$, delete the vertex of H^* corresponding to the outer face of H, and delete all edges incident to it. Moreover, add edges $e_1 = \{v_0, v_1\}, \ldots, e_k = \{v_{k-1}, v_k\}$ (where $v_k := v_0$). This makes the graph $G = (V', E')$.

The condition in the theorem states that $d := Nm \geq 0$. For each edge e of G define $l(e) := d(\{e_g, e_h\})$ if e corresponds to an edge in H which is on the line-segment $\overline{w_g w_h}$, while $l(e) := \infty$ (or big enough, or $m(\{v_{i-1}, v_i\})$) if $e = e_i = \{v_{i-1}, v_i\}$ for some i.

It is easy to see (using the fact that $Md = m$) that this gives a realization as required. $\square$

3. Two counterexamples

In Okamura's theorem (cf. (16)) we generally cannot accept 'mixed' ports, i.e. ports $\{r_i, s_i\}$ with $r_i \in O$ and $s_i \in I$, as is shown by the following example of

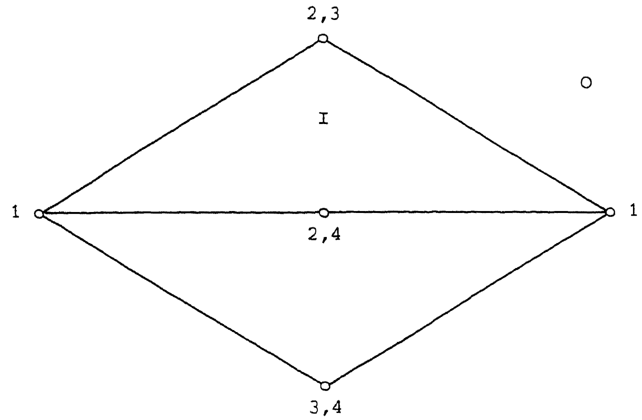

Fig. 2.

Okamura (Fig. 2). In this example (denoting r_i and s_i just by i), the cut condition (2) and the parity condition (3) are satisfied, but there are no paths as required, since each $r_i - s_i$-path has at least two edges, while there are six edges in total.

This last argument shows that there does not even exist a 'fractional' solution, in the sense of (6) (taking $c \equiv 1$, $d \equiv 1$). András Frank asked whether the

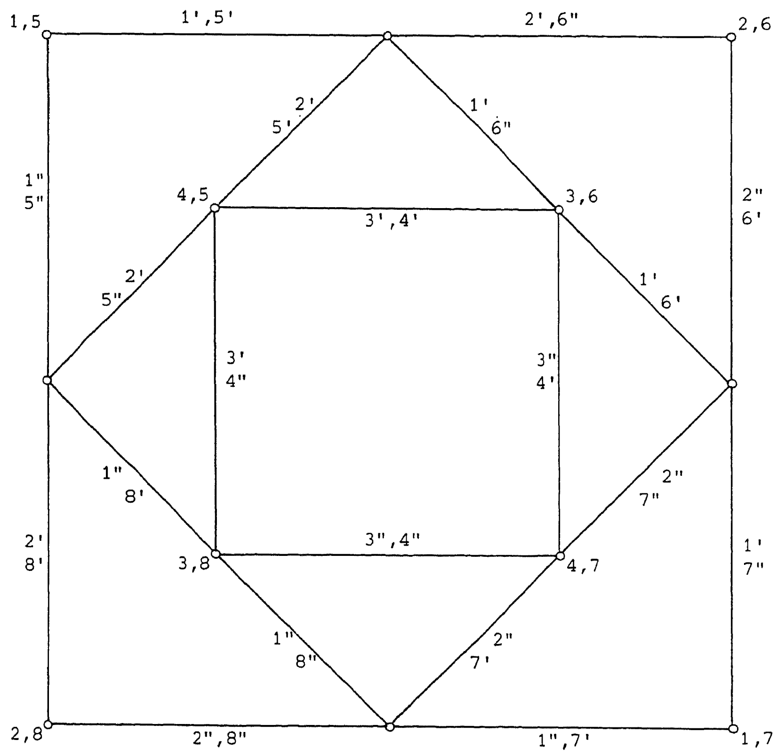

Fig. 3.

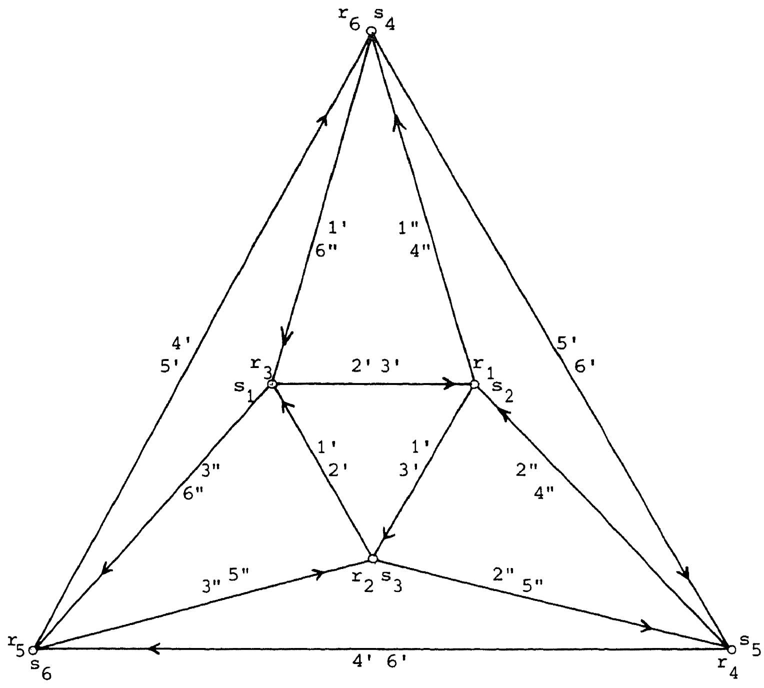

Fig. 4.

existence of such a fractional solution might imply the existence of paths as required. A negative answer is provided by the example in Fig. 3. Note that the parity condition is satisfied. For each $i = 1, \ldots, 8$, the two paths indicated by i' and i'' are $i - i$-paths. Each edge is in exactly two of these paths. So this yields a fractional solution in the sense of (7) (with all λ_i^j equal to $\frac{1}{2}$). However, there is no integer solution, i.e. (1) is not fulfilled. For suppose $P_1, \ldots, P_8$ are pairwise edge-disjoint paths, with P_i connecting r_i and s_i $(i = 1, \ldots, 8)$. Clearly, $|P_i| \geq 4$ for $i = 1, 2$, and $|P_i| \geq 2$ for $i = 3, \ldots, 8$. Moreover, $|P_1| + \cdots + |P_8| \leq 20$, since there are 20 edges. Hence $|P_3| = |P_4| = 2$. But there do not exist edge-disjoint $r_3 - s_3$- and $r_4 - s_4$-paths, both of length 2.

The second example also answers a question of András Frank, concerning a directed analogue of Seymour's theorem (cf. (17)). Consider the directed graph shown in Fig. 4. It is easy to see that there are no pairwise arc-disjoint directed paths $P_1, \ldots, P_6$ so that P_i is an $r_i - s_i$-path $(i = 1, \ldots, 6)$. Note that in each vertex v, indegree $(v) + |\{i \mid s_i = v\}| =$ outdegree $(v) + |\{i \mid r_i = v\}|$ (the analogue of the parity condition). There exists a 'fractional' solution: for $i = 1, \ldots, 6$, the paths indicated by i' and i'' form two $r_i - s_i$-paths, while each arc is in exactly two of these paths (it follows that the directed analogue of the cut condition is satisfied).

4. Some further notes

We mention some questions. Is there a common generalization of the Okamura and the Van Hoesel–Schrijver theorem (cf. (16) and (18))? Or can one be derived from the other? Note that in order to derive the Okamura theorem from (18) it suffices to show that, given the input of the Okamura theorem, one can specify curves connecting r_i and s_i $(i = 1, \ldots, k)$ in $\mathbb{R}^2 \setminus (I \cup O)$ so that the condition mentioned in (18) is satisfied. We do not see a direct way (i.e. one not using the Okamura theorem itself) to derive this.

In [13] Theorem 1 is extended to the case where we also allow that both v' and v'' belong to some other fixed face I. This corresponds to the Okamura theorem, in the same way as Theorem 1 corresponds to the Okamura–Seymour theorem. Karzanov [5] observed that a similar result with respect to Seymour's theorem (cf. (17)) can be derived from Seymour's results on 'sums of circuits' [15].

The Van Hoesel–Schrijver theorem (18) cannot be extended in the obvious way to the case where there are more 'holes', as is shown by the example in Fig. 5.

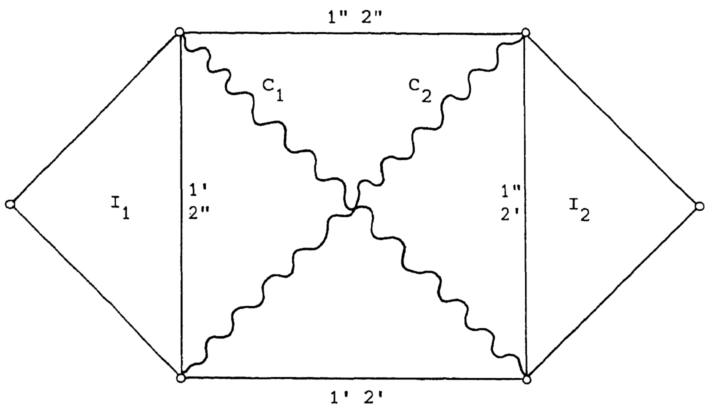

Fig. 5.

Here the "dual curve condition" given in (18) is satisfied, but there are no edge-disjoint paths P_1 and P_2, where P_i is homotopic to C_i in the space $\mathbb{R}^2$ $(O \cup I_1 \cup I_2)$. However, there is a 'fractional' solution, by taking each of the paths $1', 1'', 2', 2''$ with multiplicity $\frac{1}{2}$. In Oberwolfach, Professor Crispin Nash-Williams asked whether the dual curve condition implies the existence of a fractional solution (in any planar graph with any number of holes). This question can be answered affirmatively, as will be shown in a forthcoming paper [14].

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