Polyhedra and Algorithms

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We give an introduction to polyhedral combinatorics and its relations to polynomial-time algorithms. This paper is based on a lecture given at the *Wintersymposium "Mathematics and Economics"* of the *Wiskundig Genootschap,* November 1984 in Tilburg. No special expertise of discrete mathematics, complexity theory or operations research is assumed.

1. Prologue: The assignment problem

The assignment problem is a special case of the "transportation problem" introduced by Kantorovich [1939] and Koopmans [1948]:

$$\textit{Assignment problem}: \text{Given an } n\times n - \text{matrix } C=(c_{ij})\,, \tag{1}$$
$$\text{find a permutation } \pi \text{ of } \{1,\cdots,n\} \text{ such that } \sum_{i=1}^{n} c_{i\pi_i} \text{ is minimal}\,.$$

This problem shows up in several situations, like when one has to assign jobs to workers or machines, rooms to guests, etc. Since there exist $n!$ permutations, solving (1) by enumerating all permutations is not recommendable.

In fact, Kantorovich and Koopmans studied the following *linear programming problem:*

$$\text{minimize } \sum_{i,j=1}^{n} c_{ij}x_{ij} \tag{2}$$

$$\left.\begin{array}{lll} \text{subject to: } (i) & x_{ij}\geqslant 0 & (i,j=1,\cdots,n),\\ (ii) & \sum_{j=1}^{n} x_{ij}=1 & (i=1,\cdots,n),\\ (iii) & \sum_{i=1}^{n} x_{ij}=1 & (j=1,\cdots,n). \end{array}\right\} \tag{3}$$

The equivalence of (1) and (2) follows from the following theorem of Birkhoff [1946]. An $n\times n$-matrix $A=(a_{ij})$ is called *doubly stochastic* if A is nonnegative, and each row sum and each column sum is equal to 1. So A is doubly stochastic if and only if A satisfies the constraints (3) (taking $x_{ij}=a_{ij}$). A *permutation matrix* is a 0,1-matrix having in each row and in each column exactly one 1.

THEOREM 1. *A matrix is doubly stochastic if and only if it is a convex combination of permutation matrices.*

PROOF. Since each permutation matrix is doubly stochastic, the 'if' part of the theorem is clear.

The 'only if' part is proved by induction on the order n. Let P be the polytope (in $\mathbb{R}^{n\times n}$) of doubly stochastic matrices. Let Q be the convex hull of all permutation matrices (so $Q\subseteq\mathbb{R}^{n\times n}$). Having to show $P\subseteq Q$, assume $P\not\subseteq Q$. Then P has a vertex $X^*=(x^*_{ij})$ which is not contained in Q. Since X^* is a vertex of P, X^* satisfies n^2 linearly independent constraints among (3) with equality. As the $2n$ constraints (ii) and (iii) in (3) are dependent, we know that at least $n^2-(2n-1)$ constraints among (3) (i) are satisfied by X^* with equality.

So X^* has at least n^2-2n+1 zeros. Hence X^* has a row with $n-1$ zeros, and 1 one. Without loss of generality, $x^*_{11}=1$. Hence $x^*_{i1}=0\ (i\neq 1)$ and $x^*_{1j}=0\ (j\neq 1)$. So the submatrix B of X^* obtained by deleting the first row and the first column of X^*, is doubly stochastic. By our induction hypothesis, B is a convex combination of permutation matrices. Therefore, as $x^*_{11}=1$, X^* itself is a convex combination of permutation matrices. So X^* belongs to Q, contradicting our assumption. □

The relation between problems (1) and (2) will now be clear: We may assume that (2) achieves its minimum value in a vertex of the polytope defined by (3), which is, by Theorem 1, a permutation matrix. The corresponding permutation is an optimum solution for the assignment problem (1).

As a consequence, DANTZIG's simplex method [1951a, 1951b] for linear programming contains as a special case an algorithm for the assignment problem (note that the simplex method always gives an optimum *vertex* solution).

VON NEUMANN [1953] raised the question of solving the assignment problem *in polynomial time*. More precisely, if C is integral, can we solve problem (1) in time bounded by

$$p\left(\sum_{i,j=1}^{n}\log(|c_{ij}|+1)\right) \tag{4}$$

for some fixed polynomial p? The sum in (4) is about the space needed to state the input of the assignment problem in binary notation (the logarithms have base 2). A polynomial-time algorithm may mean a considerable reduction of computing time over checking all $n!$ permutations.

Von Neumann's problem was solved by KUHN [1955], giving a polynomial-time algorithm for the assignment problem. This method is based on earlier work of EGERVÁRY [1931] and is therefore called the *Hungarian method.*

Only recently one was able to prove that also (a variant of) the simplex method gives a polynomial-time method for the assignment problem (BERTSEKAS [1981], BALINSKI [1985]).

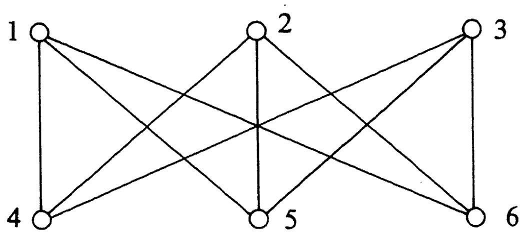
2. A MORE COMBINATORIAL SETTING: THE PERFECT MATCHING POLYTOPE OF A BIPARTITE GRAPH

This section contains just a reformulation of parts of Section 1 in terms of *graphs*. Let, for $n \in \mathbb{N}$,

$$V_{2n} := \{1, \cdots, 2n\}, \tag{5}$$
$$E_{n,n} := \{\{i,j\} \mid i=1, \cdots, n; j=n+1, \cdots, 2n\}.$$

The pair $(V_{2n}, E_{n,n})$ is called a *complete bipartite graph*, and is denoted by $K_{n,n}$. The elements of V_{2n} and of $E_{n,n}$ are called the *vertices* and *edges*, respectively, of the graph.

One may represent the vertices by points, or small circles, in the plane, and the edges by line segments connecting the two points which they contain. So $K_{3,3}$ may be represented by:



(6)

A subset M of $E_{n,n}$ is called a *perfect matching* if each vertex is contained in exactly one edge in M. Consequently, $|M|=n$. E.g., $\{\{1,4\}, \{2,6\}, \{3,5\}\}$ is a perfect matching in $K_{3,3}$.

One easily sees that the assignment problem (1) is equivalent to:

$$\text{given } c: E_{n,n} \rightarrow \mathbb{Q}, \text{ find a perfect matching } M \text{ minimizing } \textstyle\sum_{e \in M} c_e . \tag{7}$$

Now let for any subset Y of a set X, the *incidence vector* χ^Y be the vector (or function) in $\mathbb{R}^X$ satisfying $\chi^Y(x)=1$ if $x \in Y$ and $=0$ otherwise. Define $\mathfrak{M}_n$ to be the collection of incidence vectors of perfect matching in $K_{n,n}$ (so $\mathfrak{M}_n \subseteq \mathbb{R}^{E_{n,n}} \cong \mathbb{R}^{n \times n}$). Then the assignment problem (1) becomes:

$$\min \{c^T x \mid x \in \mathfrak{M}_n\}. \tag{8}$$

This is clearly equivalent to

$$\min \{c^T x \mid x \in \text{conv.hull}\,(\mathfrak{M}_n)\}. \tag{9}$$

The set conv. hull $(\mathfrak{M}_n)$ is called the *perfect matching polytope* of $K_{n,n}$. In terms of the perfect matching polytope, Birkhoff's theorem (Theorem 1) says:

COROLLARY 1a. *The perfect matching polytope of $K_{n,n}$ is equal to the set of vectors x in $\mathbb{R}^{E_{n,n}}$ satisfying:*

$$\begin{aligned} & x_e \geq 0 && (e \in E), \\ & \sum_{\substack{e \in E_{n,n} \\ v \in e}} x_e = 1 && (v \in V). \end{aligned} \tag{10}$$

PROOF. Directly from Theorem 1. □

Equivalently, (8) has the same optimum value as

$$\min \{c^T x | x \text{ satisfies (10)}\} , \tag{11}$$

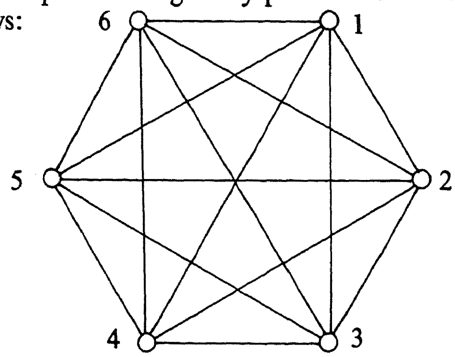
thus describing again the assignment problem as a linear programming problem.

3. THE "NONBIPARTITE" CASE

We now go into a more difficult situation. Define, for $n \in \mathbb{N}$,

$$V_n := \{1, \cdots, n\} , \tag{12}$$
$$E_n := \{\{i,j\} | i,j, \cdots, n ; i \neq j\} .$$

The pair (V_n, E_n) is called a *complete graph*, and denoted by K_n. Again the elements of V_n and E_n are called the *vertices* and *edges*, respectively, of the graph, and may be represented again by points and line segments in the plane. So K_6 is as follows:



(13)

Again, a subset M of E_n is called a *perfect matching* if each vertex is in exactly one edge in M. So $|M| = \frac{1}{2}n$. E.g., $\{\{1,2\}, \{3,5\}, \{4,6\}\}$ is a perfect matching in K_6. For even n, the graph K_n contains $(n-1)(n-3)(n-5)...3.1$ different perfect matchings.

The *nonbipartite assignment problem* now is:

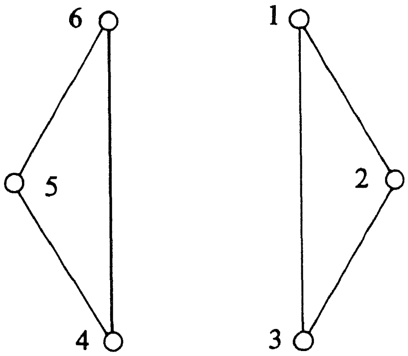
$$\text{given } c \in \mathbf{Q}^{E_n}, \text{ find a perfect matching } M \text{ in } K_n \text{ minimizing } \textstyle\sum_{e \in M} c_e . \tag{14}$$

This problem occurs when we want to split a set optimally into pairs, e.g. aircraft crews or room mates. It also shows up as subproblem in certain methods for routing problems, e.g. the travelling salesman problem and the Chinese postman problem (cf. Section 7).

In order to solve (14), we can try to formulate it as a linear programming problem. Let the *perfect matching polytope* P of K_n be the convex hull of the incidence vectors of all perfect matchings in K_n. So $P \subseteq \mathbb{R}^{E_n}$. Then the nonbiassignment problem is equivalent to:

$$\min\{c^T x | x \text{ is the incidence of a perfect matching in } K_n\} = \min\{c^T x | x \in P\}. \qquad (15)$$

an describe P as the solution set of a system of linear inequalities, we (14) as an LP-problem. Note that the system: $x_e \geqslant 0 (e \in E_n)$, $= 1 (v \in V_n)$ is not enough, as is shown for K_6 by taking $x_e = \frac{1}{2}$ for dge e drawn in



(16)

$= 0$ for all other edges. Then x satisfies the constraints mentioned, but x n the perfect matching polytope.

stem fully describing the perfect matching polytope was given in a paper of EDMONDS [1965]:

M 2. *The perfect matching polytope of $K_n = (V,E)$ is equal to the set of x in $\mathbf{R}^E$ satisfying:*

$$\begin{array}{lll} (i)\ x_e \geqslant 0 & (e \in E), & \\ (ii)\ \sum_{e \ni v} x_e = 1 & (v \in V), & (17) \\ (iii)\ \sum_{e \in \delta(W)} x_e \geqslant 1 & (W \subseteq V, |W| \text{ odd}). & \end{array}$$

(W) denotes the set $\{\{i,j\} \in E | i \in W,\ j \notin W\}$.)

The proof extends the proof of Theorem 1. Let Q be the perfect ng polytope of K_n and let P be the solution set of (17). We have to nat $P = Q$. Since each incidence vector of a perfect matching clearly (17), we have $Q \subseteq P$. The converse inclusion is shown by induction on ly, we may assume that n is even: for n odd both P and Q are empty npty as inequality (iii) in (17) is unsatisfiable for $W = V$).

ose $P \not\subseteq Q$, and let $x^* = (x_e^* | e \in E)$ be a vertex of P not contained in Q. s a vertex, there exist $\binom{n}{2}$ linearly independent constraints in (17) which sfied by x^* with equality. We consider two cases.

No constraint in (17) (iii), with $3 \leqslant |W| \leqslant n-3$, is satisfied by x^* with . Note that if $|W| = 1$ or if $|W| = n-1$, constraint (iii) follows from (ii).

So it follows that there are $\binom{n}{2}$ linearly independent constraints among (i) and (ii) satisfied by x^* with equality. at least $\binom{n}{2}-n$ constraints in (i) are satisfied with equality. Let

$$F := \{e \in E | x_e^* > 0\} . \tag{18}$$

So $|F| \leq n$. Note that by (ii), $V = \cup F$. Case 1 now splits into two cases.

CASE 1A. There exists a vertex, say n, which is contained in exactly one edge in F, say in $\{n-1,n\} \in F$. Then $x^*_{\{n-1,n\}} = 1$, and hence by (ii), $x^*_{\{i,n-1\}} = x^*_{\{i,n\}} = 0$ for $i = 1, \cdots, n-2$. Then $\tilde{x} := (x_e^* | e \in E_{n-2})$ satisfies (17) for the case $n-1$, as one easily checks. Hence, by our induction hypothesis, $\tilde{x}$ is a convex combination of incidence vectors of perfect matchings in K_{n-2}. Therefore, x^* itself belongs to Q, contradicting our assumption.

CASE 1B. Each vertex is contained in at least two edges in F. Since $|F| \leq n$, it follows that each vertex is in exactly two edges in F. So we may assume that F contains $\{1,2\},\{2,3\},\{3,4\}, \ldots, \{k-1,k\},\{k,1\}$ for some $k = 1, \cdots, n$ (possibly after renaming $1, \cdots, n$). Then k is even (otherwise (17) (iii) is violated for $W := \{1, \cdots, k\}$). However, by resetting:

$$\begin{aligned} x_e^* &:= x_e^* + \epsilon && \text{for } e = \{1,2\},\{3,4\},\{5,6\}, \cdots, \{k-1,k\} ; \\ &:= x_e^* - \epsilon && \text{for } e = \{2,3\},\{4,5\},\{6,7\}, \cdots, \{k-2,k-1\},\{k,1\} ; \\ &:= x_e^* && \text{for all other edges ,} \end{aligned} \tag{19}$$

for each ϵ near enough to 0, gives again a solution of (17). This contradicts the fact that x^* is a vertex of P.

CASE 2. There exists a constraint in (17) (iii) which is satisfied by x^* with equality, with $3 \leq |W| \leq n-3$. Without loss of generality, $W = \{1, \cdots, t\}$, with t odd. Then define $y = (y_e | e \in E_{t+1})$ as follows:

$$\begin{aligned} y_e &:= x_e^* && \text{if } e \subseteq \{1, \cdots, t\} \\ y_e &:= \Sigma_{j=t+1}^{n} x^*_{\{i,j\}} && \text{if } e = \{i,t+1\} \text{ for some } i \in \{1, \cdots, t\}. \end{aligned} \tag{20}$$

It follows that y satisfies (17) for the case K_{t+1}. (Note that $\Sigma_{e \ni t+1} y_e = 1$, as x^* satisfies (17) (iii) for W with equality.) So by our induction hypothesis, y is a convex combination of incidence vectors of perfect matchings in K_{t+1}, say

$$y = \sum_{M \text{ perfect matching in } K_{t+1}} \lambda_M \chi^M , \tag{21}$$

with $\lambda_M \geq 0$ and $\Sigma \lambda_M = 1$.

Similarly, define $z = (z_e | e \in E, e \subseteq \{t, \cdots, n\})$ as follows:

$$\begin{aligned} z_e &:= x_e^* && \text{if } e \subseteq \{t+1. \cdots, n\}, \\ z_e &:= \sum_{i=1}^{t} x^*_{\{i,j\}} && \text{if } e = \{t,j\} \text{ for some } j \in \{t+1, \cdots, n\}. \end{aligned} \tag{22}$$

Then z satisfies (17) for the case K_{n-t+1} (taking $V = \{t, \cdots, n\}$). Again, by induction, z is a convex combination of incidence vectors of perfect matchings on $\{t, \cdots, n\}$, say

$$z = \sum_{M \text{ perfect matching on } \{t, \cdots, n\}} \mu_M \chi^M , \tag{23}$$

with $\mu_M \geqslant 0$ and $\Sigma \mu_M = 1$.

Now let for each perfect matching M on $\{1, \cdots, n\}$ with $|M \cap \delta(W)| = 1$, say $M \cap \delta(W) = \{i', j'\}$, with $1 \leqslant i' \leqslant t, t+1 \leqslant j' \leqslant n$, the perfect matchings M' and M'' on $\{1, \cdots, t+1\}$ and on $\{t, \cdots, n\}$, respectively, be defined as follows:

$$M' := \{\{i,j\} \in M \mid \{i,j\} \subseteq \{1, \cdots, t\}\} \cup \{\{i', t+1\}\} , \tag{24}$$
$$M'' := \{\{i,j\} \in M \mid \{i,j\} \subseteq \{t+1, \cdots, n\}\} \cup \{\{t, j'\}\} .$$

One easily checks that

$$x^* = \sum_{e \in \delta(W)} \sum_{\substack{M \text{ perfect matching} \\ M \cap \delta(W) = \{e\}}} \frac{\lambda_{M'} \mu_{M''}}{x_e^*} \chi^M \tag{25}$$

and

$$\sum_{e \in \delta(W)} \sum_{\substack{M \text{ perfect matching} \\ M \cap \delta(W) = \{e\}}} \frac{\lambda_{M'} \mu_{M''}}{x_e^*} = 1 . \tag{26}$$

Thus we have written x^* as a convex combination of incidence vectors of perfect matchings on $\{1, \cdots, n\}$, contradicting our assumption. □

In fact, Edmonds obtained this theorem as a by-product of a polynomial-time algorithm for the nonbipartite assignment problem. Note that, by the theorem, the nonbipartite assignment problem can be formulated as a linear programming problem. One should however be careful in applying LP-methods too directly: there are exponentially many constraints in (17), so writing down the program explicitly cannot be done in polynomial time. We shall see in the following section that this problem can be overcome.

4. The Ellipsoid method

In the previous sections we studied optimization problems of the form:

$$\min\{c^T x \mid x \in S\} , \tag{27}$$

where S is a finite set of vectors in $\mathbf{Q}^n$, and where $c \in \mathbf{Q}^n$. In fact, each set S consisted of 0,1-vectors.

Since (27) is equal to

$$\min\{c^T x \mid x \in \text{conv.hull}(S)\} , \tag{28}$$

we characterized conv. hull (S) by means of linear inequalities. We now shall see that this approach is, at least implicitly, unavoidable: roughly speaking, problem (27) is solvable in polynomial time, if and only if we can describe conv. hull (S) appropriately by linear inequalities. This is a result following with the *ellipsoid method*.

KHACHIYAN [1979] showed that the ellipsoid method solves linear programming in polynomial time, thus solving a long-standing open problem (the simplex method, though fast in practice, has exponential running time in the worst case).

To apply the ellipsoid method in combinatorial optimization, let us define a *graph property* G to be a sequence $(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots)$ where

$$\mathfrak{F}_n \subseteq \mathfrak{P}(E_n) \tag{29}$$

for $n = 1,2,3, \cdots$. Recall that $E_n = \{\{i,j\} \mid i,j \in \{1, \cdots, n\}, i \neq j\}$. So each $\mathfrak{F}_n$ is a collection of subsets of E_n. Examples of graph properties are obtained by taking for $n = 1,2,3, \cdots$:

$$\mathfrak{F}_n := \{M \mid M \text{ is a perfect matching in } K_n\} \tag{30}$$

(so $\mathfrak{F}_n = \varnothing$ if n is odd);

$$\mathfrak{F}_n := \{T \mid T \text{ is a spanning tree in } K_n\} \tag{31}$$

(T is called a *spanning tree* in K_n if for all $i,j = 1, \cdots, n$ there exist $i_1, \cdots, i_t$ so that $\{i,i_1\}, \{i_1,i_2\}, \cdots, \{i_{t-1},i_t\}, \{i_t,j\} \in T$ (a *path*) and if T contains no 'circuit' $\{i_1,i_2\}, \cdots, \{i_{t-1},i_t\}, \{i_t,i_1\}$ with $t \geqslant 3$ and $i_1, \cdots, i_t$ all distinct);

$$\mathfrak{F}_n := \{H \mid H \text{ is a Hamiltonian circuit in } K_n\} \tag{32}$$

(H is called a *Hamiltonian circuit* in K_n if we can permute $1, \cdots, n$ to $i_1, \cdots, i_n$ so that $H = \{i_1,i_2\}, \cdots, \{i_{n-1},i_n\}, \{i_n,i_1\}\}$).

The following now is a special case of a theorem of GRÖTSCHEL, LOVÁSZ and SCHRIJVER [1981, 1987]:

THEOREM 3. *Let $G = (\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots)$ be a graph property. Then the following are equivalent:*

(i) *there exists a polynomial-time algorithm for the following ("optimization") problem: Given $n \in \mathbb{N}$ and $c \in \mathbf{Q}^{E_n}$, find $F \in \mathfrak{F}_n$ minimizing $\sum_{e \in F} c_e$;*

(ii) *there exists a polynomial-time algorithm for the following ("separation") problem: Given $n \in \mathbb{N}$ and $x \in \mathbf{Q}^{E_n}$, decide if x belongs to* conv. hull *$\{\chi^F \mid F \in \mathfrak{F}_n\}$, and if not, find a separating hyperplane.*

The statements (i) and (ii) are in a sense dual to each other: (i) has as input a vector c in dual space and output x in 'primal' space, while (ii) has input x in primal space and output (in the 'if not' case) in dual space.

The proof of Theorem 3 uses the ellipsoid method combined with simultaneous diophantine approximation based on Lenstra, Lenstra and Lovász's basis reduction method. A complete proof would require several tedious details. We restrict ourselves here to giving a very rough sketch.

Suppose we have a polynomial-time algorithm for the problem in (ii) (a "separation" algorithm), and we have input $n \in \mathbb{N}$ and $c \in \mathbb{Q}^{E_n}$ for the optimization problem. We construct ellipsoids $L_0, L_1, L_2, \cdots$ in $\mathbb{R}^{E_n}$, each satisfying

$$L_i \supseteq \text{conv.hull } \{\chi^F \mid F \in \mathcal{F}_n,\ F \text{ attains } \min\{\sum_{e \in F} c_e \mid F \in \mathcal{F}_n\}\}. \tag{33}$$

L_0 is the ball around the origin of radius n^2. If L_i has been constructed, let it have center z_i. Check with our polynomial-time separation algorithm if z_i belongs to $P := \text{conv. hull } \{\chi^F \mid F \in \mathcal{F}_n\}$.

CASE 1: z_i belongs to P. Let L_{i+1} be the ellipsoid of smallest volume satisfying

$$L_{i+1} \supseteq L_i \cap \{x \mid c^T x \leq c^T z_i\}. \tag{34}$$

CASE 2: z_i does not belong to P, and the separation algorithm gives us a vector $a \in \mathbb{Q}^{E_n}$ so that $a^T x < a^T z_i$ for all x in P. Let L_{i+1} be the ellipsoid of smallest volume satisfying

$$L_{i+1} \supseteq L_i \cap \{x \mid a^T x \leq a^T z_i\}. \tag{35}$$

(It is not difficult to derive the parameters determining L_{i+1} from those determining L_i and from c and a, respectively.)

One easily checks, by induction, that (33) holds for each i. Since vol (L_i) can be shown to decrease with exponential convergence speed, we may hope that the z_i converge quickly to χ^F, where F is an optimum solution for the optimization problem. Indeed, with the help of simultaneous diophantine approximation the algorithm can be adapted to produce such an optimum solution.

The converse implication $(i) \Rightarrow (ii)$ is shown similarly by considering dual space.

Since Edmonds showed that the nonbipartite assignment problem is solvable in polynomial time, we know that (i) of Theorem 3 holds for the graph property defined in (30). So by (ii) of Theorem 3, the system (17) can be tested in polynomial time, although there are exponentially many constraints.

Actually, PADBERG and RAO [1982] gave a direct polynomial-time method to test if a given vector satisfies (17). So, using the converse implication in Theorem 3, this gives an alternative proof of the polynomial-time solvability of the nonbipartite assignment problem. (In fact, it is a simpler way of showing the polynomial-time solvability of the nonbipartite assignment problem, but it yields a less practical algorithm.)

Similarly, for the graph property defined in (31) (spanning trees), we know that (i) of Theorem 3 holds. That is, there is a polynomial-time algorithm for the following *minimum spanning tree problem:* given $n \in \mathbb{N}$ and $c \in \mathbb{Q}^{E_n}$ find a spanning tree T of minimum 'length' $\Sigma_{e \in T} c_e$. The so-called *greedy algorithm,* designed by BORŮVKA [1926], solves this problem in polynomial time. Theorem 3 then implies that also the convex hull of the incidence vectors of spanning tress can be characterized appropriately, viz. in the sense of (ii) of Theorem 3.

For several other graph properties the convex hull of the incidence vectors is characterized, which yields by Theorem 3 a polynomial-time algorithm for the corresponding optimization problem.

We can also use Theorem 3 in the negative. It is generally believed that the so-called *traveling salesman problem:*

$$\text{given } n \in \mathbb{N} \text{ and } c \in \mathbb{Q}^{E_n}\text{, find a Hamiltonian circuit } H \text{ minimizing } \sum_{e \in H} c_e\,, \tag{36}$$

cannot be solved in polynomial time (cf. Section 6). If this belief is justified, Theorem 3 then would imply that also the convex hull of the incidence vectors of Hamiltonian circuits cannot be characterized, in the sense of (ii) of Theorem 3.

5. Cutting planes

One may conclude from the above that it is important to have a method to find, for any given set S of integral vectors in, say, $\mathbb{R}^n$, a system $Mx \leqslant d$ of linear inequalities so that

$$\text{conv.hull}(S) = \{x \mid Mx \leqslant d\}\,. \tag{37}$$

Usually, it is not difficult to represent S as

$$S = \{x \mid Ax \leqslant b;\ x \text{ integral}\}\,, \tag{38}$$

for some rational matrix A and rational column vector b, E.g., the set of incidence vectors of perfect matchings in K_n is equal to the set of integral vectors x in $\mathbb{R}^{E_n}$ satisfying (i) and (ii) in (17). As an intermezzo, we now describe a general procedure to derive from any rational matrix A and column vector b, a rational matrix M and a column vector d satisfying

$$\text{conv.hull}\ \{x \mid Ax \leqslant b;\ x \text{ integral}\} = \{x \mid Mx \leqslant d\}\,. \tag{39}$$

This procedure is based on Gomory's *cutting plane method* for integer linear programming (cf. Chvátal [1973], Schrijver [1980]).

Define for each polyhedron $P \subseteq \mathbb{R}^n$, the set P_I by

$$P_I := \text{conv.hull}\ (P \cap \mathbb{Z}^n)\,. \tag{40}$$

Moreover, let

$$P' := \bigcap_{H \supseteq P} H_I \tag{41}$$

where the intersection ranges over all 'affine half-spaces' $H = \{x \mid c^T x \leqslant \delta\}$, where c is a rational vector, and δ is so that $P \subseteq H$. We may assume that the components of c are relatively prime integers; this implies that $H_I = \{x \mid c^T x \leqslant \lfloor \delta \rfloor\}$, where $\lfloor\ \rfloor$ denotes lower integer part. (The set $\{x \mid c^T x = \lfloor \delta \rfloor\}$ is called a *cutting plane.)* So

$$P' = \{x \mid c^T x \leqslant \lfloor \delta \rfloor \text{ for each integral vector } c \text{ such that } c^T z \leqslant \delta \text{ for each } z \text{ in } P\}\,. \tag{42}$$

Since $P \subseteq H$ implies $P_I \subseteq H_I$, it follows that $P_I \subseteq P'$.

If P is a rational polyhedron (i.e., defined by rational linear inequalities), then P' is a rational polyhedron again. Applying the operation to P' we obtain $P''(=(P')')$, which can be different from P'. As an example, if $P \subseteq \mathbb{R}^2$ with $P=\{(x,y)^T \mid y \geqslant 0; \quad 3x+y \leqslant 3; -3x+y \leqslant 1\}$, then $P'=\{(x,y)^T \mid x \geqslant 0; y \geqslant 0; 2x+y \leqslant 2; -x+y \leqslant 1\}$ and $P''=\{(x,y)^T \mid x \geqslant 0; y \geqslant 0; x+y \leqslant 1\}=P_I$.

In general, we have

$$P \supseteq P' \supseteq P'' \supseteq P''' \cdots \supseteq P_I . \tag{43}$$

Denoting the $(t+1)$-th polyhedron in this sequence by $P^{(t)}$, the following was shown by CHVÁTAL [1973] and SCHRIJVER [1980]:

THEOREM 4. *For each rational polyhedron P there exists a number t such that $P^{(t)}=P_I$.*

This theorem yields a procedure for finding all linear inequalities defining P_I (see SCHRIJVER [1986] for details). It is not difficult to derive from Theorem 2 that if P is defined by (17) (i) and (ii), then $P'=P_I$. However, in general, the number t with $P^{(t)}=P_I$ can be arbitrary large (even in dimension 2). The procedure does not yield a polynomial-time algorithm for determining P_I.

6. NP AND P

We shall now discuss on which the 'general belief' that the traveling salesman problem is not solvable in polynomial time, mentioned in Section 4, is based. It is a result of the study named 'complexity theory', centered around the notions of P and NP and the question $P=NP$?. We shall not go into defining P and NP here, but restrict ourselves to making a statement equivalent to $P=NP$.

A graph property $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots)$ is called *polynomially recognizable* if the following problem is solvable in polynomial time:

$$\text{given } n \in \mathbb{N} \text{ and } F \subseteq E_n, \text{ decide if } F \text{ belongs to } \mathcal{F}_n . \tag{44}$$

Note that each of the examples (30), (31) and (32) gives a polynomially recognizable graph property. Call a graph property $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots)$ *polynomially optimizable* if the following ("optimization") problem is solvable in polynomial time:

$$\text{given } n \in \mathbb{N} \text{ and } c \in \mathbb{Q}^{E_n}, \text{ find } F \in \mathcal{F}_n \text{ minimizing } \textstyle\sum_{e \in F} c_e . \tag{45}$$

This is (i) in Theorem 3, which is equivalent to (ii) in Theorem 3.

It is not difficult to see that each polynomially optimizable graph property is polynomially recognizable (by taking c appropriately: to check if F belongs to $\mathcal{F}_n$, solve (45) with $c_e=-1$ if $e \in F$ and $c_e=+1$ if $e \notin F$). However, the reverse implication is a big open problem, equivalent to the question $P=NP$?. Indeed, this question is characterized by:

$$P=NP \Leftrightarrow \text{each polynomially recognizable graph property is polynomially optimizable} . \tag{46}$$

There seems to be no reason to assume that each polynomially recognizable graph property is also polynomially optimizable, i.e., that $P = NP$. However, no counterexample has been found as yet. It is a general belief that there exist polynomially recognizable graph properties which are not polynomially optimizable, i.e., that $P \neq NP$.

KARP [1972], extending work of COOK [1971]. showed that if there exist a polynomially recognizable graph property which is not polynomially optimizable, then the graph property defined in (32) (the Hamiltonian circuits) is such a one.

THEOREM 5. *If graph property (32) is polynomially optimizable, then each polynomially recognizable graph property is polynomially optimizable.*

This is why the traveling salesman problem is called *NP – complete*. Several other important basic combinatorial optimization problems were shown by COOK [1971] and KARP [1972] to be *NP*-complete.

Theorem 5 is equivalent to:

COROLLARY 5a. *If $P \neq NP$, then the traveling salesman problem is not solvable in polynomial time.*

By Theorem 3 this implies:

COROLLARY 5b. *If $P \neq NP$, then the separation problem for graph property* (32) *is not solvable in polynomial time.*

(The separation problem is defined in Theorem 3.) It means that if $P \neq NP$, the convex hull of the incidence vectors of Hamiltonian circuits is difficult.

The recent successes of CROWDER and PADBERG [1980], GRÖTSCHEL [1980] and PADBERG and HONG [1980] in solving large-scale instances of the traveling salesman problem with 'branch and bound' techniques, are based on approximating the convex hull of the incidence vectors of Hamiltonian circuits. To this end, one first observes that a vector $x \in \mathbf{Q}^{E_n}$ is equal to χ^H for some Hamiltonian circuit, if and only if x satisfies:

$$\begin{aligned} &(i) \quad \sum_{e \ni i} x_e = 2 \quad (i = 1, \cdots, n), \\ &(ii) \quad \sum_{e \in \delta(W)} x_e \geq 2 \quad (W \subset \{1, \cdots, n\},\ W \neq \emptyset), \\ &(iii) \quad 0 \leq x_e \leq 1 \quad (e \in E_n), \\ &(iv) \quad x_e \text{ integer} \quad (e \in E_n). \end{aligned} \tag{47}$$

Let P be the polytope defined by (i), (ii) and (iii). So P_I is the convex hull of incidence vectors of Hamiltonian circuits. It can be obtained by applying the cutting plane procedure of Section 5. This appears to be time-consuming, if we assume $P \neq NP$. However,

$$\min\{c^T x \mid x \in P\} \tag{48}$$

is a lower bound for the minimum length of a Hamiltonian circuit. Having good lower bounds is essential in applying branch-and-bound techniques: the term 'good' here means relatively easy to compute and relatively near to the optimum value (one has to find a compromise between these two clashing goals).

It can be shown that the lower bound (48) can be computed in polynomial time. It amounts to solving a linear program whose constraints can be checked in polynomial time (although there are exponentially many constraints in (47) (iii)). Indeed, (47) (iii) can be checked by considering x as a capacity function, and by finding a cut of minimum capacity. If this cut has capacity at least 2, (47) (iii) is fulfilled, and otherwise not. The ellipsoid method then tells us that (48) can be computed in polynomial time.

The ellipsoid method is not fast in practice, and other methods are applied to determine (48) in practice. Replacing P by P' in (48) gives a better lower bound for the traveling salesman problem. If we are able to recognize quickly (part of) the cutting planes making up P', we obtain a good algorithm for determining this better lower bound. With such a combination of branch-and-bound and cutting plane techniques, Padberg recently was able to solve a traveling salesman problem with 2392 'cities'.

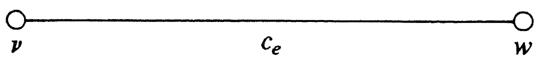
7. Perfect matchings and the traveling salesman problem

In this final section we go into some relations between perfect matching problems and routing problems. Perfect matching problems frequently show up in algorithms for routing problems. Here we restrict ourselves to giving two examples. (A well-known other example is the polynomial-time algorithm for the 'Chinese postman problem'.)

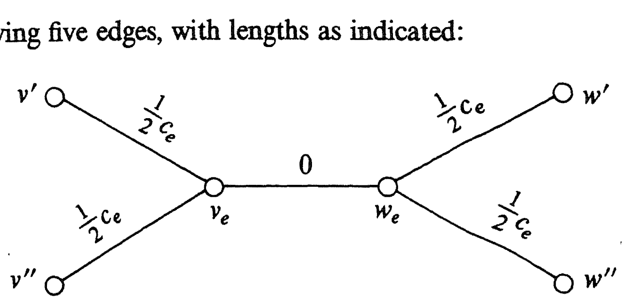
The traveling salesman problem is equivalent to the problem of minimizing $c^T x$ over $x \in \mathbb{R}^{E_n}$ satisfying (47) (i) - (iv), for any given 'length' function $c: E_n \to \mathbb{Q}$. In the previous section we observed that having good lower bounds for the traveling salesman problem is essential in branch-and-bound procedures, and we discussed the lower bound obtained by deleting (iv) in (47). An alternative lower bound is obtained by not deleting (iv), but instead deleting (ii):

$$\begin{aligned} &\text{minimize } \sum_{e \in E_n} c_e x_e && && (49)\\ &\text{subject to } \sum_{e \ni i} x_e = 2 && (i = 1, \cdots, n), \\ &\qquad\qquad x_e \in \{0,1\} && (e \in E_n). \end{aligned}$$

This lower bound again can be computed in polynomial time. Note that if we would have $=1$ instead of $=2$ in (49), it would be the problem of finding a minimum weighted perfect matching. But also problem (49) can be reduced to the perfect matching problem. In graph theoretic terms this reduction amounts to splitting each vertex v into two vertices v' and v'' and replacing each edge $e = \{v, w\}$, with length c_e:



by the following five edges, with lengths as indicated:



where v_e and w_e are two new vertices. So, starting with a graph with n vertices, we obtain a graph with $2n+2\binom{n}{2}$ vertices. To those edges not occurring in (51) we assign a length of ∞ (or some very large number). Now the problem of finding a perfect matching of minimum length in the large graph, is equivalent to solving (49), as one easily checks. It follows that the lower bound (49) can be determined in polynomial time.

Another occurrence of perfect matchings is in the 'heuristic' of CHRISTOFIDES [1976] for the traveling salesman problem, if the length function satisfies the triangle inequality:

$$c(\{i,k\}) \leqslant c(\{i,j\}) + c(\{j,k\}) \qquad i,j,k=1,\cdots,n\,. \tag{52}$$

This heuristic is a polynomial-time algorithm, not yielding (generally) a shortest Hamiltonian circuit, but one of length at most $\frac{3}{2}$ times the length of a shortest Hamiltonian circuit. So the relative error with respect to the optimum traveling salesman tour is at most $\frac{1}{2}$. (The value $\frac{1}{2}$ is the lowest relative error for which a polynomial-time algorithm is known.)

Christofides' algorithm works as follows. First find a spanning tree T of minimum length. This can be done in polynomial time with the greedy algorithm, as we mentioned in Section 4. Without loss of generality, let $\{1,\cdots,k\}$ be those vertices which are contained in an odd number of edges in T, and let $\{k+1,\cdots,n\}$ be those vertices contained in an even number of edges in T. Note that k is even. Next determine a perfect matching M of minimum length on $\{1,\cdots,k\}$ (i.e. in E_k). Now $T\cup M$ forms a connected graph on $\{1,\cdots,n\}$, in which each vertex is contained in an even number of edges in $T\cup M$ (we here count edges occurring both in T and in M for two). Then by Euler's theorem, $T\cup M$ forms a circuit $C=(\{v_0,v_1\},\{v_1,v_2\},\cdots,\{v_{m-1},v_m\})$ (with $v_0=v_m$) so that each edge in $T\cup M$ occurs exactly once in this circuit. In particular, each $i=1,\cdots,n$ occurs at least once among $v_1,\cdots,v_m$. Now in the sequence $v_1,\cdots,v_m$ we can delete each v_i for which there is a $j<i$ with $v_j=v_i$. We are left with a permutation $w_1,\cdots,w_n$ of $1,\cdots,n$, so that

$$\sum_{i=1}^{m} c(\{v_{i-1},v_i\}) \geqslant \sum_{i=1}^{n} c(\{w_{i-1},w_i\}) \tag{53}$$

(where $w_0:=w_n$), by the triangle inequality (52). The algorithm gives as output the Hamiltonian circuit $C=(\{w_0,w_1\},\{w_1,w_2\}, \cdots, \{w_{n-1},w_n\})$.

Let l^* be the length of a shortest Hamiltonian circuit C^*. We show that C^* has length at most $3l^*/2$. First, T has length less than l^*, since leaving out one edge from C^* gives a spanning tree, while T is a shortest spanning tree.

Next, M has length at most $\frac{1}{2}l^*$. Indeed, writing $C^*=\{\{u_0,u_1\}, \{u_1,u_2\}, \cdots, \{u_{n-1},u_n\}\}$ (with $u_0=u_n$), there are $i_1<i_2< \cdots <i_k$ so that $\{u_{i_1}, \cdots ,u_{i_k}\} = \{1, \cdots ,k\}$. Then

$$\text{length}\,(M) \leq \sum_{j=1}^{\frac{1}{2}k} c(\{u_{i_{2j-1}},u_{i_{2j}}\}), \text{ and} \tag{54}$$

$$\text{length}\,(M) \leq \sum_{j=1}^{\frac{1}{2}k} c(\{u_{i_{2j-2}},u_{i_{2j-1}}\})$$

(where $u_{i_0}:=u_{i_k}$), because M is a perfect matching on $\{1, \cdots ,k\}$ of minimum length. So

$$2\cdot\text{length}(M) \leq \sum_{j=1}^{k} c(\{u_{i_{j-1}},u_{i_j}\}) \leq \sum_{j=1}^{n} c(\{u_{j-1},u_j\}) = l^* . \tag{55}$$

The second inequality here follows from the triangle inequality (52). Concluding:

$$\text{length}(C) = \sum_{i=1}^{n} c(\{w_{i-1},w_i\}) \leq \sum_{i=1}^{m} c(\{v_{i-1},v_i\}) = \tag{56}$$

$$\text{length}(T) + \text{length}(M) \leq l^* + \frac{1}{2}l^* = \frac{3}{2}l^* .$$

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