SOME ASPECTS OF APPLIED ANALYSIS: ASYMPTOTICS, SPECIAL FUNCTIONS AND THEIR NUMERICAL COMPUTATION

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CONTENTS

INTRODUCTION

This dissertation consists of eight papers on aspects of asymptotics, special functions and their numerical computation. Six of the papers have appeared in scientific journals, the other two have been submitted for publication. The papers are

[1] *Analytical methods for a singular perturbation problem in a sector,* SIAM J. Math. Anal., 5, pp. 876-877, 1974.

[2] *Numerical evaluation of functions arising from transformations of formal series,* J. Math. Anal. Appl., 51, pp. 678-694, 1975.

[3] *Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function,* Math. Comp., 29, pp. 1109-1114, 1975.

[4] *On the numerical evaluation of the modified Bessel function of the third kind,* J. Comput. Phys., 19, pp. 324-337, 1975.

[5] *On the numerical evaluation of the ordinary Bessel function of the second kind,* J. Comput. Phys., 21, pp. 343-350, 1976.

[6] *Remarks on a paper of A. Erdélyi,* SIAM J. Math. Anal., 7, pp. 767-770, 1976.

[7] *Uniform asymptotic expansions of confluent hypergeometric functions,* report TW 153, prepublication, Mathematical Centre, Amsterdam, 1975, to appear in J. Inst. Maths Applics.

[8] *The asymptotic expansion of the incomplete gamma functions,* report TW 165, prepublication, Mathematical Centre, Amsterdam, 1977, to appear in SIAM J. Math. Anal.

In the Introduction and the Summaries these papers are referred to by using the above enumeration. Reference to the literature at the end of the Summaries is made by mentioning the author and the year of publication.

In the Summaries we give a short description of the papers, with an indication of their relationship, firstly however, we make some general remarks on our work.

These eight papers fall apart in two classes: papers mainly on asymptotics ([1], [3], [6], [7], [8]) and papers mainly on computation ([2], [4], [5]). In all papers, however, aspects of uniformity play a prominent part. For many special functions, asymptotic expansions with respect to one of their parameters were derived long ago, but these expansions may become invalid if other parameters approach critical values.

Let us illustrate this phenomenon by the well-known asymptotic expansion of the incomplete gamma function, a function that is considered in [3] and [8]. It is given by:

$$(1) \qquad \Gamma(a,z) = \int_z^\infty e^{-t} t^{a-1} dt,$$

where for the time being we suppose $a \in \mathbb{R}$, $z > 0$. By successive partial integration we obtain:

$$(2) \qquad \Gamma(a,z) = z^{a-1} e^{-z} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \ldots + \frac{(a-1)\ldots(a-n)}{z^n} + R_n(a,z)\right], \qquad n = 0,1,2,\ldots,$$

where for $n > a-2$ the remainder satisfies

$$|R_n(a,z)| \leq \frac{|(a-1)\ldots(a-n-1)|}{z^{n+1}} .$$

This follows from elementary standard methods in asymptotic analysis. It can be concluded that for large values of z, $n > a-2$, the remainder $R_n(a,z)$ becomes as small as we please, uniformly in a if a is restricted to any compact subset of the real line such that $a < n+2$. However if a depends on z such that $c_1|z| \leq |a|$, $c_1 > 0$, $z \to \infty$, the remainder in (2) is not small enough to furnish a good approximation for $\Gamma(a,z)$.

This phenomenon frequently occurs in asymptotic problems. Expansion (2) is a good example of a non-uniform expansion: for $z \to \infty$ it is uniformly valid for a in compact intervals, but if we allow a to range through unbounded sets the uniform character of the expansion will disappear. (In order to be more specific, we use the concept of uniform asymptotic expansion as defined by OLVER (1974, p.26)).

If one pays attention to the uniform character of asymptotic expansions, one is able to formulate important and difficult problems in special function theory. In this respect, this part of classical analysis is full of life. OLVER (1975) gives an excellent survey of developments in this field and it appears that in the last two decades many new results on uniform expansions have been established. For some important classes of special functions, however, research has hardly begun. Olver mentions, for instance, the lack of results for a commonly used well-known class of functions: the Gauss hypergeometric functions ${}_2F_1(a,b;c;z)$.

Formerly, asymptotic expansions were given in terms of elementary transcendental functions, such as the exponential, logarithmic, circular and gamma functions, but, as mentioned earlier, if extra parameters were present in the coefficients of the expansions (as frequently occurs in special functions or physical problems) the asymptotic expressions were without uniformity with respect to these parameters. Nowadays, in order to obtain uniform expansions, one uses other functions, such as error functions, Airy functions, Bessel functions, incomplete gamma functions, or parabolic cylinder functions. Of course, when expanding in terms of these basic functions we need all information on their qualitative behaviour and on methods for their numerical evaluation.

It is very curious that of the functions listed above, the incomplete gamma functions are the most neglected ones, in spite of their importance in many areas of applied mathematics, including mathematical statistics. TRICOMI (1950) starts his paper (in which important progress is made for these functions) with the remark *Seit einiger Zeit pflege ich die unvollständige Gammafunktion* $\gamma(\alpha,x) = \int_0^x e^{-t} t^{\alpha-1} dt$ *das Aschenbrödel der Funktionen zu nennen, weil es mir scheint das sie trotz ihres unbestreitbaren Intresses bis jetzt zu wenig beachtet worden ist.* With the results of this paper and [3] and [8] the Cinderella days are definitely over.

When expansions, which give a high degree of approximation in the analytical sense, are derived for special functions, the problems for the numerical analyst are not yet solved. He needs these analytic expansions, since the usual techniques of function approximation (such as Chebyshev expansions or best approximation in the Chebyshev sense) are not attractive if more (complex) parameters are involved. However, expanding the functions with respect to one variable, leaves the coefficients as functions of the remaining variables. This creates problems of effective computation, satisfactory rate of convergence, etc. Examples in point are our uniform expansions of the incomplete gamma function (of which numerical aspects are discussed in [8]) and of the expansion (based on Taylor series) for the Bessel function $K_\nu(z)$, which is treated in [4]. In both cases the coefficients of the expansion appear in indeterminate form if one of the parameters equals some critical value. Analytically, there is no problem, for the limit in question is well defined, but numerically these situations ask for special attention. In the Bessel function case this problem has been solved satisfactorily, and the method is implemented in ALGOL 60 programs. In the incomplete gamma case indications for the numerical implementation are given, but a computer program based on these results has not, as yet, been constructed.

From the point of view of this thesis the work with special functions has a low degree of abstraction and generality. In this sense, it is work on a small scale, something which is not always appreciated in modern mathematics. I should like to comment on it, as far as the study of special functions is concerned.

The main point is that useful and manageable analytic expansions and efficient and reliable algorithms cannot be constructed for large classes of functions. In this field it is necessary to be interested in individual functions. We call them by name, we know their specific character and we derive results that are applicable to one and only one function. This kind of interest in individual functions was rather common in the past. In Euler's days (1707 - 1783) a function was more or less synonymous with a formula. Such a formula was called an *expressio analytica,* an analytic expression: e.g. an integral, a polynomial, an infinite series, etc. The important functions were called by name and for each function a formula was available. These functions are the building blocks of mathematical analysis and in the 18th and 19th century hundreds of special functions were singled out for research. In the 20th century this branch of analysis was also cultivated, culminating in the monographs of WHITTAKER & WATSON (1927, first edition 1902) and WATSON (1944, first edition 1922), two imposing works of the English school.

For generalizations in special functions we are in Dutch company. In 1936 C.S. Meijer introduced the generalization of the ${}_pF_q$-functions, which was called after him the Meijer G-function. After its introduction there was a stream of publications by Meijer on this function and on functions that can be expressed as G-functions. It seemed to inaugurate a new era for western mathematicians specializing on functions, but in fact, it did not. Instead, with, (and I do not say due to,) the introduction of the Meijer G-function, interest in special functions fell into the background. This was not so in all parts of the world, however. Influenced by the British, some workers in analysis were attracted to special functions, although they had not always inherited discipline in handling them. The G-function started a real chaos of uninteresting formulas and results, which pollute sometimes the fashionable mathematical journals. The wrong problems were formulated. Special functions are interesting as long as they are considered with reference to their origin: applied mathematics, especially mathematical physics, or with reference to an interesting mathematical theory, such as, for instance, group theory. If they are studied *l'art pour l'art,* they become suspicious.

The modern analyst, with a background in function analysis has no interest in individual functions. He studies large classes in which the functions are hardly distinguishable, and certainly not recognizable; he doesn't know their names and their *expressio analytica*, but he knows their common properties. In this respect, he may feel as a sociologist, unlike his fellow analyst from special functions, who may consider himself as a biographer.

The work of Askey, Cody, Gautschi, Koornwinder and Olver demonstrates that today special functions can be studied in the right way. In all modesty, I hope that my work will be considered in the same manner.

SUMMARIES

Some of the papers are reviewed in the *Mathematical Reviews* (MR) or the *Zentralblatt für Mathematik* (Zbl.). If this is the case we give relevant volume numbers and review numbers.

[1] *Analytical methods for a singular perturbation problem in a sector,* MR 51, #6105; Zbl. 295, #35005.

The work in this paper is connected with singular perturbation problems in a quarter plane or a finite rectangular domain as treated in ECKHAUS & DE JAGER (1966), GRASMAN (1974[a]) and (1974[b]), COMSTOCK (1968), KNOWLES & MESSICK (1964), TEMME (1971) and DIEKMANN (1975). The solution of an elliptic partial differential equation (in which a small parameter ε multiplies the second order derivative) is considered asymptotically for $\varepsilon \to 0$. In this paper we treat a model problem by solving the equation explicitly and by using asymptotic methods for integrals. With the results of this paper new insight is gained on the genesis of the parabolic boundary layer. This layer arises when the sector (with sharp angle) becomes a quarter plane. When the angle becomes obtuse, the boundary layer changes its character again: it becomes an internal free layer. Also hidden free layers which become important in limiting cases can be distinguished. In the event of an almost right angle, two small asymptotic parameters are to be considered: ε and $\pi/2 - \alpha$, where α is the sector angle. COMSTOCK (1968), GRASMAN (1974[a]) and DIEKMANN (1975) also paid attention to the case of almost characteristic boundaries.

We mention three aspects that might be considered for further research.

1. In the representation (4.8) and (4.9) of the solution, finite series with exponential terms are present. For $\varepsilon \to 0$ these contributions are exponentially small (with respect to other terms in the expansion), but (4.8) and (4.9) represent the solution for all positive ε (or $\omega = 1/\varepsilon$). There must be an explanation of the role of these contributions in terms of wave fronts for a diffraction or reflection problem.

2. The asymptotic representation of the solution is not valid at the origin,

i.e., the corner of the sector. It might be interesting to construct (from the integral representation) an asymptotic expansion that is uniformly valid in parts of the sector that contain the corner.

3. If an analogous boundary value problem is considered for the interior of a circle, say $x^2 + y^2 \le 1$, then the solution can be solved explicitly in terms of a Fourier series. To be more specific, we formulate the problem with its solution. Let Φ be the solution of the partial differential equation (we write $x = r\cos\theta$, $y = r\sin\theta$)

$$(3)\qquad \varepsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x,y) - \frac{\partial\Phi(x,y)}{\partial y} = 0, \quad r < 1,$$

$$\Phi(\cos\theta, \sin\theta) = \sin\theta, \quad 0 \le \theta \le 2\pi.$$

Then

$$(4)\qquad \Phi(x,y) = -e^{\omega r\sin\theta} \sum_{n=-\infty}^{\infty} \frac{I_n'(\omega)}{I_n(\omega)} I_n(\omega r)\, e^{in(\theta+\pi/2)}, \quad \omega = 1/(2\varepsilon).$$

$I_\nu(z)$ is a modified Bessel function. The expansion of Φ (for a more general problem) is considered by GRASMAN (1971), cf. also VAN HARTEN (1976). For the model problem (3) an expansion might be derived by using (4). We expect that new insight may thus be gained in the peculiar behaviour of Φ in the neighbourhoods of the points $(x,y) = (-1,0)$ and $(x,y) = (1,0)$. Our earlier investigations did not reach the stage of publication, but we point out that it is an interesting, difficult problem to derive from (4) an asymptotic expansion of Φ for $\varepsilon \to 0$ that is uniformly valid in neighbourhoods of the above mentioned points.

[2] *Numerical evaluation of functions arising from transformations of formal series*. MR 53, #9593, Zbl. 263, #65002.

Let us consider the Laplace integral

$$(5)\qquad f(z) = z\int_0^\infty e^{-zt} F(t)\, dt,$$

where F is such that (5) is meaningful as a Riemann integral and let F be analytic at $t = 0$. Then F can be expanded in convergent series

$$(6) \qquad F(t) = \sum_{k=0}^{\infty} a_k t^k, \qquad F(t) = \sum_{k=0}^{\infty} b_k t^k/(1+t)^{k+1},$$

valid for certain t-values, and we can assign to f the formal expansions

$$(7) \qquad f(z) \sim \sum_{k=0}^{\infty} a_k k! z^{-k}, \qquad f(z) \sim \sum_{k=0}^{\infty} b_k s_k(z),$$

with

$$s_k(z) = z \int_0^{\infty} e^{-zt}\, t^k/(1+t)^{k+1}\, dt.$$

The relation between a_k and b_k is rather simple (see for instance PÓLYA & SZEGŐ (1964, I problem 224)). Under certain conditions on F the first series of (7) gives an asymptotic expansion of f for $z \to \infty$ and it may happen (see VAN WIJNGAARDEN (1964)) that the second series of (7) gives a convergent expansion of f. The second series of (7) can be considered as a transformation of the first one. LAUWERIER (1975) gives a new interpretation of this transformation. His conclusion is (formally speaking) that this transformation is the Laplace transform of the Euler transformation for series.

Our paper discusses methods for numerical evaluation of the functions s_k. For this purpose new asymptotic expansions are derived, which are uniformly valid in given domains of the complex z-plane. For small $|z|$ the computation heavily depends on the asymptotic expansion of s_k in terms of Bessel functions.

Some aspects of the paper may be reconsidered. We mention three points.

1. The choice of the parameter ν in the computation of a starting value for a backward recursion process may be based on a method introduced by Olver (1967).

2. The asymptotic expansions of $s_k(z)$ for $k \to \infty$ and bounded $|z|$ may be compared with expansions introduced by SKOVGARD (1966) in which z is allowed to range through unbounded regions.

3. From a numerical point of view it is important to have numerical bounds for the remainders in the asymptotic expansions mentioned in point 2.

[3] *Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function.* MR 52, #8513; Zbl. 313, #33002.

The integrand of the integral defining the gamma function

$$\gamma(a,x) = \int_0^x e^{-t}\, t^{a-1}\, dt, \quad x \geq 0, \quad a > 0,$$

has its maximum at $t_0 = a-1$ (if $a > 1$). The asymptotic behaviour of γ as $a \to \infty$ is completely different for the three cases $t_0 < x$, $t_0 = x$, $t_0 > x$. It is well-known that the function $P(a,x) = \gamma(a,x)/\Gamma(a)$, which is connected with statistical distribution functions, follows the behaviour predicted by the central limit theorem. Therefore, it is clear that an error function (i.e., a normal distribution function) describes the limit behaviour. The error functions (and related functions) are indeed used by TRICOMI (1950), DINGLE (1973) and in [3].

Compared with Tricomi's and Dingle's results our expansion gives the following improvements:

- the expansion (for $a \to \infty$) is uniformly valid in $x \geq 0$;
- just one term containing an error function is needed for the description of the asymptotic behaviour;
- the expansions for $P(a,x)$ and $Q(a,x) = 1 - P(a,x)$ have the same simple complementary relation as P and Q themselves.

In this paper we also give an expansion for the incomplete beta function

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\, dt, \quad 0 \leq x \leq 1,\ p > 0,\ q > 0,$$

as $q \to \infty$. The expansion is of the same type as that for the incomplete gamma function, but the range of uniformity is not as large as is desirable. (For instance, it is not valid for $x \to 0$). I have strong indications that expansions will be found, which gives a wider range of uniformity, but these will involve a function that is more complicated than the error function.

A simpler case is considered in Stelling 1 of the thesis, where q is fixed, $p \to \infty$ and $x \in [0,1]$ (uniform), but uniformity with respect to q and x in, say, $q \geq \delta > 0$, $x \in [0,1]$ is still an open problem.

[4] *On the numerical evaluation of the modified Bessel function of the third kind*. MR 53, #4488; Zbl. 334, #65013.

Originally the research for this paper was motivated by (5). A useful approach for obtaining asymptotic expansions for series is to transform the series in an integral (Watson/Sommerfeld transformation). The integral may then be evaluated by the method of residues. In the present problem it was important to compute the zeros of the function $I_\nu(\omega)$, considered as a function of the complex variable ν and fixed, large positive ω. These zeros appear in the half-plane Re $\nu < 0$, especially in the neighbourhood of non-negative integers. This follows easily from the relation

$$(8) \qquad I_{-\nu}(z) = I_\nu(z) + \frac{2}{\pi} \sin \nu\pi \; K_\nu(z)$$

and from drawing graphs of the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ for positive ν (see ABRAMOWITZ & STEGUN (1964, Fig. 9.9)). Since $I_\nu(z)$ is readily available (see GAUTSCHI (1964)), the main problem was to compute $K_\nu(z)$, especially around $\nu = n$ (positive integer), but the recurrence relation for K_ν reduces this problem to the computation of K_ν in the ν-domain $0 \leq \text{Re}\ \nu \leq 1$.

For small $|z|$ we use (8), with Taylor series for $I_{-\nu}$ and I_ν. However, for $\nu = n$ (integer) these functions are identical and this phenomenon raises non-trivial problems in the computation. These problems are solved by judicious combination of appropriate terms in the Taylor series and by using recurrence relations for remaining terms.

For large $|z|$ the function $K_\nu(z)$ is computed by considering K_ν as a special confluent hypergeometric function and by then using a recurrence relation with respect to one of the parameters of this more general function class.

[5] *On the numerical evaluation of the ordinary Bessel function of the second kind*. MR 54, #1548.

The Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ can be expressed in linear combinations of $K_\nu(iz)$ and $K_\nu(-iz)$. With the results of the previous paper an algorithm is constructed for $Y_\nu(z)$. In fact, the algorithm for $K_\nu(z)$ applies for the whole family of confluent hypergeometric functions that are singular in the origin. Notation: $U(a,b,z)$, cf. ABRAMOWITZ & STEGUN (1964). This implies that the algorithm can also serve for the computation of $U(L+1-i\eta,\ 2L+2,\ zi\rho)$, which is related to the irregular solution $G_L(\eta,\rho)$ of the Coulomb wave equation.

In the papers [4] and [5] we give ALGOL 60 procedures for the computation of $K_\nu(z)$ and $Y_\nu(z)$ for real values of the parameters. For K_ν this is the only implemented algorithm available in the literature; very recently CODY, MOTLEY & FULLERTON (1977) published their Fortran version of the computation of Y_ν. Since the rise of program libraries, it is no longer the case that all available implemented algorithms are published in the scientific journals. Some of these libraries indeed contain contributions on K_ν and Y_ν. For a review of the availability of software for special functions the reader is referred to Van der Laan's contributions in VAN DER LAAN & TEMME (1977).

The construction of sound algorithms for the computation of special functions and the construction of software based on these algorithms, are separate jobs. With the presentation of the ALGOL 60 procudures in [4] and [5] we do not claim that the implementation is perfect. We consider them as algorithmic translations of our analytic work. From reactions of workers in physics, chemistry and other branches of science, we perceived that this kind of service is greatly appreciated, but the final word rests with the software engineer.

[6] *Remarks on a paper of Erdélyi*. Zbl. 335, #41017.

This paper gives the asymptotic expansion of

$$F(z,a) = \int_a^\infty e^{-z(t-a)}\, t^{\lambda-1}\, g(t)\, dt$$

for $z \to \infty$; we suppose $a \geq 0$, $z > 0$, $0 < \lambda < 1$ and for g the existence of the first n derivatives for $t \geq 0$. The emphasis lies on the role of the parameter a; it may depend on z. An expansion is given for $z \to \infty$, which is uniformly valid in $a \geq 0$; λ is considered as a fixed parameter.

For fixed a the asymptotic problem is rather simple. The usual technique is to expand $t^{\lambda-1}g(t)$ in a power series in $t = a$ followed by integrating term by term. For $a \to 0$, however, the singularity of $t^{\lambda-1}$ (at $t = 0$) coincides with the end point of integration. This causes non-uniform behaviour in the integral.

For $g \equiv 1$, $F(z,a)$ is an incomplete gamma function, viz. $F(z,a) = z^{-\lambda} e^{az} \Gamma(\lambda, az)$. This quantity plays an important role in the expansion of the general problem.

In Stelling 1 of the thesis a non-trivial application is mentioned on incomplete beta functions.

In the preceding remarks we supposed that λ is fixed. In the paper the author observed that the asymptotic expansion is uniformly valid with respect to λ in compact subsets of the λ-interval $(0,1]$. I conjecture that the domain of uniformity can be extended to $\lambda \geq \delta$, where δ is a positive constant.

[7] *Uniform asymptotic expansions of confluent hypergeometric functions.*

The second order differential equation

$$z \frac{d^2w}{dz^2} + (b-z) \frac{dw}{dz} - aw = 0$$

is called Kummer's equation and has as solutions the Kummer functions. In ${}_pF_q$-language they are called the confluent hypergeometric functions, for they can be considered as limiting cases of the more familiar Gauss hypergeometric functions. The name Whittaker functions is also used.

Asymptotic methods can be used directly on the differential equation. Those familiar with singular perturbation problems for ordinary differential equations recognize this equation in connection with turning point problems. If b is large and z is of the same size, the middle term seems to vanish. Actually, dw/dz becomes large at the same moment, as if to compensate for the relative smallness of $(b-z)$. The result is that the solutions of the equation change suddenly from finite values to other finite values if z passes the turning point b. This effect, a "shock layer effect", can also be noticed in the case of incomplete gamma functions, or, more simply and basically, in the case of the error functions. The incomplete gamma functions are special cases of the solutions of the above equation.

In the paper we do not use the differential equation for asymptotics. We present some integral representations that generalize integrals for the incomplete gamma functions, as treated in [3]. The expansions are in terms of generalizations of the error functions: the parabolic cylinder functions.

We consider b as a large parameter and we discuss uniformity with respect to z, especially near z = b. The parameter a is fixed. It would be interesting to find out if our expansions are uniformly valid in the neighbourhood of a = 0, -1, -2,... As follows from our results and methods, these points are rather exceptional. The differential equation is in these cases solvable in terms of well-known orthogonal polynomials of the Laquerre type. Asymptotic methods for differential equations reveal resonance situations if a crosses non-positive integer values. See DE GROEN (1976). In our method the parabolic cylinder functions used as approximants degenerate into elementary functions in these cases.

[8] *The asymptotic expansion of the incomplete gamma function.*

Suppose f is analytic in a domain containing the real axis, such that

$$F(x,a) = \int_{-\infty}^{x} e^{-at^2} f(t)\, dt$$

exists for $x \in \mathbb{R}$; $a > 0$. Many statistical distribution functions can be brought into this form. We look for the asymptotic expansion of F for $a \to \infty$, which is uniformly valid with respect to x, especially near x = 0. For large a, $dF(x,a)/dx$ is very small, except for x-values near x = 0. This gives the shock effect as mentioned in the description of [7]. If $f \equiv 1$ we have simply

$$A := \int_{-\infty}^{x} e^{-at^2}\, dt = \tfrac{1}{2}(\pi/a)^{\frac{1}{2}} \operatorname{erfc}(-xa^{\frac{1}{2}}).$$

In the general case, we expect the main contribution from t = 0 (if x is close to or greater than 0) and we write

$$F(x,a) = Af(0) + \int_{-\infty}^{x} e^{-at^2}\, t\, \frac{f(t) - f(0)}{t}\, dt$$

and by partial integration we obtain

$$F(x,a) = Af(0) - \frac{1}{2a}\frac{f(x) - f(0)}{x} e^{-ax^2} + \frac{1}{2a}\int_{-\infty}^{x} e^{-at^2} f_1(t)\, dt,$$

with

$$f_1(t) = \frac{d}{dt}\left[\frac{f(t) - f(0)}{t}\right].$$

And so the process can be continued.

This method takes into account the contributions at t = 0 and t = x. It is not likely that it gives clear information on the domain of uniformity nor does it give a manageable expression for the remainder for rigorous error analysis.

In the paper we combine this method with our previous results on the incomplete gamma functions. The earlier expansion is rather intricate and the method of [3] did not give insight in the remainder. The combination however, does give this insight, together with strict error bounds. It also gives recurrence relations for the coefficients, which eases their numerical evaluation. In addition, it gives extension to complex variables. In mentioning our results OLVER (1975) stated that *Extensions to complex variables and the construction of error bounds are still needed*. With this paper the author considers this subject closed.

Although we extended the domain of the parameters to complex values, we did not reach the negative axis for the parameter a. It is still worthwhile to consider this a-domain and to derive expansions that are ready for computation. In TRICOMI (1950) one may find a first attempt in this direction.

The software specialists are invited to base computer programs on the results of [8]. In my opinion the computation of the incomplete gamma functions for large variables must be based on the asymptotic expansions given in the underlying paper.

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ANALYTICAL METHODS FOR A SINGULAR PERTURBATION PROBLEM IN A SECTOR*

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Abstract. From the exact solution of an elliptic boundary value problem in a sector, asymptotic approximations with respect to a small parameter are derived. The asymptotic expansion is uniformly valid in the boundary layers. Also the phenomena for the case of almost characteristic boundaries are discussed.

1. Introduction. In [4], the author considered a singular perturbation problem for an elliptic equation in a quarter-plane. The exact solution of the equation was represented as a contour integral and from this representation the asymptotic solution was derived by using saddle point methods.

In this paper we consider the same equation

$$\varepsilon\Delta\Phi_\varepsilon(x, y) - \frac{\partial}{\partial y}\Phi_\varepsilon(x, y) = 0, \tag{1.1}$$

the domain of definition now being an arbitrary sector shaped domain in the x, y-plane

$$A = \{r, \phi | r \geqq 0, 0 \leqq \phi \leqq \alpha\}. \tag{1.2}$$

In (1.1) ε is a small positive parameter and Δ is Laplace's operator; in (1.2) r and ϕ are polar coordinates, where

$$x = r\cos\phi, \quad y = r\sin\phi \tag{1.3}$$

and

$$0 < \alpha \leqq 2\pi. \tag{1.4}$$

The case $\alpha = \frac{1}{2}\pi$ (the quarter-plane) is discussed in [4].

Along the boundary of the sector A, the function Φ_ε is subjected to the following boundary conditions:

$$\Phi_\varepsilon(x, 0) = 0, \quad \Phi_\varepsilon(x, y) = 1 \quad \text{if } \phi = \alpha. \tag{1.5}$$

In order to investigate the asymptotic behavior for small values of ε, the exact solution of (1.1) is determined from which the asymptotic approximations are derived. Also the various types of boundary layers are discussed, for instance the "free" (i.e., internal) boundary layer in the case of an obtuse angle α. Finally, the case of an almost right angle will be considered.

2. The solution of the boundary value problem. We shall first remove the first order derivative in equation (1.1) by substituting

$$\Phi_\varepsilon(x, y) = 1 - e^{\omega y}F(x, y), \qquad \omega = 1/(2\varepsilon). \tag{2.1}$$

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Then the function F has to satisfy the following boundary value problem:

$$\begin{aligned} &\Delta F(x, y) - \omega^2 F(x, y) = 0, \\ &F(x,0) = 1, \quad F(x, y) = 0 \quad \text{if } \phi = \alpha. \end{aligned} \tag{2.2}$$

In general, the solution of an elliptic equation in an unbounded domain is not unique. But, by imposing a condition upon F concerning its growth at infinity, uniqueness can be ensured. We prove the following lemma. The function $I_0(\omega r)$ appearing in the lemma is a modified Bessel function satisfying $\Delta u - \omega^2 u = 0$, $I_0(\omega r) > 0$, $I_0(\omega r) = \exp(\omega r)/\sqrt{2\pi\omega r}\,(1 + O(r^{-1}))$ as $r \to \infty$.

LEMMA. *Assume that u is a regular function in A satisfying*: (i) $\Delta u - \omega^2 u = 0$, (ii) $u = 0$ *on the boundary of* A, (iii) $\lim_{r\to\infty} u(x, y)/I_0(\omega r) = 0$. *Then $u = 0$ in the whole domain A.*

Proof. Let $v = u/w$, with $w(x, y) = I_0(\omega r)$. The function v satisfies the elliptic equation

$$\Delta v + \frac{2}{w}(v_x w_x + v_y w_y) = 0,$$

and $v = 0$ on the boundary of A. Owing to (iii), for every positive number σ we can find R such that $r > R$ implies $|v(x, y)| < \sigma$. Consider the part Δ of A contained inside a circle with radius $R_1 > R$ and center at the origin. Then on the boundary of Δ we have $|v| < \sigma$. According to the maximum principle for elliptic equations in bounded domains, the inequality $|v| < \sigma$ holds in the whole set Δ. For an arbitrary point $(x_0, y_0) \in A$, R_1 can be chosen large enough for Δ to contain that point. Then $|v(x_0, y_0)| < \sigma$, and, since σ may be arbitrarily small, $v(x_0, y_0) = 0$ and hence $u(x_0, y_0) = 0$, which proves the lemma.

A formal solution of the Helmholtz equation in (2.2) may be written as

$$F(x, y) = \int e^{Aze^{-t} + B\bar{z}e^{t}} f(t)\, dt, \tag{2.3}$$

where $z = x + iy$, $\bar{z} = x - iy$ and A, B are constants to be specified. It can be easily verified that F in (2.3) satisfies the equation in (2.2) by writing Laplace's operator as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}}.$$

Performing the differentiation in (2.3) we obtain $\Delta F = 4ABF$. Hence, if $4AB = \omega^2$, then F satisfies the equation in (2.2). Taking $A = \frac{1}{2}i\omega$, $B = -\frac{1}{2}i\omega$, $z = re^{i\phi}$, we have

$$F(x, y) = \int e^{-i\omega r \sinh t} f(t + i\phi)\, dt. \tag{2.4}$$

By changing (2.3) into

$$\int e^{A\bar{z}e^{-t} + Bze^{t}} g(t)\, dt,$$

we obtain a representation as in (2.4), but now with $g(t - i\phi)$ in the integrand. Hence, a formal solution of the Helmholtz equation can be represented by

$$F(x, y) = \int e^{-i\omega r \sinh t} \{f(t + i\phi) + g(t - i\phi)\}\, dt.$$

For suitable choices of f and g the expression $f(t + i\phi) + g(t - i\phi)$ becomes the real (or imaginary) part of a holomorphic function of the complex variable $\zeta = t + i\phi$ (with real t and ϕ). In that case this expression is a harmonic function of the variables t and ϕ.

To solve the boundary value problem (2.2) we choose a representation of the following kind:

$$F(x, y) = \int_{-\infty}^{\infty} e^{-i\omega r \sinh t}\, U(t, \phi)\, dt,$$

where U is harmonic (but not necessarily holomorphic) in the strip

$$B = \{t, \phi | -\infty < t < \infty, 0 < \phi < \alpha\}.$$

In view of the boundary conditions of F (see (2.2)) we obtain for U the following boundary value problem:

$$\begin{aligned} &\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2 U}{\partial \phi^2} = 0 \quad \text{in } B, \\ &U(t, 0) = \delta(t), \quad U(t, \alpha) = 0. \end{aligned} \tag{2.5}$$

Solutions of the Laplace equation in the strip B with Dirichlet boundary conditions can be obtained by using the conformal mapping $\eta = \exp(\pi\zeta/\alpha)$, which gives a potential problem in a half-plane. In the underlying case we choose a more direct method.

Suppose $U(t, \phi) = \operatorname{Re} f(\zeta)$, $\zeta = t + i\phi$. Then the singularity of U in $\zeta = 0$ may be represented by $(i/\pi)(1/\zeta)$ and f may be constructed by the principle of reflection.

$$f(\zeta) = \frac{i}{\pi}\left[\frac{1}{\zeta} + \sum_{k=1}^{\infty} \frac{2\zeta}{\zeta^2 + 4k^2\alpha^2}\right] = \frac{i}{\pi}\,\frac{\mu}{e^{\mu\zeta} - 1},$$

where

$$\mu = \pi/\alpha. \tag{2.6}$$

The real part of f is then given by

$$U(t, \phi) = \frac{1}{2\alpha}\,\frac{\sin \mu\phi}{\cosh \mu t - \cos \mu\phi}. \tag{2.7}$$

Hence

$$F(x, y) = \frac{\sin \mu\phi}{2\alpha} \int_{-\infty}^{\infty} e^{-i\omega r \sinh t} \frac{dt}{\cosh \mu t - \cos \mu\phi}.$$

This function is bounded by the expression

$$\frac{\sin \mu\phi}{2\alpha} \int_{-\infty}^{\infty} \frac{dt}{\cosh \mu t - \cos \mu\phi} = \frac{\alpha - \phi}{\alpha},$$

and hence the conditions for uniqueness are fulfilled. With this function F the solution of (1.1) is

$$\Phi_\varepsilon(x, y) = 1 - \frac{\sin \mu\phi}{2\alpha} e^{\omega r \sin \phi} \int_{-\infty}^{\infty} e^{-i\omega r \sinh t} \frac{dt}{\cosh \mu t - \cos \mu\phi}. \tag{2.8}$$

This representation of the solution of the singular perturbation problem (1.1) will be the starting point of the investigations on the asymptotic behavior of $\Phi_\varepsilon(x, y)$ for $\varepsilon \to 0$ (i.e., $\omega \to \infty$). The integral in (2.8) will be evaluated by saddle point methods. The saddle points of the function $e^{-i\omega r \sinh t}$ are located at the zeros of cosh t, i.e., at $t_n = i(\frac{1}{2}\pi + n\pi)$, n being an integer. The steepest descent lines are lines parallel to the real t-axis through t_n. If convergence is not disturbed, the path of integration of the integral in (2.8) may be shifted towards a steepest descent line. With this condition, only the saddle point at $-\frac{1}{2}i\pi$ can be considered.

By shifting the path of integration of (2.8) downwards to the line $\operatorname{Im} t = -\frac{1}{2}\pi$, singularities of the integrand may be passed. The singularities in this case are poles due to the zeros of

$$\cosh \mu t - \cos \mu\phi = 2 \sin (\tfrac{1}{2}\mu(\phi + it)) \sin (\tfrac{1}{2}\mu(\phi - it)). \tag{2.9}$$

The zeros are $t_k = -i(\phi + 2\alpha k)$ and $\bar{t}_k$ (the complex conjugate of t_k) for integer values of k. The following poles are important in our problem:

$$\begin{aligned} t_k &= -i(\phi + 2\alpha k) \quad \text{for } k = 0, 1, 2, \cdots, \\ \bar{t}_k &= i(\phi + 2\alpha k) \quad \text{for } k = -1, -2, -3, \cdots. \end{aligned} \tag{2.10}$$

Only these poles may be located in $[0, -\frac{1}{2}i\pi]$, the number of which is dependent on α. We consider two different cases: $\frac{1}{2}\pi < \alpha < 2\pi$ and $0 < \alpha < \frac{1}{2}\pi$. The first case is simpler than the second one, and will be considered first.

3. The case $\frac{1}{2}\pi < \alpha < 2\pi$. For $0 < \phi < \alpha$, only the pole $t_0 = -i\phi$ of (2.10) may be located in the interval $[0, -\frac{1}{2}i\pi]$. For values of ϕ close to $\frac{1}{2}\pi$, the pole t_0 lies close to the saddle point at $t = -\frac{1}{2}i\pi$. In order to obtain an asymptotic expansion of Φ_ε which holds uniformly for all values of ϕ in $[0, \alpha]$, we use the same method as in [4].

Essential to this method is the regularization of the integrand in (2.8) by an appropriate function. This will be done by determining a constant (i.e., independent of t) c such that the function

$$\frac{\sin \mu\phi}{2\alpha} \frac{1}{\cosh \mu t - \cos \mu\phi} - \frac{c}{\sinh \frac{1}{2}(t + i\phi)} \tag{3.1}$$

is regular at $t = -i\phi$. By calculating the residues at $t = -i\phi$ of both members of (3.1), we infer $c = i/4\pi$. The function Φ_ε of (2.8) can now be written as

$$(3.2)\quad \Phi_\varepsilon(x, y) = 1 + \frac{e^{\omega y}}{4\pi i}\int_{-\infty}^{\infty} e^{-i\omega r \sinh t} \frac{dt}{\sinh \frac{1}{2}(t + i\phi)} - e^{\omega y}\int_{-\infty}^{\infty} e^{-i\omega r \sinh t} g(t)\, dt,$$

where

$$(3.3)\quad g(t) = U(t, \phi) + \frac{1}{4\pi i \sinh \frac{1}{2}(t + i\phi)},$$

and U is defined in (2.7).

The first integral in (3.2) can be evaluated by means of the following formula:

$$(3.4)\quad F(r, \gamma) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-r\cosh u} \frac{du}{\sinh \frac{1}{2}(u - i\gamma)} = e^{-r\cos\gamma} \operatorname{erfc}(\sqrt{2r} \sin \tfrac{1}{2}\gamma), \quad 0 < \gamma < 2\pi,$$

where

$$(3.5)\quad \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}\, dt.$$

Formula (3.4) can be found in Lauwerier's papers [3] and is also used in [4]. A proof of (3.4) is easily obtained by verifying that

$$\frac{\partial}{\partial r}\{e^{r\cos\gamma} F(r, \gamma)\} = -2\sqrt{\frac{2\pi}{r}} \sin \tfrac{1}{2}\gamma\, e^{-r(1-\cos\gamma)}.$$

Now, letting $u = t + \frac{1}{2}i\pi$, $\gamma = 5\pi/2 - \phi$ and using

$$\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z),$$

we obtain

$$(3.6)\quad 1 + \frac{e^{\omega r \sin\phi}}{4\pi i}\int_{-\infty}^{\infty} e^{-i\omega r \sinh t} \frac{dt}{\sinh \frac{1}{2}(t + i\phi)} = \tfrac{1}{2}\operatorname{erfc}(z),$$

where

$$(3.7)\quad z = \sqrt{2\omega r} \sin \tfrac{1}{2}(\tfrac{1}{2}\pi - \phi).$$

Formula (3.6) holds for $\frac{1}{2}\pi < \phi < 2\pi$. But by considering complex values of ϕ and by using analytic continuation, (3.6) can be shown to hold for $0 < \operatorname{Re} \phi < 2\pi$.

The function g of (3.3) is regular for $t \in [0, -\frac{1}{2}i\pi]$ and $0 < \phi < \alpha$ $(\frac{1}{2}\pi < \alpha < 2\pi)$. Hence, by shifting the path of integration in the second integral of (3.2) downwards to the line $\operatorname{Im} t = -\frac{1}{2}\pi$, we obtain

$$(3.8)\quad \Phi_\varepsilon(x, y) = \tfrac{1}{2}\operatorname{erfc}(z) - e^{\omega y}\int_{-\infty}^{\infty} e^{-\omega r \cosh t} g(t - \tfrac{1}{2}i\pi)\, dt.$$

So far, large values of ω (i.e., small values of ε) have not been considered. The representation (3.8) of the solution of the boundary value problem is the exact representation. In order to get an asymptotic expansion of Φ_ε, we expand g

into a series

$$g(t - \tfrac{1}{2}i\pi) = \cosh\tfrac{1}{2}t \sum_{k=0}^{\infty} c_k(\sinh \tfrac{1}{2}t)^k. \tag{3.9}$$

Substitution of this series in (3.8) and interchanging the order of summation and integration yields

$$\Phi_\varepsilon(x, y) \simeq \tfrac{1}{2}\operatorname{erfc}(z) - 2\, e^{-\omega r(1-\sin\phi)} \sum_{k=0}^{\infty} c_{2k}\frac{\Gamma(k+\frac{1}{2})}{(2\omega r)^{k+1/2}} \tag{3.10}$$

as $\omega r \to \infty$, uniformly with respect to ϕ, $0 \leqq \phi \leqq \alpha$.

The expansion in (3.10) breaks down if $\alpha \to \frac{1}{2}\pi$. Namely, the function $g(t - \frac{1}{2}i\pi)$ has a pole at $i(\phi + \frac{1}{2}\pi - 2\alpha)$. For $\phi = \alpha$, this pole is located at $i(\frac{1}{2}\pi - \alpha)$ and if $\alpha \to \frac{1}{2}\pi$ this pole approaches the origin. As a consequence, the coefficients c_k in (3.9) and (3.10) tend to infinity if $\alpha \to \frac{1}{2}\pi$. For this question the reader is referred to § 5.

The most important term in the asymptotic expansion (3.10) is

$$\Phi_\varepsilon^{(0)}(x, y) \equiv \tfrac{1}{2}\operatorname{erfc}(z)$$

with z defined in (3.7). This term exhibits the behavior of Φ_ε in the neighborhood of $\phi = \frac{1}{2}\pi$, i.e., along the y-axis in the x, y-plane. Just as in the quarter-plane case, this term leads to a parabolic boundary layer, situated along the positive y-axis. In this domain, for large values of ωr, the function $\Phi_\varepsilon^{(0)}$ (and hence Φ_ε) rapidly changes from the value $\frac{1}{2}$ to very small values ($x > 0$) or to values close to 1 ($x < 0$). This boundary layer is called a "free" or "internal" boundary layer, since it is not located along the boundary of the domain A for which the boundary value problem (1.1) is defined.

Along the boundary $\phi = \alpha$ boundary layers do not occur, as can be seen from (3.10). Namely, if $\phi \simeq \alpha$ ($\alpha > \frac{1}{2}\pi$),

$$\Phi_\varepsilon(x, y) - \Phi_\varepsilon^{(0)}(x, y) = O((\omega r)^{-N})$$

as $\omega r \to \infty$, for any positive N and all ϕ, $\frac{1}{2}\pi + \delta \leqq \phi \leqq \alpha$, where δ is a small positive constant. For these values of ϕ we also have

$$\Phi_\varepsilon^{(0)}(x, y) - 1 = O((\omega r)^{-N}),$$

as follows from

$$\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$$

and from the well-known asymptotic formula

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}z} e^{-z^2}(1 + O(z^{-2}))$$

as $z \to +\infty$.

4. The acute angle. In this section we consider values of ϕ and α in the range

$$0 \leqq \phi \leqq \alpha < \tfrac{1}{2}\pi. \tag{4.1}$$

First we determine the number of poles (2.14) located on the imaginary t-axis between 0 and $-\frac{1}{2}i\pi$.

We introduce

$$\lambda \equiv \frac{2\alpha}{\pi}\left(=\frac{2}{\mu}\right), \tag{4.2}$$

so that $0 < \lambda < 1$. Consequently, we can choose an integer $n \geqq 2$, satisfying

$$n - 1 < 1/\lambda \leqq n. \tag{4.3}$$

We distinguish two cases:

(a) If in (4.3) n is odd, then we have with $k_0 = \frac{1}{2}(n-1)$,

$$2\alpha k_0 < \tfrac{1}{2}\pi \leqq (2k_0 + 1)\alpha. \tag{4.4}$$

Therefore, the pole

$$t_{k_0} = -i(\phi + 2\alpha k_0) \tag{4.5}$$

passes through $-\frac{1}{2}i\pi$ when ϕ changes from 0 to α. If $\phi + 2\alpha k_0 < \frac{1}{2}\pi$, then $t_{k_0} \in [0, -\frac{1}{2}i\pi]$; if $\phi + 2\alpha k_0 > \frac{1}{2}\pi$, then $t_{k_0} \notin [0, -\frac{1}{2}i\pi]$. For all values of ϕ $(0 \leqq \phi \leqq \alpha)$, we have

$$t_k, \bar{t}_l \in [0, -\tfrac{1}{2}i\pi] \quad \text{for } k = 0, 1, \cdots, k_0 - 1, \quad l = 1, 2, \cdots, k_0.$$

(b) If in (4.3) n is even, we have with $l_0 = \frac{1}{2}n$,

$$(2l_0 - 1)\alpha < \tfrac{1}{2}\pi \leqq 2\alpha l_0. \tag{4.6}$$

Therefore, the pole

$$\bar{t}_{-l_0} = i(\phi - 2\alpha l_0) \tag{4.7}$$

passes the point $-\frac{1}{2}i\pi$ if ϕ changes from 0 to α. If $\phi + \frac{1}{2}\pi > 2\alpha l_0$, then $\bar{t}_{-l_0} \in [0, -\frac{1}{2}i\pi]$; if $\phi + \frac{1}{2}\pi < 2\alpha l_0$, then $\bar{t}_{-l_0} \notin [0, -\frac{1}{2}i\pi]$. For all values of ϕ $(0 \leqq \phi \leqq \alpha)$,

$$t_k, \bar{t}_{-l} \in [0, -\tfrac{1}{2}i\pi] \quad \text{for } k = 0, 1, \cdots, l_0 - 1, \quad l = 1, 2, \cdots, l_0 - 1.$$

As in § 3, the poles t_{k_0} and $\bar{t}_{-l_0}$ ((4.5) and (4.7) respectively) can be split off. In this way error functions are introduced. Afterwards the path of integration will be shifted downwards to the line $\operatorname{Im} t = -\frac{1}{2}\pi$. The residues of the poles being passed turn out to be exponential functions. A simple calculation gives the following results. (We distinguish again the two cases (a) and (b).)

(a)

$$\begin{aligned} \Phi_\varepsilon(x, y) = {} & \sum_{k=1}^{k_0} e^{-\omega r\{\sin(2k\alpha - \phi) - \sin\phi\}} - \sum_{k=1}^{k_0 - 1} e^{-\omega r\{\sin(2k\alpha + \phi) - \sin\phi\}} \\ & - \tfrac{1}{2}\operatorname{erfc}(z) e^{-\omega r\{\sin(2\alpha k_0 + \phi) - \sin\phi\}} - e^{\omega r \sin\phi} \int_{-\infty}^{\infty} e^{-\omega r \cosh t} g(t)\, dt, \end{aligned} \tag{4.8}$$

where k_0 is specified in (4.4), $z = \sqrt{2\omega r}\sin\frac{1}{2}\gamma$,

$$g(t) = U(t - \tfrac{1}{2}i\pi, \phi) + \frac{1}{4\pi i \sinh\frac{1}{2}(t - i\gamma)}, \tag{4.9}$$

and $\gamma = \frac{1}{2}\pi - \phi - 2\alpha k_0$.

(b)

$$\Phi_\varepsilon(x, y) = \sum_{k=1}^{l_0-1} e^{-\omega r\{\sin(2\alpha k - \phi) - \sin\phi\}} - \sum_{k=1}^{l_0-1} e^{-\omega r\{\sin(2\alpha k + \phi) - \sin\phi\}}$$

(4.10)

$$+ \tfrac{1}{2}\operatorname{erfc}(z) e^{-\omega r\{\sin(2\alpha l_0 - \phi) - \sin\phi\}} - e^{\omega r \sin\phi} \int_{-\infty}^{\infty} e^{-\omega r \cosh t} g(t)\, dt,$$

where l_0 is specified in (4.6), $z = \sqrt{2\omega r} \sin \frac{1}{2}\gamma$,

(4.11)
$$g(t) = U(t - \tfrac{1}{2}i\pi, \phi) - \frac{1}{4\pi i \sinh \frac{1}{2}(t - i\gamma)}$$

and $\gamma = \frac{1}{2}\pi + \phi - 2\alpha l_0$.

The representation (4.8) (resp. (4.10)) of $\Phi_\varepsilon(x, y)$ is the exact solution. In order to get an asymptotic expansion of Φ_ε for small values of ε, the function g in (4.9) (resp. (4.11)) may be expanded in the same way as was done for g in (3.9). The asymptotic expansion obtained by interchanging the order of summation and integration (cf. (3.10)) is uniformly valid in $0 \leqq \phi \leqq \alpha$. Just as in the foregoing section, if $\alpha \to \frac{1}{2}\pi$, the expansion must be reconsidered (see § 5).

We conclude this section with some remarks concerning the boundary layer. If $\phi \simeq \alpha$, the asymptotic behavior of Φ_ε is determined by the first term of the first finite series in (4.8) (resp. (4.10)); the other terms are of lower order. Hence

(4.12)
$$\Phi_\varepsilon(x, y) \simeq e^{-\omega r\{\sin(2\alpha - \phi) - \sin\phi\}}$$

as $\omega r \to \infty$, $\phi \to \alpha$, $\phi \leqq \alpha$. If $\phi < \alpha$, the right-hand side of (4.12) is very small, explicitly

$$\Phi_\varepsilon(x, y) = O((\omega r)^{-N}),$$

where N is an arbitrary positive number. This estimate, however, is not uniformly valid in ϕ. If $\phi \to \alpha$, the exponential function in (4.12) may not be small at all. We can determine the locus in the x, y-plane on which the argument of the exponential function in (4.12) is constant. We infer from

$$-\omega r\{\sin(2\alpha - \phi) - \sin\phi\} = -c \qquad (c > 0)$$

that the locus is a straight line

(4.13)
$$y = x \tan\alpha - c/\omega,$$

which is parallel to the boundary $y = x \tan\alpha$ of the sector A. From these aspects we conclude that along the line $y = x \tan\alpha$ a boundary layer of thickness $O(\varepsilon)$ is located.

The term with the error function in (4.8) (resp. 4.10)) is asymptotically of lower order than the term in (4.12). The error function part, however, is of great importance. The error function changes rapidly at $\phi = \frac{1}{2}\pi - 2\alpha k_0$ (resp. $2\alpha l_0 - \frac{1}{2}\pi$), but the effect is damped by the exponential function contained in this term. This term is gaining in influence if $\alpha \to \frac{1}{2}\pi$ and the (hidden) internal boundary layer due to the error function comes to light if $\alpha \to \frac{1}{2}\pi$ (see § 5).

5. The almost right angle. In §§ 3 and 4 we discussed the asymptotic behavior of Φ_ε $(\varepsilon \to 0)$ for values of α larger, respectively smaller than $\frac{1}{2}\pi$. However, the expansion in (3.9) and the expansions which can be derived from (4.8) (resp. (4.10)) in an analogous way, are not valid if ϕ, $\alpha \to \frac{1}{2}\pi$, since the function $g(t)$ has a singularity, which tends to zero for ϕ, $\alpha \to \frac{1}{2}\pi$. In this section we shall give the asymptotic representation of Φ_ε holding for all $\alpha \in [\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta]$, where $0 < \delta < \frac{1}{4}\pi$.

Suppose first that $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$. In this case the results of § 4 can be used. In (4.2) we have $\frac{1}{2} < \lambda < 1$ and in (4.3) we have $n = 2$. Hence, the (b)-case applies and from (4.6) it follows that $l_0 = 1$. Thus (4.8) becomes

$$\Phi_\varepsilon(x, y) = \tfrac{1}{2}\,\mathrm{erfc}\,(z)\,e^{-\omega r\{\sin(2\alpha - \phi) - \sin\phi\}} + e^{\omega r \sin\phi} \int_{-\infty}^{\infty} e^{-\omega r \cosh t} g(t)\,dt, \tag{5.1}$$

where $z = \sqrt{2\omega r}\sin\frac{1}{2}\gamma$, $\gamma = \phi + \frac{1}{2}\pi - 2\alpha$ and g is defined in (4.9). The function g has a pole in $i(\frac{1}{2}\pi - \phi)$, corresponding to t_k in (2.14) with $k = 0$. In (4.8) this pole has no influence since α is constant. If, alternatively, $\alpha \simeq \frac{1}{2}\pi$, this singularity is close to the origin for values of ϕ close to α. This pole can be split off and so another error function is introduced.

Suppose next $\frac{1}{2}\pi < \alpha < 2\pi$. The function g in (3.3) has a pole in $i(\phi + \frac{1}{2}\pi - 2\alpha)$, corresponding to $\bar{t}_k$ in (2.14) with $k = -1$. Again, for $\alpha \simeq \frac{1}{2}\pi$, this pole is close to the origin for values of ϕ close to α.

Combining the two cases, we have

$$\begin{aligned} \Phi_\varepsilon(x, y) &= \tfrac{1}{2}\,\mathrm{erfc}\,(\zeta) + \tfrac{1}{2}\,\mathrm{erfc}\,(z)\,e^{-\omega r\{\sin(2\alpha - \phi) - \sin\phi\}} \\ &\quad - e^{\omega r \sin\phi} \int_{-\infty}^{\infty} e^{-\omega r \cosh t} h(t)\,dt, \end{aligned} \tag{5.2}$$

where

$$h(t) = U(t - \tfrac{1}{2}i\pi, \phi) - \frac{1}{4\pi i \sinh\frac{1}{2}(t - i\gamma)} + \frac{1}{4\pi i \sinh\frac{1}{2}(t - i(\frac{1}{2}\pi - \phi))},$$

$$\gamma = \phi + \tfrac{1}{2}\pi - 2\alpha, \quad z = \sqrt{2\omega r}\sinh\tfrac{1}{2}\gamma, \quad \zeta = \sqrt{2\omega r}\sin\tfrac{1}{2}(\tfrac{1}{2}\pi - \phi).$$

The asymptotic expansion of Φ_ε for large ωr may be derived by expanding h in the same way as was done for g in (3.8). The asymptotic expansion so obtained holds uniformly in $0 \leqq \phi \leqq \alpha$, $\frac{1}{2}\pi - \delta \leqq \alpha \leqq \frac{1}{2}\pi + \delta$, where $0 < \delta < \frac{1}{4}\pi$. If $\alpha = \frac{1}{2}\pi$, (the quarter-plane, see [4]) we have $\gamma = \phi - \frac{1}{2}\pi$ and

$$\Phi_\varepsilon(x, y) = \mathrm{erfc}\,(\sqrt{2\omega r}\sin\tfrac{1}{2}(\tfrac{1}{2}\pi - \phi)) - e^{\omega r \sin\phi} \int_{-\infty}^{\infty} e^{-\omega r \cosh t} h(t)\,dt,$$

where the integral equals the corresponding integral of reference [4] (formula (4.6)).

The significant terms of (5.2) are the two terms with the error functions. For $\alpha < \frac{1}{2}\pi$, the second term may be connected with the "linear" boundary layer along $\phi = \alpha$ and the (hidden) internal layer at $\phi = 2\alpha - \frac{1}{2}\pi$. The first one may be connected with an external parabolic boundary layer outside the sector A. This boundary layer has no influence since it is situated outside the domain of definition. If however $\alpha \to \frac{1}{2}\pi$ ($\alpha < \frac{1}{2}\pi$) this boundary layer enters the domain A, and coalesces (in the limit $\alpha = \frac{1}{2}\pi$) with both the "linear" boundary layer and the (hidden) internal layer at $\phi = 2\alpha - \frac{1}{2}\pi$ (see Fig. 1).

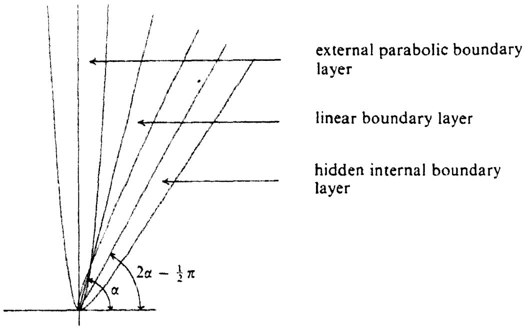

FIG. 1. $\alpha < \frac{1}{2}\pi$

For $\alpha > \frac{1}{2}\pi$, the first term in (5.2) may be associated with the internal parabolic boundary layer inside the sector at $\phi = \frac{1}{2}\pi$. The second one may be connected with a boundary layer outside A, which is situated at $\phi = 2\alpha - \frac{1}{2}\pi$ and which enters the domain if $\alpha \to \frac{1}{2}\pi$. In the limit ($\alpha = \frac{1}{2}\pi$) the two types of boundary layers pass into the parabolic boundary layer along $\phi = \alpha = \frac{1}{2}\pi$ (see Fig. 2).

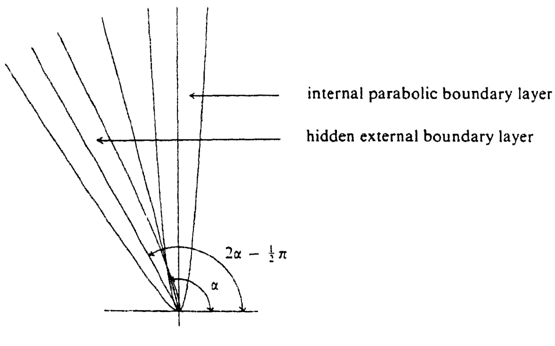

FIG. 2. $\alpha > \frac{1}{2}\pi$

From our remarks on the coincidence of the boundary layers it may be established that the parabolic boundary layer of the quarter-plane is a particular case of parabolic boundary layers for the almost right angle. For other aspects, the reader is referred to Grasman [2], where the case of almost characteristic boundaries is treated with coordinate-stretching techniques.

6. An analogous problem. An analogous, but much simpler, problem is encountered in looking for the asymptotic expansion of the solutions of the boundary value problem

$$\varepsilon\Delta\Phi_\varepsilon(x,y) - \mu\frac{\partial}{\partial x}\Phi_\varepsilon(x,y) - \lambda\frac{\partial}{\partial y}\Phi_\varepsilon(x,y) = 0 \tag{6.1}$$

in the quarter-plane $A = \{x, y | x \geqq 0, y \geqq 0\}$ with boundary conditions

$$\Phi_\varepsilon(0,y) = 1, \quad \Phi_\varepsilon(x,0) = 0.$$

In (6.1), μ and λ are numbers independent of x and y. The characteristics of the reduced equation ($\varepsilon = 0$ in (6.1))

$$\mu\frac{\partial\phi}{\partial x} + \lambda\frac{\partial\phi}{\partial y} = 0 \tag{6.2}$$

are the lines $y = (\lambda/\mu)x + c$. For small values of μ, the characteristics of (6.2) are nearly parallel to the boundary line $x = 0$. Therefore, for small values of μ it is expected that again two error functions appear in the asymptotic expansion of Φ_ε (for $\varepsilon \to 0$). As can be verified by the methods of § 2, the function Φ_ε can be written down as follows:

$$\Phi_\varepsilon(x,y) = e^{\omega r \sin(\phi+\beta)} \int_{-\infty}^{\infty} e^{-i\omega r \sinh t}\, U(t,\phi)\, dt,$$

where

$$x = r\cos\phi, \quad y = r\sin\phi, \quad \lambda = \rho\cos\beta, \quad \mu = \rho\sin\beta, \quad \omega = \rho/(2\varepsilon),$$

$$U(t,\phi) = \frac{1}{\pi}\operatorname{Re}\{\tan\tfrac{1}{2}(it + \phi + \beta) + \tan\tfrac{1}{2}(it + \phi - \beta)\}$$

$$= \frac{1}{\pi}\frac{\sin(\phi+\beta)}{\cosh t + \cos(\phi+\beta)} + \frac{1}{\pi}\frac{\sin(\phi-\beta)}{\cosh t + \cos(\phi-\beta)}.$$

7. Concluding remarks. In this paper we used analytical methods which only can be applied on singular perturbation problems with simple differential operators, boundary values and suitable domains of definition. The methods cannot easily be generalized for other problems. In treating the relatively simple problems, however, we have a different aim.

For instance, our approach of the problem gives results which are not easily noticed by using the usual singular perturbation techniques. We allude to the existence of the hidden boundary layer along the line $\phi = 2\alpha - \frac{1}{2}\pi$ (see Fig. 1 and the conclusion of § 4). This aspect is not discussed in boundary layer techniques, since the function Φ_ε is asymptotically of order zero in the neighborhood of this internal layer. In order to obtain a uniform asymptotic expansion with respect to ϕ (in $0 \leqq \phi \leqq \alpha < \frac{1}{2}\pi$), the error function corresponding with this layer cannot be omitted.

Further we shall point to the case of an almost characteristic boundary (see § 5). In a clear and simple way the asymptotic behavior of Φ_ε can be described, using our methods. Also the way in which the various boundary layers pass into each other is apparent.

An important disadvantage of our methods is the following. The asymptotic expansions are derived for large values of ωr. Hence, the results of our paper do not hold in an ε-neighborhood of the origin. This domain is very small but it is very interesting, since in this part of the x, y-plane the boundary layers arise. It is possible to give expansions which represent the behavior of Φ_ε close to the origin, but it seems better to us to tackle this problem with coordinate stretching techniques. This aspect, however, falls outside the scope of this paper.

Our results can successfully be applied in general singular perturbation problems, which yield reduced problems with relatively simple differential operators, boundary values and domains of definition. With these reduced problems the local behavior of the solutions are investigated.

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Numerical Evaluation of Functions Arising From Transformations of Formal Series*

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An algorithm is given for the numerical evaluation of a class of functions of the confluent hypergeometric type. The method of computation is based on the well-known Miller algorithms and on asymptotic expansions.

1. Introduction

In 1953, A. van Wijngaarden wrote a paper on transformations of formal series [6]. He discussed a general transformation of the asymptotic expansion of certain integrals for large parameter values. Special attention was paid to a transformation from which the following functions arose

$$s_k(z) = z \int_0^\infty e^{-zt} t^k (1+t)^{-k-1}\, dt, \qquad k = 0, 1, 2, \ldots, \quad \operatorname{Re} z > 0. \tag{1.1}$$

This transformation can be described in different ways. One way is the following. If in the Laplace integral

$$f(z) = z \int_0^\infty e^{-zt} F(t)\, dt \tag{1.2}$$

the function F is expanded in powers of t and the order of summation and integration is interchanged, then a formal series

$$f(z) \sim \sum_{k=0}^{\infty} F^{(k)}(0)\, z^{-k} \tag{1.3}$$

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results. When, however, $F(t)$ is expanded in the following way

$$F(t) = \sum_{k=0}^{\infty} c_k t^k (1+t)^{-k-1}, \tag{1.4}$$

then we obtain by termwise integration

$$f(z) \sim \sum_{k=0}^{\infty} c_k s_k(z), \tag{1.5}$$

with s_k defined in (1.1). The series in (1.5) can be considered as a transformation of the series in (1.3).

In [3], H. A. Lauwerier considers Van Wijngaarden's transformation from a different point of view.

Van Wijngaarden announced the construction of tables for the functions $s_k(z)$ for complex values of z. The construction of these tables turned out to be a heavy task and the tables did not reach the stage of publication. Nowadays, with large scale computer systems at our disposal, tabular values are not as interesting as methods of computing.

The aim of this paper is to give information about the numerical evaluation of $s_k(z)$ for $|\arg z| < \pi$ and $k = 0, 1, \ldots, K$, where K is some positive integer.

In the next section some elementary properties of the functions s_k are discussed. In fact s_k can be expressed as a confluent hypergeometric function (Whittaker function). In Section 3 the asymptotic behavior of s_k is determined. With these results the convergence of the algorithm in Section 4 is proven. In Section 5 the computation for small values of $|z|$ is discussed. Also in that case asymptotic expansions are of fundamental importance. Our methods of computation apply to a more general class of functions, in fact to the whole class of confluent hypergeometric functions to which the functions s_k belong. Information on that point will be given in Section 7.

2. The Functions $s_k(z)$

The functions $s_k(z)$, defined by (1.1) for $\operatorname{Re} z > 0$, $k = 0, 1, 2, \ldots$, can be expressed in terms of confluent hypergeometric functions. Using the notation of Abramowitz and Stegun [1, Chap. 13], we have

$$s_k(z) = z k!\, U(k+1, 1, z). \tag{2.1}$$

Relevant properties of $s_k(z)$ can be derived from well-known properties of $U(a, b, z)$, Equation (1.1) defines $s_k(z)$ in the halfplane $\operatorname{Re} z > 0$. The domain of definition can be extended to $|\arg z| < 3\pi/2$ by rotating the path of integration. The function $s_k(z)$ is a many-valued function of z. We will consider its principal branch in the plane cut along the negative real axis, this branch being determined by the condition that $s_k(z)$ is real and positive if z is real and positive.

For convenience we will denote

$$u_k(z) = s_k(z)/z, \tag{2.2}$$

$k = 0, 1, 2, \ldots$. Then u_k satisfies the confluent hypergeometric differential equation

$$zu_k'' + (1 - z)u_k' - (k + 1)u_k = 0. \tag{2.3}$$

A second solution of this equation, linearly independent of u_k, is the function

$$y_k(z) = M(k + 1, 1, z); \tag{2.4}$$

$y_0(z) = e^z$, $y_1(z) = (1 + z)e^z$. $M(a, b, z)$ is known as Kummer's function. In the notation of hypergeometric functions the function $M(a, b, z)$ is defined by

$$M(a, b, z) = {}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b + n)} \frac{z^n}{n!}. \tag{2.5}$$

A corresponding series-representation for $s_k(z)$ can be derived from a known representation of $U(a, b, z)$ in [1, 13.1.6]. For $k = 0, 1, 2, \ldots$, we have

$$s_k(z) = -\frac{z}{k!} \sum_{n=0}^{\infty} \frac{\Gamma(k + n + 1)z^n}{n!\, n!} \{\ln z + \Psi(k + n + 1) - 2\Psi(n + 1)\}, \tag{2.6}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$. The series converges for all z in the finite z plane.

From the contiguous relations of the confluent hypergeometric functions we derive

$$(k + 1)\, s_{k+1}(z) - (2k + 1 + z)\, s_k(z) + ks_{k-1}(z) = 0. \tag{2.7}$$

This formula can also be obtained by partial integration of (1.1). The recurrence formula (2.7) can be considered as a homogeneous linear difference equation of the second order, of which of course u_k is also a solution, but u_k is not linearly independent of s_k. An independent solution turns out to be the function y_k defined in (2.4).

3. Asymptotic Expansions

In this section we will study the asymptotic behavior of s_k and y_k for large values of k and for various values of z, $|\arg z| < \pi$.

For small values of $|z|$, k fixed, the asymptotic behavior follows from (2.5) and (2.6), viz

$$y_k(z) = 1 + O(z), \tag{3.1}$$

$$s_k(z) = -z(\ln z + \Psi(k+1)) + O(|z^2 \ln z|). \tag{3.2}$$

For bounded values of $|z|$, say $|z| \leqslant M$, and large values of k we will use the differential equation (2.3) and a theorem due to Olver [4]. First we give a transformation of the dependent variable. If $u_k(z)$ is a solution of (2.3) then

$$w(z) = z^{1/2}e^{-(1/2)z}u_k(z) \tag{3.3}$$

satisfies the equation

$$w'' - [(k+\tfrac{1}{2})/z - 1/(4z^2) + \tfrac{1}{4}]w = 0. \tag{3.4}$$

The transformation of the independent variable

$$z = t^2 \tag{3.5}$$

and the substitution

$$k + \tfrac{1}{2} = \tfrac{1}{4}\lambda^2 \tag{3.6}$$

yield

$$w'' - w'/t - [\lambda^2 - t^{-2} + t^2]w = 0, \tag{3.7}$$

the differentiation in this equation being with respect to t.

For large values of λ and uniformly bounded values of $|t|$ the solutions of (3.7) will behave like the solutions of the so-called basic-equation

$$w'' - w'/t - [\lambda^2 - t^{-2}]w = 0. \tag{3.8}$$

The solutions of this equation are $tK_0(\lambda t)$ and $tI_0(\lambda t)$, where K_0 and I_0 are modified Bessel functions. By direct substitution in (3.7) it can be verified that the formal series

$$w_1(t) \sim tK_0(\lambda t) \sum_{n=0}^{\infty} \frac{A_n(t)}{\lambda^{2n}} - \frac{t}{\lambda} K_1(\lambda t) \sum_{n=0}^{\infty} \frac{B_n(t)}{\lambda^{2n}} \tag{3.9}$$

$$w_2(t) \sim tI_0(\lambda t) \sum_{n=0}^{\infty} \frac{A_n(t)}{\lambda^{2n}} + \frac{t}{\lambda} I_1(\lambda t) \sum_{n=0}^{\infty} \frac{B_n(t)}{\lambda^{2n}} \tag{3.10}$$

formally satisfy (3.7). The functions A_n and B_n are polynomials in t, recursively given by

$$A_0(t) = 1$$

$$2B_n(t) = -A_n'(t) + \int_0^t \{x^2 A_n(x) - A_n'(x)/x\}\, dx \qquad (3.11)$$

$$2A_{n+1}(t) = B_n(t)/t - B_n'(t) + \int_0^t x^2 B_n(x)\, dx + a_{n+1},$$

the integration constant a_{n+1} being arbitrary. The first few coefficients will be given in Section 5. By application of Olver's theorem it can be shown that the series in (3.9), (3.10) are asymptotic expansions (in Poincaré's sense) of two linearly independent solutions w_1 and w_2 of (3.7) for large values of λ. These expansions hold uniformly in a closed bounded z-domain which includes the origin.

After these preliminary results we return to the functions u_k and y_k, introduced in the foregoing section. The functions

$$F(t) = \exp(-\tfrac{1}{2}t^2)\, t u_k(t^2), \qquad G(t) = \exp(-\tfrac{1}{2}t^2)\, t y_k(t^2)$$

are solutions of (3.7). Hence

$$F(t) = \alpha_1 w_1(t) + \alpha_2 w_2(t), \qquad G(t) = \beta_1 w_1(t) + \beta_2 w_2(t),$$

where α_1, α_2, β_1, β_2 are independent of t. To evaluate these coefficients we need the following well-known properties of the Bessel functions.

$$I_0(x) = 1 + O(x), \qquad K_0(x) = -\ln x + O(1), \qquad x \to 0. \qquad (3.12)$$

$$I_0(x) = (2\pi x)^{-1/2} e^x[1 + O(1)], \quad K_0(x) = (\pi/2x)^{1/2} e^{-x}(1 + O(1)), \quad x \to \infty. \qquad (3.13)$$

The formulas in (3.12) hold for arbitrary values of $\arg x$, those of (3.13) hold for $|\arg x| < \pi/2$.

For real z and for all values of k considered we have $0 < u_k(z) < 1/z$; this follows from (1.1) and (1.2). Hence $\alpha_2 = 0$ for all values of k. Because of the uniform property of the expansions (3.9) and (3.10) we may keep $\lambda = 2(k + \tfrac{1}{2})^{1/2}$ fixed and let $t \to 0$ through positive values. Since $y_k(z)$ is bounded if $z \to 0$ (see (3.1)) it is obvious that $\beta_1 = 0$. Finally, from (3.1), (3.2), and (3.12) it follows that $\alpha_1 = 2$, $\beta_2 = 1$. Moreover, all integration constants in (3.11) have to vanish ($n = 0, 1, 2, \ldots$). Hence, using the various transformations we obtain

$$s_k(z) \sim 2ze^{(1/2)z}\left\{K_0(\zeta)\sum_{n=0}^{\infty}\frac{A_n(z^{1/2})}{(4k+2)^n} - \frac{K_1(\zeta)}{(4k+2)^{1/2}}\sum_{n=0}^{\infty}\frac{B_n(z^{1/2})}{(4k+2)^n}\right\}, \tag{3.14}$$

$$y_k(z) \sim e^{(1/2)z}\left\{I_0(\zeta)\sum_{n=0}^{\infty}\frac{A_n(z^{1/2})}{(4k+2)^n} + \frac{I_1(\zeta)}{(4k+2)^{1/2}}\sum_{n=0}^{\infty}\frac{B_n(z^{1/2})}{(4k+2)^n}\right\}, \tag{3.15}$$

for $k \to \infty$, where

$$\zeta = 2[z(k+\tfrac{1}{2})]^{1/2}, \qquad \zeta > 0 \qquad \text{if} \quad z > 0. \tag{3.16}$$

The expansions are uniformly valid with respect to z in a bounded domain of the z-plane, which contains the origin.

COROLLARY. *For fixed values of* $|z|$ *it follows from* (3.14), (3.15), *and* (3.13) *that*

$$s_k(z) \sim \pi^{1/2}z^{3/4}k^{-1/4}\exp[\tfrac{1}{2}z - 2(zk)^{1/2}], \tag{3.17}$$

$$y_k(z) \sim \tfrac{1}{2}\pi^{-1/2}(zk)^{-1/4}\exp[\tfrac{1}{2}z + 2(zk)^{1/2}] \tag{3.18}$$

as $k \to \infty$. *The restrictions on* z *in* (3.17) *and* (3.18) *are*

$$|\arg z| < \pi, \qquad z \textit{ fixed}, \quad z \neq 0. \tag{3.19}$$

It has to be pointed out that (3.17) and (3.18) are not valid when both k and z are large. Representations, which are valid for large k uniformly in $|z|$ for $|z| \geqslant \delta > 0$, can be derived by applications of theorem A in Olver [4]. This will now be done.

Again, the starting point is the differential equation (3.4). The transformation

$$z = 2\lambda t, \qquad \lambda = 2k + 1 \tag{3.20}$$

yields

$$(d^2w/dt^2) - [\lambda^2(t+1)/t - 1/t^2]w = 0 \tag{3.21}$$

and a further transformation

$$w = [t/(1+t)]^{1/4}v, \qquad x = \ln[t^{1/2} + (1+t)^{1/2}] + [t(1+t)]^{1/2} \tag{3.22}$$

results in

$$(d^2v/dx^2) - \{\lambda^2 + f(x)\}v = 0. \tag{3.23}$$

The function $f(x)$ cannot be given explicitly in terms of x, but in the variable t we have

$$f(x) = (3 + 8t)/[16t(1 + t)^3] - 1/[t(1 + t)];$$

the relation between x and t is given in (3.22). According to Olver, for large values of λ the solutions of (3.23) behave like the solutions $\exp(\pm\lambda x)$ of the basic-equation $v'' - \lambda^2 v = 0$. As a consequence, we have for two linearly independent solutions w_1 and w_2 of (3.21)

$$w_1(t) \sim [t/(1 + t)]^{1/4} \exp[\lambda\{\ln[t^{1/2} + (1 + t)^{1/2}] + [t(1 + t)]^{1/2}\}] \tag{3.24}$$

$$w_2(t) \sim [t/(1 + t)]^{1/4} \exp[-\lambda\{\ln[t^{1/2} + (1 + t)^{1/2}] + [t(1 + t)]^{1/2}\}], \tag{3.25}$$

as $\lambda \to \infty$.

Using Olver's theorem, we can prove that these formulas hold uniformly in t in the domain $|t| \geqslant \delta$, $|\arg t| \leqslant \pi - \epsilon$, where ϵ and δ are fixed positive numbers ($\epsilon < \pi$). As in (3.9) and (3.10), we can construct asymptotic series which formally satisfy (3.23), but here we are interested in the first order approximation only.

To give the results for y_k and s_k we proceed as in the foregoing case. First we remark that

$$z^{1/2}e^{-(1/2)z}u_k(z) = (2\lambda t)^{-1/2}\, e^{-\lambda t} s_k(2\lambda t) = F(t)$$
$$z^{1/2}e^{-(1/2)z}y_k(z) = (2\lambda t)^{1/2}\, e^{-\lambda t} y_k(2\lambda t) = G(t)$$

(where $\lambda = 2k + 1$) are solutions of (3.20). Hence

$$F(t) = \alpha_1 w_1(t) + \alpha_2 w_2(t), \qquad G(t) = \beta_1 w_1(t) + \beta_2 w_2(t),$$

where α_1, α_2, β_1, β_2 are independent of t. To evaluate these coefficients we use (3.17) and (3.18). If in (3.20) z is fixed and λ is large, then t is small. In this case it follows from (3.24) and (3.25)

$$w_1(t) \sim t^{1/4} \exp(2\lambda t^{1/2}) \sim z^{1/4}2^{-1/2}k^{-1/4} \exp[2(kz)^{1/2}]$$
$$w_2(t) \sim t^{1/4} \exp(-2\lambda t^{1/2}) \sim z^{1/4}2^{-1/2}k^{-1/4} \exp[-2(kz)^{1/2}]$$

as $k \to \infty$. Taking into account (3.17) and (3.18) we have $\alpha_1 = \beta_2 = 0$ and $\alpha_2 = (2\pi)^{1/2}$, $\beta_1 = 1/(2\pi)^{1/2}$. Hence

$$s_k(z) = (2\pi z)^{1/2}\, e^{(1/2)z} w_2(t)$$
$$y_k(z) = 1/(2\pi z)^{1/2}\, e^{(1/2)z} w_1(t),$$

and using (3.24) and (3.25) we have

$$s_k(z) \sim (2\pi z \tanh \alpha)^{1/2} \exp\{\tfrac{1}{2}z - (k + \tfrac{1}{2})(2\alpha + \sinh 2\alpha)\} \tag{3.26}$$

$$y_k(z) \sim (\tanh \alpha/2\pi z)^{1/2} \exp\{\tfrac{1}{2}z + (k + \tfrac{1}{2})(2\alpha + \sinh 2\alpha)\} \tag{3.27}$$

as $k \to \infty$, where α is defined by

$$z = 4(k + \tfrac{1}{2}) \sinh^2 \alpha; \tag{3.28}$$

$\sinh \alpha$ is real and positive if z is positive. In (3.26), (3.27) the restrictions on z are

$$|\arg z| \leqslant \pi - \epsilon, \qquad 0 < \epsilon < \pi, \quad |z| \geqslant \delta > 0. \tag{3.29}$$

In (3.26) and (3.27) we can fix k and let $z \to \infty$. Taking limiting forms of the functions of α for large α we obtain

$$s_k(z) \sim k!/z^k$$
$$y_k(z) \sim z^k e^z/k!,$$

as $z \to +\infty$, k fixed. These formulas correspond to well-known results of the confluent hypergeometric functions. The formula for s_k can be derived direct from (1.1). The formula for y_k then follows from (3.26), (3.27), $y_k(z)\, s_k(z) \sim \tanh \alpha e^z \sim e^z$.

The formulas (3.17) and (3.26) may also be derived from (1.1) by using saddle point techniques.

The asymptotic expansion in (3.14) will be used for the numerical evaluation of $s_k(z)$ for large values of k (and small values of $|z|$). See Section 5. Formula (3.17) gives information about the rate of convergence of series with $s_k(z)$, for instance series of type (1.5).

4. Method of Computation

The recurrence relation (2.7) is an important tool for generating a sequence of values $s_k(z)$ for fixed z and $k = 0, 1, 2, \ldots, K$. If the values of $s_k(z)$ are known for two consecutive values of k, then the functions may be computed for other values of k by successive application of the recurrence relation.

In [2], Gautschi investigates the problem of numerical instability for general three-term recurrence relations. In this connection he introduces the concepts of minimal solution and dominant solution of a recurrence relation. Starting from two initial values, an application of the recurrence in the forward direction (i.e., in the direction of increasing order) yields a disastrous build-up of errors for the minimal solution, whereas the computation of the dominant solution remains numerically stable.

If the recurrence relation

$$y_{n+1} + a_n y_n + b_n y_{n-1} = 0 \tag{4.1}$$

has two linearly independent solutions f_n and g_n having the property

$$\lim_{n\to\infty} f_n/g_n = 0 \tag{4.2}$$

then f_n is called a minimal solution and g_n is called the dominant solution of (4.1). From (3.17) and (3.18) it follows

$$s_k(z)/y_k(z) \sim 2z\pi \exp[-4(zk)^{1/2}], \tag{4.3}$$

$k \to \infty$, under the conditions in (3.19). Consequently, in our case, $s_k(z)$ is a minimal solution of (2.7) and $y_k(z)$ is a dominant solution.

Gautschi's paper concentrates mainly on the development of an algorithm for the computation of minimal solutions. This algorithm is based on Miller's method which enables computation without any knowledge of starting values for large k.

To describe the algorithm for the computation of the minimal solution f_n ($n = 0, 1,..., N$) of (4.1) let

$$\sum_{m=0}^{\infty} \lambda_m f_m = t, \qquad t \neq 0, \tag{4.4}$$

$$r_n = f_{n+1}/f_n\,, \tag{4.5}$$

$$t_n = f_n^{-1} \sum_{m=n+1}^{\infty} \lambda_m f_m\,, \tag{4.6}$$

where t and $\lambda_0, \lambda_1,...$ are given quantities and the series (3.4) is known to converge. At first we suppose that r_n and t_n are known for some value $n = \nu \geqslant N$. From (4.1) and (4.5) there follows

$$r_{n-1} = -b_n/(a_n + r_n), \qquad n = \nu, \nu - 1,..., 1, \tag{4.7}$$

and from (4.6) and (4.7)

$$t_{n-1} = r_{n-1}(\lambda_n + t_n), \qquad n = \nu, \nu - 1,..., 1. \tag{4.8}$$

Hence r_n and t_n can be obtained recursively for $0 \leqslant n < \nu$; in particular we have, using (4.4)

$$t_0 = f_0^{-1}(t - \lambda_0 f_0), \tag{4.9}$$

and so

$$f_0 = t/(\lambda_0 + t_0), \tag{4.10}$$

giving the initial value of the minimal solution. The remaining values can be obtained from

$$f_n = r_{n-1} f_{n-1}, \qquad n = 1, 2, \ldots, N. \tag{4.11}$$

When the algorithm is executed with the (incorrect) starting values $r_\nu^{(\nu)} = 0$, $t_\nu^{(\nu)} = 0$, we have the following set of recursions

$$\left.\begin{aligned} r_\nu^{(\nu)} &= 0, \quad & r_{n-1}^{(\nu)} &= -b_n/(a_n + r_n^{(\nu)}) \\ t_\nu^{(\nu)} &= 0, \quad & t_{n-1}^{(\nu)} &= r_{n-1}^{(\nu)}(\lambda_n + t_n^{(\nu)}) \end{aligned}\right\} \quad n = \nu, \nu - 1, \ldots, 1, \tag{4.12}$$

$$f_0^{(\nu)} = t/(\lambda_0 + t_0^{(\nu)}), \qquad f_n^{(\nu)} = r_{n-1}^{(\nu)} f_{n-1}^{(\nu)}, \qquad n = 1, 2, \ldots, N.$$

Gautschi showed that the set of recursions (4.12) is numerically stable and that

$$\lim_{\nu\to\infty} f_n^{(\nu)} = f_n \qquad (n = 0, 1, \ldots, N) \tag{4.13}$$

if and only if

$$\lim_{\nu\to\infty} \frac{f_{\nu+1}}{g_{\nu+1}} \sum_{m=0}^{\nu} \lambda_m g_m = 0, \tag{4.14}$$

where g_n is a dominant solution of (4.1).

Under the restriction on z given in (3.29) the functions $s_k(z)$ ($k = 0, 1, \ldots, K$) can be computed with Gautschi's algorithm (4.12). For the series in (4.4) the following series can be used

$$\sum_{k=0}^{\infty} s_k(z) = 1. \tag{4.15}$$

This formula may be proved by substitution of (1.1). Hence

$$t = \lambda_n = 1, \qquad a_n = -(2n + 1 + z)/(n + 1), \qquad b_n = n/(n + 1),$$

(cf. (4.1) and (2.7)). We can choose ν so large that

$$|[f_k^{(\nu)} - s_k(z)]/s_k(z)| < \epsilon, \qquad \epsilon > 0, \quad k = 0, 1, \ldots, K, \tag{4.16}$$

if and only if condition (4.14) is fulfilled.

In our case it reads

$$\lim_{\nu\to\infty} \frac{s_{\nu+1}(z)}{y_{\nu+1}(z)} \sum_{m=0}^{\nu} y_m(z) = 0 \tag{4.17}$$

for the values of z specified in (3.19).

To verify condition (4.17) we compute the finite sum in this formula. We have

$$\sum_{m=0}^{\nu} y_m(z) = (\nu + 1)[y_{\nu+1}(z) - y_\nu(z)]/z. \tag{4.18}$$

This formula can be derived by using (2.7) and mathematical induction with respect to ν. With (4.18), (4.3), and (3.18) it is easy to prove that (4.17) holds for $|\arg z| < \pi$, $z \neq 0$, z fixed.

The positive integer ν in (4.12), which indicates the starting-point of the backward recurrence, can be chosen so that (4.16) is fulfilled. The number ν depends on ϵ, z and K; ν is large if $|z|$ is small, even $\nu \to \infty$ if $z \to 0$. This can be recognized by observing that for $z \to 0$ the series in (4.15) converges poorly. Besides, and this is the main point, the dominance of y_k over s_k becomes rather weak, as can be seen from (4.3).

Therefore, for small values of $|z|$ the algorithm becomes less attractive, and as a consequence, for small values of $|z|$ we need accurate starting values of $s_k(z)$ for two consecutive values of k.

Throughout this paper we will fix the dividing line in the z-plane for the two methods at $|z| = 1$. An optimal choice of a boundary may be found by numerical methods. From our experience, $|z| = 1$ is a convenient choice.

5. The Computation for $|z| < 1$, $z \neq 0$

The algorithm described in (4.12) provides us with a numerical procedure for the computation of $s_k(z)$ ($k = 0, 1, \ldots, K$) which converges and which is numerically stable for every z in the z-plane satisfying $|\arg z| < \pi$, $z \neq 0$. However for small values of $|z|$, the number ν may be considerably large when (4.16) has to be satisfied and therefore the procedure converges slowly, unless correct values of $r_\nu^{(\nu)}$ and $t_\nu^{(\nu)}$ are substituted in (4.12). In that case we need two starting values $s_\nu(z)$ and $s_{\nu+1}(z)$, $\nu \geqslant N$.

The series-representation (2.6) converges for all finite values of z. However, when k is a large integer cancellation of significant digits occurs when the series is summed numerically. Besides, convergence is rather poor when k is large.

Since $s_\nu(z)$ has to be evaluated for large values of ν it is more attractive to use an asymptotic expansion of $s_\nu(z)$ valid for large values of ν, while small values of $|z|$ do not invalidate the approximation. In Section 3 we derived an expansion satisfying these requirements, namely (3.14). This expansion gives an excellent approximation for $s_k(z)$ for large k and fixed values of $|z|$, while the approximation improves as $|z|$ becomes smaller.

The first few values of the polynomials A_n and B_n are

$$
\begin{aligned}
A_0(t) &= 1,\\
A_1(t) &= t^2(t^4 - 12)/72,\\
A_2(t) &= t^4(5t^8 - 1128t^4 + 27216)/1\,55520,\\
A_3(t) &= t^2(35t^{16} - 31500t^{12} + 58\,59216t^8 - 2067\,63840t^4\\
&\qquad + 5486\,74560)/11757\,31200,\\
A_4(t) &= t^4(5t^{20} - 11472^{16} + 70\,68384t^{12} - 13282\,03008t^8\\
&\qquad + 6\,51177\,79200t^4 - 49\,98537\,21600)/33\,86105\,85600,\\
B_0(t) &= t^3/6,\\
B_1(t) &= t(5t^8 - 432t^4 + 2160)/6480,\\
B_2(t) &= t^3(7t^{12} - 3528t^8 + 2\,98224t^4 - 30\,48192)/65\,31840,\\
B_3(t) &= t(5t^{20} - 7560t^{16} + 27\,76896t^{12} - 2642\,28480t^8\\
&\qquad + 47953\,03680t^4 - 65840\,94720)/70543\,87200.
\end{aligned}
$$

By substitution of polynomial representations of A_n and B_n in (3.11), recurrence relations for the coefficients of these polynomials may be derived in order to compute A_n and B_n for other values of n.

Some remarks on the computation of the modified Bessel functions $K_0(\zeta)$ and $K_1(\zeta)$ will follow in Section 7.

6. The Estimation of ν

In this section we give an estimate of the starting value ν to be used in algorithm (4.12), given the relative accuracy desired. Gautschi [2] obtained an approximation for the relative error which in our notation reads as follows (see Gautschi's discussions around 3.18 and 5.11 in [2])

$$[f_k^{(\nu)} - s_k(z)]/s_k(z) \sim s_{\nu+1}(z) - [s_{\nu+1}(z)\, y_k(z)/(y_{\nu+1}(z)\, s_k(z))] \tag{6.1}$$

ν large, $k = 0, 1, \ldots, K$.

Our aim is to determine ν such that

$$|[f_k^{(\nu)} - s_k(z)]/s_k(z)| < \epsilon \tag{6.2}$$

holds for $k = 0, 1, \ldots, K$, $|z| \geqslant 1$, $|\arg z| < \pi$, where ϵ is the relative accuracy desired. Since $|y_k(z)/s_k(z)|$ ultimately grows rapidly with k, see (4.3), it is plausible to expect that when (6.2) holds for $k = K$ it will also be valid when $k < K$, particularly when K is large. We therefore consider the simplified problem of bounding

$$|s_{\nu+1}(z) - [s_{\nu+1}(z)\, y_k(z)]/[y_{\nu+1}(z)\, s_k(z)]|. \tag{6.3}$$

We assume K, and thus ν, so large that the functions in (6.3) may be replaced by approximations of these functions holding for large values of ν and K. The asymptotic expressions in (3.17) and (3.18) are not suitable for large $|z|$, therefore it is necessary to use the more intricate formulas (3.26) and (3.27).

Applications of (3.26) and (3.27) to (6.3) give (if a few unimportant coefficients have been omitted)

$$|[f_k^{(\nu)} - s_k(z)]/s_k(z)| \leqslant \exp \operatorname{Re}\{\tfrac{1}{2}z - \nu f(\alpha)\} + \exp 2\operatorname{Re}\{Kf(\beta) - \nu f(\alpha)\}, \tag{6.4}$$

$$f(\alpha) = 2\alpha + \sinh 2\alpha,$$

$$\sinh^2 \alpha = z/4\nu, \qquad \sinh \alpha > 0 \qquad \text{if} \quad z > 0,$$

$$\sinh^2 \beta = z/4K, \qquad \sinh \beta > 0 \qquad \text{if} \quad z > 0.$$

The positive integer ν can be chosen so large (if z, K, and ϵ are known) that simultaneously

$$\exp \operatorname{Re}\{\tfrac{1}{2}z - \nu f(\alpha)\} < \tfrac{1}{2}\epsilon, \qquad \exp 2 \operatorname{Re}\{Kf(\beta) - \nu f(\alpha)\} < \tfrac{1}{2}\epsilon. \tag{6.5}$$

The proper value of ν can be found by inverting the function $\operatorname{Re}\{\nu f(\alpha)\}$. Some properties of this function will now be given.

Let

$$\sinh^2 \alpha = z/4\nu = re^{i\theta}/4\nu, \qquad r > 0, \quad -\pi < \theta < \pi,$$

$$\alpha = \gamma + i\delta, \qquad \gamma > 0, \quad -\pi/2 < \delta < \pi/2.$$

Then, by eliminating δ,

$$\operatorname{Re}\{\nu f(\alpha)\} = \frac{r}{2 \sinh^2 2\gamma} [2\gamma\{\cosh 2\gamma + \cos \theta\} + \sinh 2\gamma\{1 + \cos \theta \cosh 2\gamma\}] \tag{6.6}$$

and

$$\nu = [r/2 \sinh^2 2\gamma]\{\cosh 2\gamma + \cos \theta\}.$$

Denoting (6.6) by $\phi(\gamma)$ then

$$\lim_{\gamma\to 0}\phi(\gamma) = \infty, \qquad \lim_{\gamma\to\infty}\phi(\gamma) = \tfrac{1}{2}r\cos\theta$$

and

$$\phi'(\gamma) = -2\gamma r\{\cosh^2 2\gamma + 2\cos\theta\cosh 2\gamma + 1\}/\sinh^3 2\gamma.$$

Thus, $\phi(\gamma)$ is a monotone decreasing function of γ and the equation

$$\phi(\gamma) = p \tag{6.8}$$

has a unique solution $\gamma = \phi^{-1}(p)$ for all $p > \frac{1}{2}r\cos\theta$.

The inequalities in (6.5) are equivalent with $\phi(\gamma) > \frac{1}{2}r\cos\theta + \ln(2/\epsilon)$, $\phi(\gamma) > \phi(\lambda) + \frac{1}{2}\ln(2/\epsilon)$ where λ is implicitly given by (cf. (6.7))

$$K = \frac{r}{2\sinh^2 2\lambda}\{\cosh 2\lambda + \cos\theta\}. \tag{6.9}$$

This equation may be inverted to give λ explicitly, viz.

$$\lambda = \tfrac{1}{2}\operatorname{arc\,cosh}[y + (y^2 + 2y\cos\theta + 1)^{1/2}], \tag{6.10}$$

where

$$y = r/(4K).$$

If we set

$$\gamma = \phi^{-1}\{\max(\tfrac{1}{2}r\cos\theta + \ln(2/\epsilon), \phi(\lambda) + \tfrac{1}{2}\ln(2/\epsilon))\} \tag{6.11}$$

then the number ν given by (6.7) may be used for starting the algorithm described in (4.12).

For real values of z approximations of ϕ^{-1} are easily obtained. By inverting (6.8) for large values of p we have

$$\phi^{-1}(p) \sim \frac{r}{p}\left(1 + \frac{1}{45}\left(\frac{r}{p}\right)^4\right), \qquad p/r \to \infty, \tag{6.12}$$

and by inverting for values of p close to $\frac{1}{2}z$ we obtain

$$\phi^{-1}(p) \sim \gamma_0\{1 + 2/[1 + \gamma_0(q + 1/q)]\}, \qquad q \to 1, \quad q > 1, \tag{6.13}$$

where $q = 2p/r$, $\gamma_0 = \frac{1}{2}\ln[(q+1)/(q-1)]$. Using (6.12) for $q \geqslant 2$ and (6.13) for $1 < q < 2$, we have a suitable approximation of $\phi^{-1}(p)$ for all $p > 1$.

For real values of z ($\theta = 0$, $z = r$) we will give the successive steps in the computation of ν. The three quantities z, K, and ϵ are given:

(1) compute λ from (6.10): $\lambda = \ln[y^{1/2} + (y+1)^{1/2}]$; $y = z/4K$;

(2) compute $\phi(\lambda) = z(2\lambda + \sinh 2\lambda)/4\sinh^2\lambda$;

(3) compute $p = \max\{\frac{1}{2}z + \ln(2/\epsilon), \phi(\lambda) + \frac{1}{2}\ln(2/\epsilon)\}$;

(4) compute $\gamma = \phi^{-1}(p)$ from (6.12) or (6.13);

(5) compute ν from (6.7).

The estimated value of ν can be compared with the smallest value of ν empirically found, for a given set of values z, K, and ϵ. Empirical values were found by running algorithm (4.12) with $\nu = K + 15$, $K + 20$, $K + 25, \ldots$ until for the first time the $K + 1$ values $f_k^{(\nu)}$, ($k = 0, 1, \ldots, K$) agreed within a relative accuracy of ϵ with the respective values of $f_k^{(\nu-5)}$.

In Table I we give some values of the starting number ν for $\epsilon = 10^{-10}$.

TABLE I

z \ K	15	30	50	80	100
1.0	155(147)	155(146)	160(168)	210(221)	245(253)
3.0	60(53)	80(79)	110(110)	150(152)	180(180)
5.0	45(42)	65(66)	95(94)	135(134)	160(160)
10.0	35(33)	55(54)	80(80)	115(117)	140(141)
25.0	30(26)	45(44)	70(68)	105(102)	125(125)
50.0	30(22)	45(40)	65(63)	95(96)	120(117)
80.0	30(21)	45(38)	65(60)	95(92)	115(114)

The empirical values may be compared with the values between brackets which are obtained from the asymptotic relations. For small values of K the estimated values appear to be less accurate.

If $|z| < 1$ the choice of ν depends on the number of terms used in the asymptotic series (3.14). We used this expansion with $n = 4$ for the A-series and with $n = 3$ for the B-series and $\nu \geqslant \max(K, 100)$. In this way we found nine correct significant digits in $s_k(z)$, $k = 0, 1, \ldots, K$.

7. Generalizations

When instead of (1.2) integrals of the type

$$f(z) = z\int_0^\infty t^\lambda e^{-zt} F(t)\, dt$$

are considered, $-1 < \operatorname{Re} \alpha < 0$, the analysis proceeds in the same manner. The function $F(t)$ is expanded as in (1.4) and the functions $s_k^{(\alpha)}$, now depending on α, can be written as

$$\begin{aligned} s_k^{(\alpha)}(z) &= z \int_0^\infty e^{-zt} t^{k+\alpha} (1+t)^{-k-1}\, dt \\ &= z\Gamma(k+\alpha+1)\, U(k+\alpha+1, \alpha+1, z). \end{aligned}$$

These functions may be computed by using analogous methods as described in Sections 4 and 5. In fact it is possible to evaluate the whole class of hypergeometric functions $u_k = U(a+k, b, z)$, $k = 0, 1,\ldots$, in this way. The asymptotic properties of u_k and s_k are not essentially different, since it suffices to consider $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$. (For $n = 0, 1, 2,\ldots$, the functions $U(a, b+n, z)$ can be computed with starting values for $n = 0, 1$; in this case computation in the forward direction is numerically stable). The asymptotic series for u_k (analogous to the representation of s_k in (3.14)) follows from Olver's paper. The coefficients A_n and B_n will depend on a and b. The Bessel functions have to be replaced by K_{b-1} and K_b.

The Bessel function $K_a(z)$ may also be computed by the methods of Section 4, at least if $|z|$ is not too small, say $|z| \geqslant 1$. Namely, we can write $K_a(z)$ as a confluent hypergeometric function,

$$K_a(z) = \pi^{1/2}(2z)^a e^{-z} U(a + \tfrac{1}{2}, 2a+1, 2z).$$

For small values of $|z|$ the calculation can be attempted by using the well-known formula

$$K_a(z) = (\pi/2 \sin a\pi)[I_{-a}(z) - I_a(z)].$$

In [5] we worked out some new ideas on the numerical evaluation of the Bessel function $K_a(z)$.

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Uniform Asymptotic Expansions of the Incomplete Gamma Functions and the Incomplete Beta Function

By N. M. Temme

Abstract. New asymptotic expansions are derived for the incomplete gamma functions and the incomplete beta function. In each case the expansion contains the complementary error function and an asymptotic series. The expansions are uniformly valid with respect to certain domains of the parameters.

1. Introduction. The *incomplete gamma functions* are defined by

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt, \quad \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1}\, dt. \tag{1.1}$$

The parameters may be complex; but here we suppose a and x to be real, where $a > 0$ and $x \geqslant 0$. Auxiliary functions are

$$P(a, x) = \gamma(a, x)/\Gamma(a), \quad Q(a, x) = \Gamma(a, x)/\Gamma(a), \tag{1.2}$$

and from the definitions it follows that

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a), \quad P(a, x) + Q(a, x) = 1. \tag{1.3}$$

For large values of x we have the well-known asymptotic expansion,

$$\Gamma(a, x) \sim x^{a-1} e^{-x} \{1 + (a-1)/x + (a-1)(a-2)/x^2 + \ldots\}.$$

See for instance Dingle [1] or Olver [3]. If both x and a are large, this expansion is not useful, unless $a = o(x)$. For large values of a, we can better use the function $\gamma(a, x)$. From (1.1) we obtain the elementary result

$$\gamma(a, x) = e^{-x} x^a \Gamma(a) \sum_{n=0}^{\infty} x^n/\Gamma(a + n + 1).$$

This series converges for every finite x. It is useful for $a \longrightarrow \infty$ and $x = o(a)$, since under this condition the series has an asymptotic character.

Expansions with a more uniform character are given by Tricomi [4], who found among others

$$\begin{aligned}\gamma(a + 1, a + y(2a)^{1/2})/\Gamma(a + 1) = \tfrac{1}{2} \operatorname{erfc}(-y) - \tfrac{1}{3}(2/a\pi)^{1/2}(1 + y^2)\exp(-y^2)\\ + O(a^{-1}), \quad y, a \text{ real}, a \longrightarrow +\infty.\end{aligned} \tag{1.4}$$

This expansion is uniformly valid in y on compact intervals of $\mathbf{R}$. The function erfc is the *complementary error function* defined by

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(1.5) $$\operatorname{erfc}(x) = 2\pi^{-1/2}\int_x^\infty e^{-t^2}\,dt.$$

It is a special case of $\Gamma(a, x)$, namely $\operatorname{erfc}(x) = \pi^{-1/2}\Gamma(\tfrac12, x^2)$. Some results of Tricomi are corrected and used by Kölbig [2] for the construction of approximations of the zeros of the incomplete gamma function $\gamma(a, x)$.

An important book with many results on asymptotic expansions of the incomplete gamma functions is the recent treatise of Dingle [1]. Apart from elementary expansions, Dingle gives also uniform expansions and, in particular, he generalizes the results of Tricomi (p. 249 of [1]). Dingle does not specify the term "uniform", but it can be verified that the same restrictions on y must hold as for (1.4).

In Section 2 we give new asymptotic expansions for $\gamma(a, x)$ and $\Gamma(a, x)$, holding uniformly in $0 \leqslant x/a$ for $a \longrightarrow \infty$ and/or $x \longrightarrow \infty$. In Section 3 an analogous result for the incomplete beta function is given.

A recent result of Wong [5] may be connected with our results. Wong considers integrals of which the endpoint is near by a saddle point of the integrand, and he applies his methods to the function $S_n(x)$ defined by

$$e^{nx} = \sum_{r=0}^{n} (nx)^r/r! + (nx)^n S_n(x)/n!.$$

The function S_n is a special case of the incomplete gamma function and the asymptotic expansion of $S_n(x)$ for $n \longrightarrow \infty$, $x \sim 1$ is expressed by Wong in terms of the error function. (Wong interpreted his results only for $0 \leqslant x \leqslant 1$, but not across the transition point at $x = 1$.)

2. Uniform Asymptotic Expansions. The integrals (1.1) are not attractive for deriving uniform expansions. Therefore, we write P as

(2.1) $$P(a, x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xs}s^{-1}(s+1)^{-a}\,ds, \qquad c > 0,$$

in which $(s + 1)^{-a}$ will have its principal value which is real for $s > -1$. Formula (2.1) can be found in Dingle's book. Here we derive it by observing that the Laplace transform of $dP(a, x)/dx$ is $(s + 1)^{-a}$, from which it follows that $(s + 1)^{-a}s^{-1} = L(P(a, \cdot))$, which can be inverted to obtain (2.1). Taking into account the residue at $s = 0$, the contour in (2.1) can be shifted to the left of the origin; and so a similar integral for Q can be given. With (1.3) and some further modifications we arrive at

(2.2) $$Q(a, x) = \frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{a\phi(t)}\frac{dt}{\lambda - t}, \qquad 0 < c < \lambda,$$

where

(2.3) $$\phi(t) = t - 1 - \ln t, \qquad \lambda = x/a.$$

The contour in (2.2) will be deformed into a path in the s-plane which crosses the saddle point of the integrand. The saddle point t_0 follows from $\phi'(t_0) = 0$. Hence $t_0 = 1$, $\phi(t_0) = \phi'(t_0) = 0$ and $\phi''(t_0) = 1$.

The steepest descent path follows from $\operatorname{Im}\phi(t) = \operatorname{Im}\phi(t_0) = 0$, and, by writing $t = \sigma + i\tau$ $(\sigma, \tau \in \mathbf{R})$ we obtain

(2.4) $$\sigma = \tau \operatorname{ctg} \tau, \qquad -\pi < \tau < \pi.$$

Let temporarily $\lambda > 1$, that is $x > a$. Then the contour in (2.2) may be shifted into the contour L in the t-plane defined by (2.4). According to Cauchy's theorem, the integral in (2.2) remains unaltered; and on L, the values of $\phi(t)$ are real and negative. Next we define the mapping of the t-plane into the u-plane by the equation,

$$-\tfrac{1}{2}u^2 = \phi(t), \tag{2.5}$$

with the condition $t \in L$ corresponds with $u \in \mathbf{R}$, and $u < 0$ if $\tau < 0$, $u > 0$ if $\tau > 0$.

The result is

$$Q(a, x) = \frac{e^{-a\phi(\lambda)}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} \frac{dt}{du} \frac{du}{\lambda - t}, \qquad \lambda > 1. \tag{2.6}$$

The presence of the pole at $t = \lambda$ in the integrand of (2.6) is somewhat disturbing, but we will get rid of it by writing

$$\frac{dt}{du}\frac{1}{\lambda - t} = \frac{dt}{du}\frac{1}{\lambda - t} + \frac{1}{u - u_1} - \frac{1}{u - u_1}, \tag{2.7}$$

where u_1 is the point in the u-plane corresponding to the point $t = \lambda$ in the t-plane. That is, $-\tfrac{1}{2}u_1^2 = \phi(\lambda)$, hence $u_1 = \pm i\phi(\lambda)^{\frac{1}{2}}$. There still is an ambiguity in the sign. However, the correct sign follows from the conditions imposed on the mapping defined in (2.5). In fact, we have

$$u_1 = i(1 - \lambda)\{2(\lambda - 1 - \ln \lambda)/(1 - \lambda)^2\}^{\frac{1}{2}}, \tag{2.8}$$

where the square root is positive for positive values of the argument. The first two terms at the right-hand side of (2.7) constitute a regular function at $t = \lambda$, and with this partition we obtain

$$Q(a, x) = -\frac{e^{-a\phi(\lambda)}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} \frac{du}{u - u_1} + R(a, x), \tag{2.9}$$

$$R(a, x) = \frac{e^{-a\phi(\lambda)}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} \left\{\frac{dt}{du}\frac{1}{\lambda - t} + \frac{1}{u - u_1}\right\} du, \tag{2.10}$$

and the integral in (2.9) can be expressed in terms of the complementary error function defined in (1.5), so that

$$Q(a, x) = \tfrac{1}{2}\operatorname{erfc}(a^{\frac{1}{2}}\zeta) + R(a, x), \qquad \zeta = iu_1 2^{-\frac{1}{2}}. \tag{2.11}$$

From $\operatorname{erfc}(x) + \operatorname{erfc}(-x) = 2$ it follows that

$$P(a, x) = \tfrac{1}{2}\operatorname{erfc}(-a^{\frac{1}{2}}\zeta) - R(a, x). \tag{2.12}$$

So far, the results in (2.11) and (2.12) are exact, since no approximations were used. In order to obtain asymptotic expansions for $P(a, x)$ and $Q(a, x)$, the function $R(a, x)$ will be expanded in an asymptotic series. The integrand of $R(a, x)$ is a holomorphic function in the finite u-plane for every $\lambda \geqslant 0$. If we put the expansion,

$$\frac{dt}{du}\frac{1}{\lambda - t} + \frac{1}{u - u_1} = \sum_{k=0}^{\infty} c_k(\lambda) u^k, \tag{2.13}$$

in (2.10), and, if we reverse the order of summation and integration, by Watson's lemma [3], we obtain the expansion

$$R(a, x) \sim \frac{e^{-a\phi(\lambda)}}{2\pi i} \sum_{k=0}^{\infty} c_{2k}(\lambda)\Gamma(k + \tfrac{1}{2})(\tfrac{1}{2}a)^{-k-\frac{1}{2}}. \tag{2.14}$$

(The conditions for Watson's lemma are certainly satisfied since the function in (2.13) is bounded for large real values of u.)

Each coefficient $c_k(\lambda)$ is an analytic function of λ near $\lambda = 1$, and the expansion (2.14) is not only valid near $\lambda = 1$ but for all $\lambda \geqslant 0$. That is to say, we can fix x and let a tend to infinity, or conversely. Also, x and a may grow dependently or independently of each other.

The first few coefficients are

$$c_0(\lambda) = \frac{i}{\lambda - 1} - \frac{1}{u_1}, \qquad c_0(1) = -\frac{i}{3},$$

$$c_2(\lambda) = \frac{-i(\lambda^2 + 10\lambda + 1)}{12(\lambda - 1)^3} - \frac{1}{u_1^3}, \qquad c_2(1) = -\frac{i}{540}.$$

Our expansion is more powerful than those of Tricomi and Dingle. Tricomi's formula (1.4) follows from our expansion by expanding (2.12) for small values of $1 - \lambda$. Moreover, for the complete expansion, Tricomi and Dingle obtained an infinite series, of which each term contains functions related to the error function. In our expansion, the information about the nonuniform behavior of the incomplete gamma functions is contained in just one error function. Besides, we obtain expansions for both P and Q. Of course, the coefficients $c_{2k}(\lambda)$ in (2.14) are more complicated than the coefficients in the other expansions.

As remarked before, the expansion (2.15) is also valid for fixed a and $x \longrightarrow \infty$, in spite of the nature of the series containing terms with negative powers of a. The coefficients, however, depend on x and a; and, in fact, we can say that the sequence $\{d_k\}$, $d_k = c_{2k}(\lambda)a^{-k}$, is an asymptotic sequence. That is, $d_{k+1} = o(d_k)$ if one (or both) of the parameters a and x is (are) large uniformly in $x/a \geqslant 0$.

3. The Incomplete Beta Function. The *incomplete beta function* is defined by

$$I_x(p, q) = \frac{1}{B(p, q)} \int_0^x t^{p-1}(1 - t)^{q-1}\, dt \tag{3.1}$$

with $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, $0 \leqslant x \leqslant 1$, and

$$B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p + q). \tag{3.2}$$

The function in (3.2) is called the *beta function*. Again, we consider real variables x, p and q, and we will derive an asymptotic expansion of $I_x(p, q)$ for large p and q uniformly valid for $0 < \delta \leqslant x \leqslant 1$.

We first give an integral representation of I_x which resembles those for the incomplete gamma function. Formula (3.1) is equivalent to

$$I_x(p, q) = \frac{1}{B(p, q)} \int_{-\ln x}^{\infty} e^{-pt}(1 - e^{-t})^{q-1}\, dt; \tag{3.3}$$

and also, we have

$$B(p, q) = \int_0^{\infty} e^{-pt}(1 - e^{-t})^{q-1}\, dt, \tag{3.4}$$

from which follows, by using the same technique as in the foregoing section,

$$I_x(p,q) = \frac{1}{2\pi i B(p,q)} \int_{c-i\infty}^{c+i\infty} e^{(p-s)\ln x} \frac{B(s,q)}{p-s}\, ds, \qquad 0 < c < p.$$

This expression can be written as

$$(3.5) \qquad I_x(p,q) = \frac{x^p(1-x)^q}{2\pi i} \frac{\Gamma(q)e^q q^{-q}}{B(p,q)} \int_{c-i\infty}^{c+i\infty} e^{q\psi(t)} F_q(t) \frac{dt}{t_1 - t},$$

with $\psi(t) = t\ln(t/x) - (1+t)\ln(1+t) - \ln(1-x)$,

$$(3.6) \quad F_q(t) = \frac{\Gamma(qt)e^{qt}(qt)^{-qt}}{\Gamma(q+qt)e^{q(1+t)}(q+qt)^{-q(1+t)}}, \qquad t_1 = p/q, \text{ and } 0 < c < t_1.$$

$F_q(t)$ is a slowly varying function as $q \to \infty$, on compact subsets of $|\arg t| < \pi$, $t \neq 0$. Of course, its construction is based on the Stirling approximation of the gamma function. For $|\arg t| < \pi$, $t \neq 0$, we have

$$(3.7) \qquad F_q(t) = \{(1+t)/t\}^{1/2}(1 + O(q^{-1})), \qquad q \to \infty, t \text{ fixed}.$$

With

$$(3.8) \qquad t_0 = x/(1+x), \qquad x_0 = p/(p+q),$$

t_0 is a saddle point of ψ, and if $x = x_0$, this saddle point coincides with the pole at t_1.

The calculation of the saddle point t_0 is based on the assumption that the gamma functions in $F_q(t)$ in (3.6) have large arguments. Hence, for small values of x, which correspond to small values of t_0, the calculation is based on false assumptions. Therefore we only consider positive values of x. It is not necessary to have a uniform bound from zero of x. We are even allowing those values of x with $qx \to \infty$.

From now on, details will be omitted, since the method is exactly the same as the one used in the foregoing section. We put $-\frac{1}{2}u^2 = \psi(t)$ and the results are

$$(3.9) \qquad I_x(p,q) = \tfrac{1}{2}\,\mathrm{erfc}(-(q/2)^{1/2}\eta) + S_x(p,q),$$

$$(3.10) \qquad \eta = (x - x_0)[2q^{-1}\{p\ln(x_0/x) + q\ln((1-x_0)/(1-x))\}/(x-x_0)^2]^{1/2}.$$

The square root is positive for positive values of its argument. The function S_x is defined by

$$(3.11) \qquad S_x(p,q) = \frac{x^p(1-x)^q\Gamma(q)e^{-q}q^q}{2\pi i B(p,q)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}qu^2} G(u)\,du,$$

$$(3.12) \qquad G(u) = \frac{F_q(t)}{t_1 - t}\frac{dt}{du} + \frac{F_q(t_1)}{u - u_1}.$$

For u_1 we have $-\frac{1}{2}u_1^2 = \psi(t_1)$, $u_1 = i\eta$.

The role of the parameter λ of the foregoing section is now played by $(x - x_0)$. We have

$$\eta = -\frac{p+q}{q}\left(\frac{p+q}{p}\right)^{1/2}(x - x_0)\left\{1 + \frac{1}{3}\frac{p^2 - q^2}{pq}(x - x_0) + O(x - x_0)^2\right\}$$

for $x \longrightarrow x_0$. An asymptotic expansion of S_x is obtained by expanding $G(u) = \Sigma\, d_k u^k$, giving

$$(3.13)\qquad S_x(p, q) \sim \frac{x^p(1-x)^q}{2\pi i}\frac{\Gamma(p+q)}{\Gamma(p)}\, e^q q^{-q} \sum_{k=0}^{\infty} d_{2k}\Gamma(k+\tfrac{1}{2})(\tfrac{1}{2}q)^{-k-\frac{1}{2}},$$

$$(3.14)\qquad d_0 = i\frac{F_q(t_0)}{t_1 - t_0}\frac{x^{\frac{1}{2}}}{1-x} - \frac{F_q(t_1)}{u_1}.$$

The expansion holds for $p \longrightarrow \infty$ and/or $q \longrightarrow \infty$, uniformly in $\delta \leqslant x \leqslant 1$, where δ may depend on q, such that $q\delta \longrightarrow \infty$.

A more transparent first approximation for $S_x(p, q)$ is obtained by replacing the functions F_q in (3.14) by the approximation (3.7). The result is

$$S_x(p, q) = \{p/[2\pi q(p+q)]\}^{\frac{1}{2}}(x/x_0)^p\{(1-x)/(1-x_0)\}^q$$

$$\cdot\, \{(1-x_0)/(x_0-x) + \eta^{-1}x_0^{-\frac{1}{2}}\}(1 + O(q^{-1})), \qquad q \longrightarrow \infty.$$

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On the Numerical Evaluation of the Modified Bessel Function of the Third Kind

N. M. TEMME

I. *Introduction*

I.1. *Definitions and relevant properties.* The modified Bessel function of the third kind can be defined by the integral

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t} \cosh \nu t \, dt, \qquad \operatorname{Re} z > 0. \tag{1.1}$$

Its definition can also be given by using the modified Bessel function of the first kind,

$$I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{2}z)^{2k}}{\Gamma(\nu + k + 1)\, k!}. \tag{1.2}$$

In terms of this function we have

$$K_\nu(z) = \tfrac{1}{2}\pi[(I_{-\nu}(z) - I_\nu(z))/\sin \nu\pi]. \tag{1.3}$$

Since $I_{-n}(z) = I_n(z)$, $n = 0, 1, 2,\ldots$, the right-hand side of (1.3) appears in indeterminate form if $\nu = n$. However, the limit of this form as $\nu \to n$ exists and agrees with $K_n(z)$ given in (1.1). Clearly we have

$$K_\nu(z) = K_{-\nu}(z). \tag{1.4}$$

Furthermore, if z and ν are real, $z > 0$,

$$K_\nu(z) > 0, \qquad K_\nu'(z) < 0. \tag{1.5}$$

The Bessel functions of half-integral order can be expressed in terms of elementary functions. For $\nu = \frac{1}{2}, \frac{3}{2}$ we have

$$K_{1/2}(z) = (\pi/2z)^{1/2}\, e^{-z}, \qquad K_{3/2}(z) = (\pi/2z)^{1/2}\, e^{-z}(1 + 1/z). \tag{1.6}$$

The functions $I_\nu(z)$ and $e^{i\nu\pi}K_\nu(z)$ are two solutions of the difference equation

$$y_{\nu+1} + (2\nu/z)y_\nu - y_{\nu-1} = 0. \tag{1.7}$$

Explicitly, we have

$$I_{\nu+1}(z) + (2\nu/z)\, I_\nu(z) - I_{\nu-1}(z) = 0, \tag{1.8}$$

$$K_{\nu+1}(z) - (2\nu/z)\, K_\nu(z) - K_{\nu-1}(z) = 0. \tag{1.9}$$

Formula (1.9) can be used to compute $K_{\nu+n}$ for $n = 2, 3,\ldots$ when K_ν and $K_{\nu+1}$ are given. In the forward direction the recurrence formula for K_ν is numerically stable (see Gautschi [5]).

From (1.2) and (1.3) the following asymptotic formulas are obtained. For small $|z|$ we have

$$I_\nu(z) \sim (z/2)^\nu/\Gamma(\nu+1), \qquad K_\nu(z) \sim \tfrac{1}{2}(z/2)^{-\nu}\,\Gamma(\nu), \qquad \operatorname{Re}\nu > 0. \tag{1.10}$$

These formulas also hold for the case that z is fixed and $\nu \to \infty$. Hence, I_ν and $e^{i\nu\pi}K_\nu$ are two linearly independent solutions of the difference equation (1.7).

When ν is fixed and $z \to \infty$ we have the well-known expansions

$$I_\nu(z) = (2\pi z)^{-1/2}\, e^z[1 + O(z^{-1})], \qquad K_\nu(z) = (\pi/2z)^{1/2}\, e^{-z}[1 + O(z^{-1})], \tag{1.11}$$

the first relation holding for $|\arg z| < \tfrac{1}{2}\pi$, and the second one for $|\arg z| < 3\pi/2$.

I.2. *Contents of the paper.* We give algorithms for the computation of $K_\nu(z)$ and $K_{\nu+1}(z)$. On account of (1.4) and (1.9) and the stability of (1.9) it suffices to consider values of ν with $-\tfrac{1}{2} \leqslant \operatorname{Re}\nu \leqslant \tfrac{1}{2}$. In Section II we describe an algorithm for the computation of $K_\nu(z)$ for small values of $|z|$. This algorithm is based on representations (1.3) and (1.2). Also, the evaluation of the gamma function is discussed; some special approximations of this function are needed in the algorithm for small $|z|$.

Section III is devoted to the computation of $K_\nu(z)$ for moderate or large values of $|z|$. In this case the algorithm is based on a combination of algorithms due to J. C. P. Miller and F. W. J. Olver.

In Section IV the algorithms are described in terms of ALGOL 60 procedures.

There is a vast literature concerning the computation of this Bessel function (see for example Luke [8, 9]), especially for large values of $|z|$, whereas the computation for small values is rather neglected. Moreover, the methods are usually restricted to $K_\nu(z)$ for integer values of ν. The algorithms described in this paper may also be used for the class of confluent hypergeometric functions denoted by $U(a, b, z)$.

For the computation of the Bessel function $I_\nu(z)$ the reader is referred to Amos [2] and Gautschi [5, 6].

Thanks are due to Gert Jan Laan, who tested the ALGOL 60 procedures.

II. *The Computation for Small Values of* $|z|$

II.1. *Series representations.* Substitution of (1.2) into (1.3) leads to

$$K_\nu(z) = \sum_{k=0}^{\infty} c_k f_k, \tag{2.1}$$

$$f_0 = \frac{\pi}{2\sin\nu\pi}\{(z/2)^{-\nu}/\Gamma(1-\nu) - (z/2)^{\nu}/\Gamma(1+\nu)\}, \tag{2.2}$$

and for general k,

$$f_k = \frac{\pi}{2\sin\nu\pi}\{(z/2)^{-\nu}/\Gamma(k+1-\nu) - (z/2)^{\nu}/\Gamma(k+1+\nu)\}, \tag{2.3}$$

$$c_k = (z^2/4)^k/k!\,. \tag{2.4}$$

By using the well-known property of the gamma function $\Gamma(z+1) = z\Gamma(z)$ we have for $k = 1, 2, 3,...$ the recurrence relations

$$f_k = (kf_{k-1} + p_{k-1} + q_{k-1})/(k^2 - \nu^2), \tag{2.5}$$

$$p_0 = \tfrac{1}{2}(z/2)^{-\nu}\,\Gamma(1+\nu), \qquad p_k = p_{k-1}/(k-\nu), \tag{2.6}$$

$$q_0 = \tfrac{1}{2}(z/2)^{\nu}\,\Gamma(1-\nu), \qquad q_k = q_{k-1}/(k+\nu). \tag{2.7}$$

In order to compute $K_{\nu+1}(z)$ we write (using (1.3) and (1.8))

$$K_{\nu+1}(z) = \frac{2}{z}\frac{\nu\pi}{2\sin\nu\pi}I_{-\nu}(z) + \frac{\pi}{2\sin\nu\pi}(I_{\nu+1}(z) - I_{-\nu+1}(z)). \tag{2.8}$$

By substitution of (1.2) we obtain

$$K_{\nu+1}(z) = \frac{2}{z}\sum_{k=0}^{\infty} c_k(p_k - kf_k). \tag{2.9}$$

If an algorithm for the gamma function is available, f_0, p_0, and q_0 can be computed, and the remaining values f_k, p_k, and q_k can be obtained by recursion.

It should be pointed out that we wish to compute K_ν, $K_{\nu+1}$ for $-\frac{1}{2} \leqslant \operatorname{Re}\nu \leqslant \frac{1}{2}$, and inspection of (2.2) shows that, if $\nu \to 0$, an indeterminate form appears in f_0 (and in all f_k, but by using (2.5) only f_0 has to be considered). Analytically, f_0 can be defined in the limit $\nu = 0$. However, for small $|\nu|$, numerical evaluation of f_0 from representation (2.2) will cause a loss of correct significant digits. If $|\nu|$ is small, f_0 might be expanded in a series $\sum a_k(z)\,\nu^k$; in fact this method is suggested by Goldstein and Thaler [7]. This series converges for $|\nu| < 1$ (because of the singularity at $\nu = 1$), but the coefficients $a_k(z)$ are not easily obtained. Moreover, for small $|z|$ convergence of the series is rather poor.

In order to avoid these troubles we propose the following representation of f_0.

$$f_0 = \frac{\nu\pi}{\sin \nu\pi} [\Gamma_1(\nu) \cosh \mu + \Gamma_2(\nu) \ln(2/z) \sinh(\mu)/\mu], \tag{2.10}$$

where

$$\begin{aligned} \Gamma_1(\nu) &= [1/\Gamma(1 - \nu) - 1/\Gamma(1 + \nu)]/(2\nu), \\ \Gamma_2(\nu) &= [1/\Gamma(1 - \nu) + 1/\Gamma(1 + \nu)]/2, \end{aligned} \tag{2.11}$$

and $\mu = \nu \ln(2/z)$. The cancellation for small ν may now occur in Γ_1 and sinh but for these functions the cancellation is better controlled than in f_0. For the computation of Γ_1 and Γ_2 the reader is referred to Subsection II.3.

II.2. *Stability of computation.* If $|z|$ is not too large, the series in (2.1) and (2.9) converge rapidly. The convergence is of the same rate as that of (1.2). For large values of $|z|$ the method described above is not attractive. Many terms in the series are needed. But there is another important reason. For large values of $|z|$, the Bessel functions behave as indicated in (1.11). Hence, if $|z|$ is not small the subtraction in (1.3) will again cause a large relative error. This time the loss of significant digits cannot easily be avoided. A rough indication of the loss of digits, say q, can be obtained from

$$10^q \sim \tfrac{1}{2}e^{2\,\mathrm{Re}\,z}.$$

Hence, for $\mathrm{Re}\, z > 5$ at least four digits are lost.

For the case of real z and ν, $z > 0$, $|\nu| \leqslant \frac{1}{2}$, the loss of relative accuracy can be elaborated somewhat further. In this case $K_\nu(z)$ is positive (see (1.5)). If $f_0 > 0$, then, as follows from (2.5), (2.6), and (2.7), all terms in (2.1) are positive, and the summation in (2.1) is stable.

But f_0 is negative if z is large. The equation $f_0 = 0$ defines a curve in the (z, ν)-plane given by

$$z(\nu) = 2[\Gamma(1 + \nu)/\Gamma(1 - \nu)]^{1/(2\nu)},$$

with $z(0) = 1.1229...$ and $z(\frac{1}{2}) = 1$. Some computations show $z(\frac{1}{2}) \leqslant z(\nu) \leqslant z(0)$ for $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$. It follows that if $0 < z \leqslant 1$, $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$, $K_\nu(z)$ can be safely computed by using (2.1).

As for $K_{\nu+1}(z)$, the situation is more complicated. If $f_0 \geqslant 0$, then all p_k and f_k in (2.9) are nonnegative. By using (2.5), (2.6), and (2.7), it follows that for $k \geqslant 1$,

$$p_k - kf_k = (\nu p_{k-1} - k^2 f_{k-1} - kq_{k-1})/(k^2 - \nu^2).$$

If $-\frac{1}{2} \leqslant \nu < 0$, then the right-hand side is negative and so all terms in (2.9) except p_0 are negative. But $K_{\nu+1}(z) > 0$ and, in summing the series, cancellation may occur.

However, for small values of z we have from (1.2), (1.10), and (2.6),

$$K_{\nu+1}(z) \sim p_0 \sim z^{-1}(\nu\pi/\sin \nu\pi)\, I_{-\nu}(z).$$

So, for small z, p_0 dominates the remaining terms in (2.9), which are $o(1)$ for $z \to 0$. Hence it may be expected that for z sufficiently small, no cancellation in (2.9) will occur.

From numerical experiments it follows that

$$p_0 > 2 \left| \sum_{k=1}^{\infty} c_k(p_k - kf_k) \right|$$

for $0 < z \leqslant 1$ and $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$. As a consequence, for these values of z and ν, $K_{\nu+1}(z)$ can be safely computed by using (2.9).

As indicated in Subsection II.1, if $\frac{1}{2} < \nu \leqslant 1$, then the functions $K_{\nu-1}(z)$ and $K_\nu(z)$ are computed. $K_{\nu+1}(z)$ follows then from (see (1.9))

$$K_{\nu+1}(z) = (2\nu)/zK_\nu(z) + K_{\nu-1}(z),$$

in which both terms on the right are positive.

II.3. *The computation of the gamma function.* Since in the literature no approximations for the odd and even parts (with respect to ν) of the function $1/\Gamma(1 - \nu)$ are available, a description of our method is given here for the case of real ν.

The starting point is the expansion

$$1/\Gamma(\nu) = \sum_{k=1}^{\infty} c_k \nu^k, \qquad |\nu| < \infty. \tag{2.12}$$

The first 26 coefficients c_k are tabulated in Abramowitz and Stegun [1] (16 digits), and the first 41 in Wrench [15] (31 digits). From (2.12) and $\Gamma(\nu + 1) = \nu\Gamma(\nu)$ we easily obtain

$$1/\Gamma(1 - \nu) = \sum_{n=0}^{\infty} (-1)^n\, c_{n+1}\nu^n. \tag{2.13}$$

From this representation the odd and even parts may be obtained and so the values of $\Gamma_1(\nu)$ and $\Gamma_2(\nu)$ defined in (2.11). In the Bessel function algorithm we need Γ_1 and Γ_2 for $-\frac{1}{2} \leqslant \operatorname{Re} \nu \leqslant \frac{1}{2}$. To give a satisfactory numerical approximation on the real interval $[-\frac{1}{2}, \frac{1}{2}]$, we expand $1/\Gamma(1-\nu)$ in the Chebyshev polynomials $T_n(x) = \cos(n \arccos(x))$,

$$1/\Gamma(1-\nu) = \sum_{n=0}^{\infty} (-1)^n c_{n+1} 2^{-n}(2\nu)^n = \sum_{n=0}^{\infty}{}' a_n T_n(2\nu). \tag{2.14}$$

(The notation $\sum'$ means that the first term in the series is to be halved.)

The coefficients a_n in (2.14) can be computed by rearranging the Taylor series in (2.14). This method is described in Clenshaw [4]. The powers of (2ν) are replaced by their expansions in Chebyshev polynomials, and the series is rearranged in the form $\sum' a_n T_n(2\nu)$. The first few coefficients a_n are given in Table I. A check on these coefficients can be performed by evaluating (2.14) for $\nu = 0, \frac{1}{2}, -\frac{1}{2}$. We must have

$$\sum_{n=0}^{\infty}{}' (-1)^n a_{2n} = 1, \qquad \sum_{n=0}^{\infty}{}' a_n = \pi^{-1/2}, \qquad \sum_{n=0}^{\infty}{}' (-1)^n a_n = 2\pi^{-1/2}.$$

The functions Γ_1 and Γ_2 defined in (2.11) may now be written as

$$\nu\Gamma_1(\nu) \simeq \sum_{n=0}^{6} a_{2n+1} T_{2n+1}(2\nu), \qquad \Gamma_2(\nu) \simeq \sum_{n=0}^{7}{}' a_{2n} T_{2n}(2\nu),$$

and an appropriate summation method (see Clenshaw [4]) gives Γ_1 and Γ_2.

III. *The Computation for Large or Moderate Values of* $|z|$

For large $|z|$ we have the well-known asymptotic expansion

$$K_\nu(z) \sim (\pi/2z)^{1/2} e^{-z} \sum_{m=0}^{\infty} (\nu, m)(-2z)^{-m},$$

TABLE I

n	a_{2n}			a_{2n+1}		
0	+1.84374	05873	00906	−0.28387	65422	76024
1	−0.07685	28408	44786	+0.00170	63050	71096
2	+0.00127	19271	36655	+0.00007	63095	97586
3	−0.00000	49717	36704	−0.00000	08659	20800
4	−0.00000	00331	26120	+0.00000	00017	45136
5	+0.00000	00002	42310	+0.00000	00000	09161
6	−0.00000	00000	00170	−0.00000	00000	00034
7	−0.00000	00000	00001			

where (ν, m) is Hankel's symbol given by

$$(\nu, m) = (m!)^{-1}\,\Gamma(\tfrac{1}{2}+\nu+m)/\Gamma(\tfrac{1}{2}+\nu-m)$$
$$= (\pi m!)^{-1}\,(-1)^m \cos \nu\pi\, \Gamma(\tfrac{1}{2}+\nu+m)\,\Gamma(\tfrac{1}{2}-\nu+m). \qquad (3.1)$$

The series diverges for all finite values of $|z|$, but it can be used very successfully if $|z|$ is large. To give an indication, for real z, $z > 15$, the asymptotic series can be used to give an approximation, which is correct up to 13 significant digits. For intermediate values of $|z|$ we have to resort to other techniques. In this section we will discuss a method which enables computation of $K_\nu(z)$ for $|z| \geqslant 1$.

III.1. *The Miller algorithm.* We need some properties of the confluent hypergeometric functions. We use the notation of Abramowitz and Stegun [1].

The Bessel function $K_\nu(z)$ can be written as

$$K_\nu(z) = \pi^{1/2}(2z)^\nu\, e^{-z}\, U(\nu+\tfrac{1}{2}, 2\nu+1, 2z), \qquad (3.2)$$

where $U(a, b, z)$ is a confluent hypergeometric function, which for $\operatorname{Re} z > 0$ and $\operatorname{Re} a > 0$ may be defined by

$$\Gamma(a)\; U(a, b, z) = \int_0^\infty e^{-zt} t^{a-1}(1+t)^{b-a-1}\, dt. \qquad (3.3)$$

The function

$$k_n(z) = (-1)^n\,(\nu, n)\; U(\nu+\tfrac{1}{2}+n, 2\nu+1, 2z), \qquad n = 0, 1, 2, \ldots, \qquad (3.4)$$

with (ν, n) given in (3.1), satisfies the recurrence relation

$$k_{n+1}(z) - b_n k_n(z) + a_n k_{n-1}(z) = 0, \qquad (3.5)$$

with

$$a_n = [(n-\tfrac{1}{2})^2 - \nu^2]/(n^2+n), \qquad b_n = 2(n+z)/(n+1), \qquad n = 1, 2, \ldots . \qquad (3.6)$$

The function

$$y_n(z) = \Gamma(n+\nu+\tfrac{1}{2})\; {}_1F_1(\nu+\tfrac{1}{2}+n; 2\nu+1; 2z)/n! \qquad (3.7)$$

also satisfies (3.5). ${}_1F_1(a; b; z)$ is the hypergeometric function defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)}\,\frac{\Gamma(b)}{\Gamma(b+n)}\,\frac{z^n}{n!}\,. \qquad (3.8)$$

The functions k_n and y_n are two linearly independent solutions of the difference

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equation (3.5), as follows from the behavior of these solutions for large values of n, viz.

$$k_n(z) \sim \pi^{-1/2} \cos \nu\pi \, 2^{1/4} n^{-1/2} z^{-\nu-(1/4)} \exp[z - 2(2nz)^{1/2}], \tag{3.9}$$

$$y_n(z) \sim \pi^{-1/2} 2^{-\nu-(3/4)} n^{-1/2} z^{-\nu-(1/4)} \Gamma(2\nu + 1) \exp[z + 2(2nz)^{1/2}], \tag{3.10}$$

$$k_n(z)/y_n(z) \sim 2^{\nu+1} \cos \nu\pi \exp[-4(2nz)^{1/2}]/\Gamma(2\nu + 1). \tag{3.11}$$

Formulas (3.9) and (3.10) may be derived from results in Buchholtz [3]. Buchholtz derived his results for real z/n by using saddle point techniques. We can show, however, by using other methods (see Slater [13] and Temme [14]) that (3.9), (3.10), and (3.11) are valid under the restrictions

$$n \to \infty, \qquad z \text{ fixed}, \qquad z \neq 0, \qquad |\arg z| < \pi. \tag{3.12}$$

We will now describe our method of computing k_0 and k_1 defined in (3.4). If these functions are evaluated then the Bessel functions K_ν and $K_{\nu+1}$ can be computed from

$$K_\nu(z) = \pi^{1/2}(2z)^\nu \, e^{-z} k_0(z),$$
$$K_{\nu+1}(z) = K_\nu(z)[\nu + z + \tfrac{1}{2} - k_1(z)/k_0(z)]/z.$$

The latter equation may be derived from (3.2) and (3.4) and some contiguous relations of the confluent hypergeometric functions (cf. Abramowitz and Stegun [1, 13.4.16 and 13.4.18]).

The functions k_0 and k_1 may be computed with Miller's algorithm. We use Gautschi's version of this algorithm, the details of which can be found in [5]. As normalization relation we use

$$\sum_{n=0}^{\infty} k_n(z) = (2z)^{-\nu-(1/2)}, \tag{3.13}$$

which follows from (3.3) and (3.4) and substitution of the integral representation of k_n in (3.13).

In Miller's algorithm a positive integer N is selected and a sequence $\bar{k}_0^{(N)}, \bar{k}_1^{(N)}, \ldots, \bar{k}_N^{(N)}$ is computed by using (3.5) in backward direction with initial values $\bar{k}_{N+1}^{(N)} = 0$, $\bar{k}_N^{(N)} = 1$. By normalizing $\bar{k}_0^{(N)}$ and $\bar{k}_1^{(N)}$ with (3.13), $k_0^{(N)}$ and $k_1^{(N)}$ are computed. Then

$$\lim_{N\to\infty} k_n^{(N)} = k_n(z), \qquad n = 0, 1. \tag{3.14}$$

Using the asymptotic estimates (3.8) and (3.9) we can readily show that the conditions of theorems in [5] are fulfilled, from which the validity of (3.14) follows. (The algorithm can be used for the computation of k_n for larger values of n, but here we only need to consider $n = 0, 1$). In [14] we applied this algorithm for the computation of $s_k(z) = zk!\, U(k+1, 1, z)$. In fact it may be used for general $U(a+n, b, z)$, $n = 0, 1, 2,\ldots$, if $|z|$ is not too small.

III.2. *Determination of the starting index N.* The relative error ϵ of $k_n^{(N)}$ with respect to $k_n(z)$ can be expressed by

$$k_n^{(N)} = k_n(z)(1+\epsilon), \tag{3.15}$$

where ϵ depends on N, z, n, and ν. On account of (3.14), $|\epsilon|$ is small for large N. For numerical applications it is necessary to have an idea how large the starting index of the Miller algorithm N has to be, in order to have a satisfactorily small $|\epsilon|$.

As in Gautschi [5], the determination of N can be based on asymptotic formulas for the functions y_n and k_n. A more satisfactory approach, however, is pointed out by Olver and Sookne [12]. Their method is based on results of Olver in [10, 11]. Beginning with $p_0 = 0$, $p_1 = 1$, Olver computes a solution p_n of (3.5) for $n = 1, 2,\ldots$. Also computed is a sequence $\{e_n\}$ defined by

$$e_0 = 1, \qquad e_n = a_n e_{n-1},$$

where, in our case, a_n is given by (3.6), giving

$$e_n = (-1)^n\,(\nu, n)/(n+1)!.$$

Next, the quantity

$$E_N = \sum_{k=N}^{\infty} e_k/(p_k p_{k+1}), \qquad N \geqslant 1,$$

is introduced and the selection of the starting index N depends on the construction of a bound of E_N.

In order to construct this bound we consider henceforth real values of z and ν. As remarked in Subsection I.2, it suffices to consider values of ν in $[-\tfrac{1}{2}, \tfrac{1}{2}]$. Furthermore, we suppose $z \geqslant 1$. Under these conditions we have $b_n \geqslant 1 + a_n$, from which $p_{n+1} \geqslant p_n$ easily follows for $n \geqslant 0$. Moreover, $e_n \geqslant 0$ for $n \geqslant 0$. Hence, E_N is dominated as follows.

$$E_N \leqslant \sum_{n=N}^{\infty} p_n^{-2} e_n = \pi^{-1} \cos \nu\pi \sum_{n=N}^{\infty} p_n^{-2} \Gamma(\tfrac{1}{2}+\nu+n)\, \Gamma(\tfrac{1}{2}-\nu+n)/[n!\,(n+1)!]. \tag{3.16}$$

The series can be bounded by using the following lemma.

LEMMA. *Let a, b, and z be real numbers such that $b \geqslant a+1 > 0$ and $z > 0$. Then*

$$\Gamma(z+a)/\Gamma(z+b) \leqslant z^{a-b}.$$

Proof. From the integral (cf. [1, 6.2.1])

$$\Gamma(b-a)\,\Gamma(z+a)/\Gamma(z+b) = \int_0^\infty e^{-(z+a+1)t} t^{b-a-1} [(1-e^{-t})/t]^{b-a-1}\,dt,$$

we obtain, by using $e^{-(a+1)t} \leqslant 1$, $(1-e^{-t})/t \leqslant 1$ $(t \geqslant 0)$,

$$\Gamma(b-a)\,\Gamma(z+a)/\Gamma(z+b) \leqslant \int_0^\infty e^{-zt} t^{b-a-1}\,dt,$$

from which the lemma follows. ∎

Applying the lemma to (3.16), we obtain

$$E_N \leqslant \pi^{-1} \cos \nu\pi \sum_{n=N}^{\infty} 1/(n^2 p_n^2). \tag{3.17}$$

The function p_n is a solution of (3.5). It can be written as a linear combination of y_n and k_n ; p_n and y_n have for large n the same asymptotic behavior up to a factor independent of n. Considering (3.10) and comparing the series in (3.17) with the integral

$$\int_N^\infty n^{-1} \exp[-4(2nz)^{1/2}]\,dn,$$

we observe that it is plausible to replace (3.17) by

$$E_N \leqslant 2\pi^{-1} \cos \nu\pi\,(2z)^{-1/2}\,N^{-3/2} p_N^{-2}. \tag{3.18}$$

To the first order of small quantities, the relative error in the Miller algorithm is in our case (cf. Olver [10, (11.11)])

$$\sigma_N = E_N \sum_{n=0}^{N} p_n + \sum_{n=N+1}^{\infty} p_n E_n\,. \tag{3.19}$$

Hence, by using (3.18) and the same argumentation for both series in (3.19) as was used for (3.17), we obtain for σ_N the bound

$$\pi^{-1} z^{-1} \cos \nu\pi\,N^{-1} p_N^{-1}. \tag{3.20}$$

The least value of $N \geqslant 1$ for which (3.20) is smaller than the prescribed relative accuracy will be taken as the starting index for the Miller algorithm.

Remark. It may be noted that (3.20) vanishes for $\nu = \pm\frac{1}{2}$. As follows from (3.4) and (3.1), the functions k_n also vanish for $n \geqslant 1$, while $k_0(z)$ equals 1 or $1/(2z)$ if $\nu = -\frac{1}{2}$ or $\nu = +\frac{1}{2}$, respectively. If the choice of N is based upon (3.20), small

TABLE II

a	eps	$5.0_{10}-06$	$5.0_{10}-09$	$5.0_{10}-12$	$5.0_{10}-14$
0.0	$d0$	$1.4_{10}-06$	$6.1_{10}-10$	$7.4_{10}-13$	$8.4_{10}-15$
	$d1$	$1.4_{10}-06$	$6.1_{10}-10$	$7.1_{10}-13$	$3.5_{10}-14$
	(n, N)	(6, 22)	(8, 50)	(9, 89)	(10, 122)
0.2	$d0$	$1.6_{10}-06$	$5.7_{10}-10$	$6.6_{10}-13$	$4.2_{10}-15$
	$d1$	$1.6_{10}-06$	$5.7_{10}-10$	$6.4_{10}-13$	$2.0_{10}-14$
	(n, N)	(6, 21)	(8, 49)	(9, 88)	(10, 120)
0.4	$d0$	$1.6_{10}-06$	$5.2_{10}-10$	$6.0_{10}-13$	$1.6_{10}-14$
	$d1$	$1.6_{10}-06$	$5.2_{10}-10$	$5.8_{10}-13$	$2.5_{10}-14$
	(n, N)	(6, 18)	(8, 44)	(9, 81)	(10, 112)
0.6	$d0$	$1.6_{10}-06$	$5.2_{10}-10$	$6.0_{10}-13$	$7.4_{10}-15$
	$d1$	$1.6_{10}-06$	$5.2_{10}-10$	$5.9_{10}-13$	$1.4_{10}-14$
	(n, N)	(7, 18)	(8, 44)	(9, 81)	(10, 112)
0.8	$d0$	$1.6_{10}-06$	$5.7_{10}-10$	$6.6_{10}-13$	$0.0_{10}+00$
	$d1$	$1.6_{10}-06$	$5.7_{10}-10$	$6.5_{10}-13$	$1.7_{10}-14$
	(n, N)	(6, 21)	(8, 49)	(9, 88)	(10, 120)
1.0	$d0$	$1.4_{10}-06$	$6.1_{10}-10$	$6.8_{10}-13$	$7.1_{10}-14$
	$d1$	$1.4_{10}-06$	$6.1_{10}-10$	$6.8_{10}-13$	$7.0_{10}-14$
	(n, N)	(6, 22)	(8, 50)	(9, 89)	(10, 122)

```
real procedure recip gamma(x, odd, even); value x; real x, odd, even;
begin integer i; real alfa, beta, x2; array b[1:12];
    b[ 1]:= −.28387 65422 76024; b[ 2]:= −.07685 28408 44786;
    b[ 3]:= +.00170 63050 71096; b[ 4]:= +.00127 19271 36655;
    b[ 5]:= +.00007 63095 97586; b[ 6]:= −.00000 49717 36704;
    b[ 7]:= −.00000 08659 20800; b[ 8]:= −.00000 00331 26120;
    b[ 9]:= +.00000 00017 45136; b[10]:= +.00000 00002 42310;
    b[11]:= +.00000 00000 09161; b[12]:= −.00000 00000 00170;
    x2:= x × x × 8; alfa:= −.00000 00000 00001; beta:= 0;
    for i:= 12 step −2 until 2 do
    begin beta:= −(alfa × 2 + beta); alfa:= −beta × x2 − alfa + b[i] end;
    even:= (beta/2 + alfa) × x2 − alfa + .92187 02936 50453;
    alfa:= −.00000 00000 00034; beta:= 0;
    for i:= 11 step −2 until 1 do
    begin beta:= −(alfa × 2 + beta); alfa:= −beta × x2 − alfa + b[i] end;
    odd:= (alfa + beta) × 2; recip gamma:=odd × x + even
end recip gamma;
real procedure sinh(x); value x; real x;
```

Table II *(continued)*

```
begin real ax, y;
    ax:= abs(x);
    if ax < .3 then
    begin y:= if ax < .1 then x × x else x × x/9;
      x:= (((1/5040 × y + 1/120) × y + 1/6) × y + 1) × x;
      sinh:= if ax < .1 then x else x × (1 + 4 × x × x/27)
    end else
    begin ax:= exp(ax); sinh:= sign(x) × .5 × (ax-1/ax) end
end sinh;

procedure besska(a, x, eps, ka, ka1); value a, x, eps; real a, x, eps, ka, ka1;
begin real a1, b, c, d, e, f, g, h, p, pi, q, s; integer n, na; boolean rec, rev;
    pi:= 4 × arctan (1);
    rev:= a < -.5; if rev then a:= -a-1;
    rec:= a ≥ .5; if rec then begin na:= entier(a+.5); a:= a - na end;
    if a = -.5 then f:= g:= sqrt(pi/x/2) × exp(-x) else
    if x < 1 then
    begin b:= x/2; d:= -ln(b); e:= a × d; c:= a × pi;
      c:= if abs(x) < ₁₀-15 then 1 else c/sin(c);
      s:= if abs(e) < ₁₀-15 then 1 else sinh(e)/e;
      e:= exp(e); a1:= (e + 1/e)/2; g:= recip gamma(a, p, q) × e;
      ka:= f:= c × (p × a1 + q × s × d); e:= a × a;
      p:= .5 × g × c; q:= .5/g; c:= 1; d:= b × b; ka1:= p;
      for n:= 1, n + 1 while h/ka + abs(g)/ka1 > eps do
      begin f:= (f × n + p + q)/(n × n - e); c:= c × d/n;
        p:= p/(n - a); q:= q/(n + a); g:= c × (p - n × f);
        h:= c × f; ka:= ka + h; ka1:= ka1 + g
      end;
      f:= ka; g:= ka1/b
    end else
    begin c:= .25 - a × a; g:= 1; f:= 0; e:= x × cos(a × pi)/pi/eps;
      for n:=1, n + 1 while h × n < e do
      begin h:= (2 × (n + x) × g - (n - 1 + c/n) × f)/(n + 1);
        f:= g; g:= h
      end;
      p:= q:= f/g; b:= x + x; e:= b - 2;
      for n:= n, n - 1 while n > 0 do
      begin p:= (n - 1 + c/n)/(e + (n + 1) × (2 - p)); q:= p × (q + 1) end;
      f:= sqrt(pi/b) × exp(-x)/(1 + q); g:= f × (a + x + .5 - p)/x
      end;
    if rec then
    begin x:= 2/x;
      for n:= 1 step 1 until na do
      begin h:= f + (a + n) × x × g; f:= g; g:= h end
    end;
    if rev then begin ka1:= f; ka:=g end else
    begin ka:= f; ka1:= g end
end besska;
```

values of N will result in ν-neighbourhoods of $\pm\frac{1}{2}$. This phenomenon will not disturb the actual algorithm. As can be verified (see the ALGOL procedure *besska*), in the limit $\nu = \pm\frac{1}{2}$ the correct values are computed.

IV. *ALGOL* 60 *Procedures*

The algorithms described in Sections II and III are given as an ALGOL 60 procedure for real values of the parameters. For convenience we write $z = x$ and $\nu = a$. The procedure *besska* computes for $x > 0$ and $a \in \mathbb{R}$ the Bessel functions $K_a(x)$ and $K_{a+1}(x)$; *besska* makes use of two nonlocal procedures *sinh* and *recipgamma*. The latter computes $1/\Gamma(1 - a)$ and the functions $\Gamma_1(a)$ and $\Gamma_2(a)$ defined in (2.11) for $-\frac{1}{2} \leqslant a \leqslant \frac{1}{2}$.

By choosing eps the procedure *besska* can be used up to any (relative) tolerance. The two procedures *recipgamma* and *sinh* are supplied with fixed relative accuracy (about 10^{-14}). By only modifying these two procedures, the set of three procedures presented here can be adapted to any computer and to any accuracy.

The procedures are tested on the CD CYBER 73 of SARA, Amsterdam. For

$$x^{\pm} = 1 \pm 2^{-47}$$

we computed the numerical values of the expressions

$$d_0 = \{K_a(x^-) - K_a(x^+)\}/K_a(x^-),$$
$$d_1 = \{K_{a+1}(x^-) - K_{a+1}(x^+)\}/K_{a+1}(x^-).$$

In Table II we give d_0, d_1, the maximum number of terms (n) used in (2.1), and the starting index N for the Miller algorithm.

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On the Numerical Evaluation of the Ordinary Bessel Function of the Second Kind

N. M. TEMME

1. Introduction

1.1. *Definitions and Relevant Properties*

The ordinary Bessel function of the first kind

$$J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{\Gamma(\nu + k + 1)\, k!} \tag{1.1}$$

and the ordinary Bessel function of the second kind

$$Y_\nu(z) = [\cos \nu\pi\, J_\nu(z) - J_{-\nu}(z)]/\sin \nu\pi \tag{1.2}$$

are two linearly independent solutions of the difference equation

$$f_{\nu+1} - (2\nu/z) f_\nu + f_{\nu-1} = 0. \tag{1.3}$$

This equation can be used to compute $Y_{\nu+n}$ for $n = 2, 3,\ldots$ when Y_ν and $Y_{\nu+1}$ are given. In the forward direction the recurrence formula (1.3) for Y_ν is numerically stable, whereas it is unstable for J_ν (see Gautschi [1]).

The ordinary Bessel functions of the third kind are the Hankel functions

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),\; H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \tag{1.4}$$

Important for the representation of the Hankel functions for large $|z|$ are the functions $P(\nu, z)$ and $Q(\nu, z)$ defined by

$$H_\nu^{(1,2)}(z) = [2/(\pi z)]^{1/2} e^{\pm i\chi}[P(\nu, z) \pm iQ(\nu, z)], \tag{1.5}$$

where the $+$ sign is used for $H_\nu^{(1)}$, the $-$ sign is used for $H_\nu^{(2)}$ and

$$\chi = z - \pi(2\nu + 1)/4. \tag{1.6}$$

For large $|z|$, P and Q are slowly varying and the oscillatory behavior of $H_\nu^{(1)}$ and

$H_\nu^{(2)}$ is contained in the exponential function in (1.5). From (1.4) and (1.5) we obtain

$$\begin{aligned} Y_\nu(z) &= [2/(\pi z)]^{1/2} [P(\nu, z) \sin \chi + Q(\nu, z) \cos \chi] \\ J_\nu(z) &= [2/(\pi z)]^{1/2} [P(\nu, z) \cos \chi - Q(\nu, z) \sin \chi]. \end{aligned} \tag{1.7}$$

Again, the oscillatory behavior of J_ν and Y_ν is fully described by the circular functions in (1.7).

The connection between the ordinary Bessel functions and the modified Bessel functions follows from

$$\begin{aligned} H_\nu^{(1)}(z) &= -2i\pi^{-1}e^{-\nu\pi i/2}K_\nu(ze^{-i\pi/2}) \qquad (-\tfrac{1}{2}\pi < \arg z \leqslant \pi), \\ H_\nu^{(2)}(z) &= 2i\pi^{-1}e^{\nu\pi i/2}K_\nu(ze^{i\pi/2}) \qquad (-\pi < \arg z \leqslant \tfrac{1}{2}\pi). \end{aligned} \tag{1.8}$$

From the Wronskian

$$J_{\nu+1}(z)\, Y_\nu(z) - J_\nu(z)\, Y_{\nu+1}(z) = 2/(\pi z)$$

and (1.7) it easily follows that

$$P(\nu, z)\, P(\nu + 1, z) + Q(\nu, z)\, Q(\nu + 1, z) = 1. \tag{1.9}$$

1.2. *Contents of the Paper*

We give algorithms for the computation of Y_ν and $Y_{\nu+1}$ and we use the methods of our previous paper on the computation of K_ν and $K_{\nu+1}$ (see Temme [6]). Our results in [6] can be used for complex values of z. Here we give the explicit results for Y_ν and $Y_{\nu+1}$ and these results follow immediately from [6] by using (1.8).

For the computation of J_ν the reader is referred to Gautschi [1], where an algorithm is given for the computation of $J_{\nu+n}(z)$, $n = 0, 1, 2,..., N$. See also Gautschi [2]. In Luke [4] rational approximations for J_ν and Y_ν are given based on Padé-representations for large $|z|$. In Luke [5] a double series of Chebyshev polynomials and values of the coefficients are given for both Y_ν, J_ν for $z \geqslant 5$. In Goldstein and Thaler [3] the computation of Y_ν is based on series expansions in ordinary Bessel functions of the first kind, but the treatment of small $|\nu|$-values is not satisfactory.

2. The Computation for Small $|z|$

In order to obtain a more symmetric representation in (1.2) we write

$$\cos \nu\pi\, J_\nu(z) - J_{-\nu}(z) = J_\nu(z) - J_{-\nu}(z) - 2 \sin^2(\nu\pi/2)\, J_\nu(z). \tag{2.1}$$

Furthermore we introduce the following notation

$$\begin{aligned}
c_k &= (-z^2/4)^k/k!,\\
p_k &= (\nu/\sin \nu\pi)\,(z/2)^{-\nu}/\Gamma(k+1-\nu),\\
q_k &= (\nu/\sin \nu\pi)\,(z/2)^{\nu}/\Gamma(k+1+\nu),\\
f_k &= (p_k - q_k)/\nu,\\
g_k &= f_k + 2\nu^{-1}\sin^2(\nu\pi/2)\,q_k\,,\\
h_k &= -kg_k + p_k\,,
\end{aligned}$$

where $k = 0, 1, \ldots$. We have for $k = 1, 2, \ldots$ the recurrence relations

$$\begin{aligned}
p_k &= p_{k-1}/(k-\nu),\ q_k = q_{k-1}/(k+\nu),\\
f_k &= (kf_{k-1} + p_{k-1} + q_{k-1})/(k^2 - \nu^2).
\end{aligned}$$

Substitution of (1.1) in (1.2) and using (2.1) yields

$$Y_\nu(z) = -\sum_{k=0}^{\infty} c_k\, g_k\,. \tag{2.2}$$

Considering (2.1) with ν replaced by $\nu + 1$ and using (1.3) we have

$$\begin{aligned}
&\cos(\nu+1)\,\pi\, J_{\nu+1}(z) - J_{-\nu-1}(z)\\
&\quad = -[J_{\nu+1}(z) - J_{-\nu+1}(z)] + (2\nu/z)\,J_{-\nu}(z) + 2\sin^2(\nu\pi/2)\,J_{\nu+1}(z).
\end{aligned}$$

We obtain by substitution of (1.1)

$$Y_{\nu+1}(z) = -(2/z)\sum_{k=0}^{\infty} c_k h_k\,. \tag{2.3}$$

As in [6], f_0 can be represented in such a way that it can be computed with a satisfactorily small relative error.

For small values of $|z|$ the series in (2.2) and (2.3) converge rapidly. But cancellation may occur in summing the series numerically. A strict error analysis, as for the modified Bessel function, can not easily be given, but from numerical experiments it turns out that for $|z| < 3$ the computation is stable.

3. The Computation for $|z| \geqslant 3$

For $|z| \geqslant 3$ we compute $P(\nu, z)$, $P(\nu + 1, z)$, $Q(\nu, z)$ and $Q(\nu + 1, z)$, by using the functions $k_n(z)$ introduced in our previous paper [6]. For K_ν and $K_{\nu+1}$ we needed $k_0(z)$ and $k_1(z)$. From (1.8) it turns out that for the P- and Q-functions the functions $k_0(\pm iz)$ and $k_1(\pm iz)$ can be used. The application of the method in [6] is straightforward. However, the determination of the starting index N for the Miller algorithm caused some trouble, since our error analysis in [6] was based on the case of real variables. But trying out the results of [6] for the P- and Q-functions we noticed that the determination of the starting index N can indeed be based upon the estimations given in [6].

4. ALGOL 60 Procedures

The algorithms for the computation of $Y_\nu(z)$ and $Y_{\nu+1}(z)$ are given as an ALGOL 60 procedure for the case of real values of ν and z, $z > 0$. For convenience we write $\nu = a$ and $z = x$.

The procedure *bessya* computes for $x > 0$ and $a \in \mathbb{R}$ the functions $Y_a(x)$ and $Y_{a+1}(x)$; *bessya* calls for three nonlocal procedures *sinh*, *recip gamma*, and *besspqa*. For the text of *sinh*, and *recip gamma* the reader is referred to [6]. In *besspqa* the functions $P(a, x)$, $P(a + 1, x)$, $Q(a, x)$ and $Q(a + 1, x)$ are computed. We supply *besspqa* as a separate procedure since it can also be used for the computation of the Bessel functions $J_a(x)$ and $J_{a+1}(x)$ (see (1.7)). In *bessya* the procedure *besspqa* is called for $x \geqslant 3$ and $|a| < .5$, but the algorithm in *besspqa* converges for all x and a $(x > 0)$. It is recommended, however, to take $x > \max(|a|, 3)$. For $|a| > x$ the recurrence relations

$$P(a + 1, x) = P(a - 1, x) - 2a/x\, Q(a, x)$$
$$Q(a + 1, x) = Q(a - 1, x) + 2a/x\, P(a, x)$$

can be used. These relations are valid for real a and x. They can be derived by substitution of (1.5) in (1.3). However, for $|a| + 1 > x$, computation of $J_a(x)$ and $J_{a+1}(x)$ by using (1.7) will cause a loss of correct significant digits.

The precision in the procedures *bessya* and *besspqa* can be controlled by using the variable *eps*. For *besspqa* its entry value corresponds to the desired relative accuracy in *pa*, *pa* 1, *qa* and *qa* 1. Also in *bessya* it corresponds to relative accuracy, except in the neighborhoods of zeros of $Y_a(x)$ or $Y_{a+1}(x)$. In that case *ya* or *ya* 1 are given with absolute accuracy *eps*.

The procedures *bessya* and *besspqa* were tested on the CD CYBER 73 of SARA, Amsterdam. For $a = 0, 0.2, 0.4$, $x = .5, 1, 2, 3, 5, 7, 10, 20, 50, 100$ and *eps* $= 10^{-15}$ we checked relation (1.9). The output of $|$ *pa.pa* 1 + *qa.qa* 1 $- 1|$ is given in Table I. The procedure *bessya* was also tested in the neighborhood of $x = 3$. For $x^\pm = 3 \pm 2^{-46}$ we computed the numerical values of the expressions

$$d_0 = \{Y_a(x^-) - Y_a(x^+)\},$$
$$d_1 = \{Y_{a+1}(x^-) - Y_{a+1}(x^+)\}.$$

In Table II we give d_0, d_1, the maximum number of terms (n) used in (2.1), and the starting index N for the Miller algorithm.

TABLE I

x \ a	0.0	0.2	0.4
0.5	$1.4_{10} - 14$	$7.1_{10} - 15$	$0.0_{10} + 00$
1.0	$0.0_{10} + 00$	$7.1_{10} - 15$	$7.1_{10} - 15$
2.0	$7.1_{10} - 15$	$2.8_{10} - 14$	$7.1_{10} - 15$
3.0	$7.1_{10} - 15$	$0.0_{10} + 00$	$0.0_{10} + 00$
5.0	$7.1_{10} - 15$	$1.4_{10} - 14$	$0.0_{10} + 00$
7.0	$7.1_{10} - 15$	$7.1_{10} - 15$	$1.4_{10} - 14$
10.0	$7.1_{10} - 15$	$7.1_{10} - 15$	$7.1_{10} - 15$
20.0	$0.0_{10} + 00$	$7.1_{10} - 15$	$0.0_{10} + 00$
50.0	$2.1_{10} - 14$	$1.4_{10} - 14$	$0.0_{10} + 00$
100.0	$2.1_{10} - 14$	$7.1_{10} - 15$	$7.1_{10} - 15$

TABLE II

	eps	$5.0_{10} - 06$	$5.0_{10} - 09$	$5.0_{10} - 12$	$5.0_{10} - 14$
a					
0.0	*d*0	$5.2_{10} - 08$	$4.3_{10} - 11$	$3.4_{10} - 14$	$5.3_{10} - 15$
	*d*1	$6.4_{10} - 08$	$1.8_{10} - 11$	$3.6_{10} - 14$	$5.3_{10} - 15$
	(n, N)	(9, 17)	(11, 37)	(13, 64)	(14, 87)
0.2	*d*0	$4.8_{10} - 08$	$5.3_{10} - 11$	$5.0_{10} - 14$	$1.8_{10} - 15$
	*d*1	$9.4_{10} - 08$	$4.9_{10} - 11$	$2.2_{10} - 14$	$1.3_{10} - 14$
	(n, N)	(9, 17)	(11, 36)	(13, 63)	(14, 86)
0.4	*d*0	$6.8_{10} - 09$	$2.2_{10} - 11$	$2.1_{10} - 14$	$8.9_{10} - 15$
	*d*1	$2.3_{10} - 08$	$1.1_{10} - 10$	$2.5_{10} - 14$	$2.3_{10} - 14$
	(n, N)	(10, 15)	(11, 33)	(13, 59)	(14, 81)
0.6	*d*0	$2.0_{10} - 07$	$8.2_{10} - 12$	$3.4_{10} - 14$	$1.6_{10} - 14$
	*d*1	$9.9_{10} - 08$	$4.8_{10} - 11$	$1.6_{10} - 14$	$2.4_{10} - 14$
	(n, N)	(8, 15)	(11, 33)	(13, 59)	(14, 81)
0.8	*d*0	$3.5_{10} - 08$	$4.7_{10} - 12$	$4.1_{10} - 14$	$1.1_{10} - 14$
	*d*1	$5.7_{10} - 08$	$4.7_{10} - 11$	$0.0_{10} + 00$	$2.1_{10} - 14$
	(n, N)	(9, 17)	(11, 36)	(13, 63)	(14, 86)
1.0	*d*0	$6.4_{10} - 08$	$1.8_{10} - 11$	$3.2_{10} - 14$	$3.6_{10} - 15$
	*d*1	$9.5_{10} - 08$	$5.5_{10} - 11$	$7.1_{10} - 15$	$1.4_{10} - 14$
	(n, N)	(9, 17)	(11, 37)	(13, 64)	(14, 87)

```
procedure bessya(a,x,eps,ya,ya1); value a,x,eps; real a,x,eps,ya,ya1;
begin real b,c,d,e,f,g,h,p,pi,q,r,s; integer n,na; Boolean rec, rev;
   pi:= 4 × arctan(1); na:= entier(a+.5); rec:= a ≥ .5;
   rev:= a < −.5; if rev ∨ rec then a:= a−na;
   if a = −.5 then
   begin p:= sqrt(2/pi/x); f:= p × sin(x); g:= −p × cos(x) end else
   if x < 3 then
   begin b:= x/2; d:= −ln(b); e:= a × d;
      c:= if abs(a) < ₁₀−15 then 1/pi else a/sin(a × pi);
      s:= if abs(e) < ₁₀−15 then 1 else sinh(e)/e;
      e:= exp(e); g:= recip gamma(a, p, q) × e; e:= (e + 1/e)/2;
      f:= 2 × c × (p × e + q × s × d); e:= a × a;
      p:= g × c; q:= 1/g/pi; c:= a × pi/2;
      r:= if abs(c) < ₁₀−15 then 1 else sin(c)/c; r:= pi × c × r × r;
      c:= 1; d:= −b × b; ya:= f + r × q; ya1:= p;
      for n:= 1, n + 1 while
      abs(g/(1 + abs(ya))) + abs(h/(1 + abs(ya1))) > eps do
      begin f:= (f × n + p + q)/(n × n − e); c:= c × d/n;
         p:= p/(n − a); q:= q/(n + a);
         g:= c × (f + r × q); h:= c × p − n × g;
         ya:= ya + g; ya1:= ya1 + h
      end;
      f:= −ya; g:= −ya1/b
   end else
   begin b:= x − pi × (a + .5)/2; c:= cos(b); s:= sin(b);
      d:= sqrt(2/x/pi);
      besspqa(a,x,eps,p,q,b,h);
      f:= d × (p × s + q × c); g:= d × (h × s − b × c)
   end;
   if rev then
   begin x:= 2/x; na:= −na − 1;
      for n:= 0 step 1 until na do
      begin h:= x × (a − n) × f − g; g:=f; f:= h end
   end else if rec then
   begin x:= 2/x;
      for n:= 1 step 1 until na do
      begin h:= x × (a + n) × g − f; f:= g; g:= h end
   end;
   ya:= f; ya1:= g
end bessya;
```

```
procedure besspqa(a,x,eps,pa,qa,pa1,qa1); value a,x,eps;
        real a,x,eps,pa,qa,pa1,qa1;
begin real b,c,d,e,f,g,p,p0,q,q0,r,s; integer n,na; Boolean rec,rev;
    rev:= a < -.5; if rev then a:= -a-1;
    rec:= a ≥ .5; if rec then
    begin na:= entier(a+.5); a:= a - na end;
    if a = -.5 then
    begin pa:= pa1:= 1; qa:= qa1:= 0 end else
    begin c:= .25 - a × a; b:= x + x; p:= 4 × arctan(1);
        e:= (x × cos(a × p)/p/eps)↑2; p:= 1; q:= -x; r:= s:= 1 + x × x;
        for n:= 2, n + 1 while r × n × n < e do
        begin d:= (n - 1 + c/n)/s; p:= (2 × n - p × d)/(n + 1);
            q:= (-b + q × d)/(n + 1); s:= p × p + q × q; r:= r × s
        end;
        f:= p:= p/s; g:= q:= -q/s;
        for n:= n, n - 1 while n > 0 do
        begin r:= (n+1) × (2-p) - 2; s:= b + (n+1) × q; d:= (n - 1 + c/n)/
            (r × r + s × s); p:= d × r; q:= d × s; e:= f;
            f:= p × (e + 1) - g × q; g:= q × (e + 1) + p × g
        end;
        f:= 1 + f; d:= f × f + g × g;
        pa:= f/d; qa:= -g/d; d:= a + .5 - p; q:= q + x;
        pa1:= (pa × q - qa × d)/x;
        qa1:= (qa × q + pa × d)/x
    end;
    if rec then
    begin x:= 2/x; b:= (a + 1) × x;
        for n:= 1 step 1 until na do
        begin p0:= pa - qa1 × b; q0:= qa + pa1 × b;
            pa:= pa1; pa1:= p0; qa:= qa1, qa1:= q0; b:= b + x
        end
    end;
    if rev then
    begin p0:= pa1; pa1:= pa; pa:= p0;
        q0:= qa1; qa1:= qa; qa:= q0
    end
end besspqa;
```

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REMARKS ON A PAPER OF A. ERDÉLYI

N. M. TEMME

Abstract. An alternative asymptotic expansion is given for an integral, which was recently considered by Erdélyi by means of fractional derivatives. The new expansion is simpler and the bounds of the remainder terms are of the same kind.

1. Introduction. In a recent paper [3], Professor Erdélyi considered integrals of the form

$$F(z, a) = \int_a^x e^{-z(t-a)} t^{\lambda - 1} g(t)\, dt, \tag{1.1}$$

where $a \geqq 0, 0 < \lambda < 1$, and z is a large parameter. In order to obtain an asymptotic expansion for $z \to \infty$, uniformly valid for $a \geqq 0$, he replaced the function $t^{\lambda-1}g(t)$ by a fractional integral $I^{\lambda-1}f(t)$, the operator I^α being defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\, ds.$$

By an integration by parts procedure, Erdélyi obtained the uniform expansion

$$F(z, a) = Q \sum_{k=0}^{n-1} \Gamma(k + \lambda) g^k(0) z^{-k}/k! + \sum_{k=1}^{n-1} z^{-k} I^\lambda f^{(k)}(a) + R_n, \tag{1.2}$$

where Q is related to the incomplete gamma function and is given by

$$Q = z^{-\lambda} e^{az} \Gamma(\lambda, az)/\Gamma(\lambda). \tag{1.3}$$

The remainder R_n is estimated uniformly in a for $a \geqq 0$. The expression $I^\lambda f^{(k)}(a)$ is explicitly given in terms of derivatives of the function $g(t)$ at $t = 0$ and $t = a$ as

$$\begin{aligned} I^\lambda f^{(k)}(a) = \sum_{m=1}^{k} \frac{a^{\lambda - m}}{(k - m)!} \Bigg[& (-1)^{m-1} \frac{\Gamma(k)\Gamma(m - \lambda)}{\Gamma(m)\Gamma(1 - \lambda)} g^{(k-m)}(a) \\ & - \frac{\Gamma(k + \lambda - m)}{\Gamma(\lambda - m + 1)} g^{(k-m)}(0) \Bigg], \qquad k = 1, 2, \cdots. \end{aligned} \tag{1.4}$$

As remarked by Erdélyi, the expansion (1.2) could have been obtained via integration by parts of (1.1), but the explicit form (1.4) in (1.2) is not easily obtained in that way.

In this note we give an alternative expansion of $F(z, a)$, which is simpler than (1.2), and in which the bounds of the remainder terms are of the same kind. Both expansions may be derived from each other by formal rearrangement of infinite series.

2. From a numerical point of view, (1.2) is not attractive because of the $I^{\lambda}f^{(k)}(a)$ in the second series. Recurrence relations for these factors based on (cf. [3, (2.3)])

$$I^{\lambda-1}f(t) = \frac{d}{dt}I^{\lambda}f(t) = \frac{f(0)}{\Gamma(\lambda)}t^{\lambda-1} + I^{\lambda}f'(t)$$

are not suitable for numerical evaluation of a sequence of $I^{\lambda}f^{(k)}(a)$, $k = 0, 1, \cdots, n$.

Furthermore, the terms $g^{(k)}(0)$ in (1.2) are somewhat surprising. Of course, the singularity at $t = 0$ due to $t^{\lambda-1}$ gives a hint that this point may significantly contribute to the asymptotic expansion, especially when a is small. But for moderate and large values of a, we cannot expect relevant information from the function values at $t = 0$.

In our opinion, the expansion (1.2) can be considerably simplified. Let us suppose that g and its first n derivatives are continuous and bounded on $[0, \infty)$. We write

$$g(t) = \sum_{k=0}^{n-1} c_k(t-a)^k + r_n(t), \qquad c_k = g^{(k)}(a)/k!.$$

Then we have

$$F(z, a) = \sum_{k=0}^{n-1} c_k F_k + R_n \tag{2.1}$$

with

$$F_k = \int_a^{\infty} e^{-z(t-a)}t^{\lambda-1}(t-a)^k\,dt, \tag{2.2}$$

$$R_n = \int_a^{\infty} e^{-z(t-a)}t^{\lambda-1}r_n(t)\,dt. \tag{2.3}$$

The first few functions F_k are easily computed. It turns out that

$$F_0 = \Gamma(\lambda)Q, \qquad F_1 = (\lambda z^{-1} - a)F_0 + a^{\lambda}z^{-1}, \tag{2.4}$$

where Q is essentially an incomplete gamma function and is defined in (1.3). By partial integration of (2.2) we obtain

$$F_{k+1} = z^{-1}[(k + \lambda - az)F_k + akF_{k-1}], \qquad k \geqq 1. \tag{2.5}$$

Hence, if F_0 is computed, the remaining F_k can be generated by (2.5).

The functions F_k are confluent hypergeometric functions. In the notation of [1], we have

$$\begin{aligned} F_k &= k!a^{k+\lambda}U(k+1, k+1+\lambda, az) \\ &= k!z^{-k-\lambda}U(1-\lambda, 1-\lambda-k, az). \end{aligned} \tag{2.6}$$

The second representation enables us to write for $0 < \lambda < 1$,

$$F_k = \frac{k!\,z^{-k-\lambda}}{\Gamma(1-\lambda)}\int_0^{\infty} e^{-azt}t^{-\lambda}(1+t)^{-k-1}\,dt, \tag{2.7}$$

from which follows, by majorizing the exponential function in the integrand by 1,

$$F_k \leqq z^{-k-\lambda}\Gamma(k+\lambda). \tag{2.8}$$

As follows from (2.2), this bound is also valid for $\lambda = 1$.

If on $[0, \infty)$ an estimate is known for $g^{(k)}$, say $|g^{(k)}(t)| \leqq a_k$, and a, λ and z are real, then R_n in (2.3) may be majorized by $|R_n| \leqq a_n F_n/n!$. Using (2.8), we obtain uniformly in a for $a \geqq 0$,

$$|R_n| \leqq a_n z^{-n-\lambda}\Gamma(n+\lambda)/n!.$$

Consequently, in the notation of [2], we have

$$F(z,a) \sim \sum c_k F_k \quad \{z^{-k-\lambda}\} \quad \text{as } z \to \infty.$$

This shows that (2.1) is an asymptotic expansion, holding uniformly in a for $a \geqq 0$, with respect to the asymptotic sequence $\{z^{-n-\lambda}\}$, which does not depend on a.

From a practical point of view, the expansion in (2.1) is more suitable than (1.2), since the coefficients c_k are simply expressed in terms of $g^{(k)}(a)$. Both expansions have the same bounds for the remainders. As a minor improvement, our expansion is also uniformly valid with respect to λ on compact subintervals of $(0, 1]$.

3. The numerical analyst may wonder if the sequence $\{F_k\}$ can be generated in a stable way by using (2.5). The answer is affirmative, as one easily deduces from the qualitative behavior of the linearly independent solutions of the second order difference equation (2.5). With

$$G_k = \int_0^a e^{-zt}(t-a)^k t^{\lambda-1}\,dt = a^{\lambda+k}(-1)^k\frac{\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+\lambda+1)}M(\lambda, k+\lambda+1, -az), \tag{3.1}$$

the functions F_n, G_n constitute a linearly independent pair of solutions of (2.5), as follows from the asymptotic behavior

$$F_n \sim n!\,z^{-n-\lambda}(1+a/n)^{\lambda+1}n^{\lambda-1}, \quad n \to \infty, \quad \text{uniformly in } a \geqq 0, \tag{3.2}$$

and from the inequality,

$$|G_n| \leqq a^{n+\lambda}\Gamma(\lambda)\Gamma(n+1)/\Gamma(n+\lambda+1), \qquad n = 0, 1, \cdots. \tag{3.3}$$

Formula (3.2) is easily derived with saddle point techniques from (2.7), and (3.3) follows from (3.1) by majorizing the exponential function by 1.

The relations (3.2) and (3.3) show that, in the sense of [4], the solution G_n is a minimal solution of (2.5) and F_n a dominant solution.

4. The relation between Erdélyi's expansion (1.2) and our expansion (2.1) can be illustrated by writing

$$F_k = P_k F_0 + Q_k a^\lambda z^{-1}, \qquad k = 0, 1, \cdots.$$

P_k and Q_k are polynomials in z^{-1} satisfying (2.5) with initial values $P_0 = 1$, $Q_0 = 0$, $P_1 = \lambda z^{-1} - a$, $Q_1 = 1$. By using the recurrence relation it can be proved that

$$P_k = z^{-k}\sum_{j=0}^{k}(-az)^{k-j}\binom{k}{j}\Gamma(\lambda+j)/\Gamma(\lambda), \qquad k = 0, 1, \cdots. \tag{4.1}$$

Hence, in a formal way, our expansion (2.1) can be written as

$$F(z, a) \sim F_0 \sum c_k P_k + a^\lambda z^{-1} \sum c_k Q_k. \tag{4.2}$$

With the substitution of (4.1) and using the (formal) expansion

$$g^{(j)}(t) = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!}(t-a)^{k-j}$$

at $t = 0$, we obtain, by interchanging the order of summation,

$$F(z, a) \sim Q \sum z^{-k}\Gamma(k + \lambda)g^{(k)}(0)/k! + a^\lambda z^{-1} \sum c_k Q_k.$$

The first series in this expression is exactly the first series of Erdélyi in (1.2). The second series is much more complicated, but probably it can be identified with the corresponding series of Erdélyi.

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Uniform asymptotic expansions of confluent hypergeometric functions

by

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ABSTRACT

New asymptotic expansions are derived for the confluent hypergeometric functions M(a,b,x) and U(a,b,x) for large b. The results are uniformly valid with respect to x in a neighbourhood containing x = b; a is a fixed parameter. The expansions contain parabolic cylinder functions and asymptotic series.

KEY WORDS & PHRASES: *confluent hypergeometric functions, asymptotic expansion, parabolic cylinder functions.*

1. INTRODUCTION

In a recent paper [7], we derived new asymptotic expansions for the incomplete gamma functions

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} \, dt, \qquad \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} \, dt$$

and for the incomplete beta function

$$I_x(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^x t^{p-1} (1-t)^{q-1} \, dt.$$

In each case, the expansion contains the complementary error function defined by

$$\text{(1.1)} \qquad \operatorname{erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty e^{-t^2} \, dt$$

and an asymptotic series. The expansions are uniformly valid with respect to certain domains of the parameters.

The incomplete gamma functions may be considered as special cases of the confluent hypergeometric functions, which, in the notation of ABRAMOWITZ & STEGUN [1], are denoted as $M(a,b,x)$ and $U(a,b,x)$. Explicitly we have

$$\text{(1.2)} \qquad \begin{aligned} \gamma(a,x) &= a^{-1} x^a M(a,a+1,-x) = a^{-1} x^a e^{-x} M(1,a+1,x), \\ \Gamma(a,x) &= x^a e^{-x} U(1,a+1,x) \quad = e^{-x} U(1-a,1-a,x). \end{aligned}$$

For large values of a and x with $x \sim a$, the functions $\gamma(a,x)$ and $\Gamma(a,x)$ exhibit a nonuniform behaviour. The expansions given in TEMME [7] describe this behaviour adequately. The same phenomena are expected for $M(a,b,x)$ and $U(a,b,x)$ for large values of x and b with $x \sim b$.

In this paper, we are concerned with the asymptotic expansions of the confluent hypergeometric functions for large positive values of b and/or x, which are uniformly valid with respect to $\lambda = x/b$ in a λ-interval containing $\lambda = 1$; a is considered as a fixed parameter.

The Whittaker functions are closely connected with the confluent hypergeometric functions. The relations are

$$M_{\kappa,\mu}(x) = e^{-x/2}\, x^{\mu+\frac{1}{2}}\, M(\tfrac{1}{2}+\mu-\kappa, 1+2\mu, x)$$
$$W_{\kappa,\mu}(x) = e^{-x/2}\, x^{\mu+\frac{1}{2}}\, U(\tfrac{1}{2}+\mu-\kappa, 1+2\mu, x).$$

There is a vast literature on confluent hypergeometric functions and Whittaker functions and on asymptotic expansions of these functions. A recent book with many references is DINGLE [3]. Apart from the well-known expansions in inverse powers of the large argument x, expansions may be found which are uniformly valid with respect to certain parameters. The theory for large x and b, however, is still incomplete.

The results in the present paper can be considered as an extension of some of the results of Dingle, who gives expansions of M(a,b,x) and U(a,b,x) for $b < x$, $b > x$ and also in a neighbourhood of the transition point $b = x$. From Dingle's expansions, we learn that the qualitative behaviour of M and U in this neighbourhood can be described by parabolic cylinder functions of which the error function in (1.1) is a special case. The parabolic cylinder functions, which are also important in our paper, are special cases of the confluent hypergeometric functions. Explicitly, we have

$$D_{\nu}(x) = 2^{\nu/2}\, e^{-x^2/4}\, U(-\tfrac{1}{2}\nu, \tfrac{1}{2}, \tfrac{1}{2}x^2).$$

As in our previous paper, the starting point of the investigations will be an integral, which can be considered as an Laplace-type inversion formula. This representation turns out to be very suitable for obtaining uniform asymptotic expansions. For $b \sim x$, the saddle point of the integrand lies near by a singularity. By expanding the integrand or by integrating by parts as suggested by BLEISTEIN [2], we obtain two types of uniform asymptotic expansions in terms of functions allied to parabolic cylinder functions.

WONG [8] gives an asymptotic expansion of the Whittaker function $W_{\kappa,\mu}(z)$ for large values of the three parameters. In his expansion, parabolic cylinder functions also occur. For $z \to \infty$, $|\arg z| < \pi - \delta$, $\kappa = o(z)$, $\mu = o(z^{\frac{1}{2}})$ the expansion is

$$W_{\kappa,\mu}(z^2) \sim 2^{\frac{1}{4}-\kappa} z^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+2\mu+n)}{n!\,\Gamma(\frac{1}{2}+2\mu-n)}\; \frac{D_{2\kappa-n-\frac{1}{2}}(z2^{\frac{1}{2}})}{(z2^{3/2})^n}\ .$$

In our expansions we can take both κ and μ of order $O(z^2)$, but we have the condition $|\mu-\kappa| \le M$ for some positive constant M.

Some of the above mentioned results of Dingle are based on the work of JORNA [5], who gave the results for the U-function only. Jorna's expansion, however, is valid for $b \to \infty$ under the restriction $x/b - 1 = o(1)$ (for $b \to \infty$), whereas in our expansions the x-variable is not restricted to a small neighbourhood of $x = b$. Moreover, Jorna's treatment is rather formal.

KAZARINOFF [6] investigated the Whittaker functions for complex variables and for large $|\kappa|$, $|\mu|$ and x unrestricted, under the hypothesis that $(\mu^2-\kappa^2)/\kappa$ be bounded. His rigorous analysis is based upon the methods of R.E. Langer for differential equations. Therefore, his method is quite different from ours, while his expansions contain first order approximations only. But our approximations are of the same kind, with essentially the same argument in the parabolic cylinder functions.

2. CONTOUR INTEGRALS

$U(a,b,x)$ and $M(a,b,x)$ are solutions of Kummer's differential equation

$$x\,y'' + (b-x)\,y' - ay = 0. \tag{2.1}$$

If $a \neq 0,-1,-2,\ldots$, M and U are linearly independent. In general, U is singular at $x = 0$, whereas M is an entire function with the expansion

$$M(a,b,x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+n)} \frac{x^n}{n!}. \tag{2.2}$$

For fixed values of a,b and as $x \to \infty$ we have

$$\begin{aligned} M(a,b,x) &= \Gamma(b)\, e^x\, x^{a-b}/\Gamma(a)\, (1+O(x^{-1})), \\ U(a,b,x) &= x^{-a}\, (1+O(x^{-1})). \end{aligned} \tag{2.3}$$

In this section, we consider integrals of the type

$$\frac{1}{2\pi i} \int_L e^s\, s^c\, (s-x)^{-a}\, ds, \tag{2.4}$$

where a,c and x are real numbers. Throughout this paper we take $x \geq 0$ and

$$c = a - b;$$

a and b are the parameters of the confluent hypergeometric functions. L is a contour either so that it is a closed circuit such that the integrand of (2.4) returns to its initial value after s has described the circuit, or so that the integrand vanishes at each limit. Of course the integral is supposed to converge on L.

LEMMA 2.1. *Let* L *be specified as above. Then the integral in* (2.4), *considered as a function of* x, *satisfies Kummer's equation* (2.1).

PROOF. Denoting the function in (2.4) by y(x), we obtain by standard methods (cf. HOCHSTADT [4, p. 100]).

$$xy'' + (b-x)\,y' - ay = \frac{-a}{2\pi i} \int_L \frac{d}{ds}\left[e^s\, s^{c+1}\, (s-x)^{-a-1}\right] ds,$$

from which the lemma follows. □

After a further specification of L, we wish to write the integral (2.4) as a linear combination of the M- and U-function. In the following three lemmas, the many-valued functions are supposed to be real for positive values of their arguments.

LEMMA 2.2. *Let* L *be given as in figure* 2.1. *On* L *the phase of* s *increases from* $-\pi$ *to* π *as* s *describes the contour.* L *encircles the point* x *in positive direction. Let the branch-cuts of* s^c *and* $(s-x)^{-a}$ *be chosen from* 0, *respectively* x, *to* $-\infty$, *such that they are both enclosed by* L. *Then*

$$(2.5)\qquad \frac{1}{2\pi i} \int_{-\infty}^{(x+)} e^s\, s^c\, (s-x)^{-a}\, ds = \frac{1}{\Gamma(b)}\, M\,(a,b,x).$$

PROOF. By considering the behaviour of U and M near x = 0, and using Hankel's integral

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^s\, s^{-z}\, ds = 1/\Gamma(z),$$

the lemma is easily verified. □

LEMMA 2.3. *Let* L *and the branch-cuts of* s^c *and* $(x-s)^{-a}$ *be as indicated in figure* 2.2. *Then*

$$(2.6)\qquad \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^s\, s^c\, (x-s)^{-a}\, ds = \frac{1}{\Gamma(-c)}\; U(a,b,x).$$

PROOF. In this case we consider (2.3) and we use Watson's lemma for loop integrals. □

LEMMA 2.4. *For* $\varepsilon = \pm 1$, *let* L_ε *be as indicated in figures* 2.3 *and* 2.4. L_ε *encloses the branch-cut of* $(s-x)^{-a}$ *and it passes above (beneath) the origin for* $\varepsilon = +1(-1)$. *Then*

$$(2.7)\qquad \frac{1}{2\pi i}\int_{L_\varepsilon} e^s\, s^c\, (s-x)^{-a} = \frac{e^{\varepsilon i\pi c+x}}{\Gamma(a)}\; U\,(-c,b,e^{-\varepsilon i\pi}x).$$

PROOF. After a shift $s \to s + x$ in (2.6), we obtain an integrand resembling that of (2.6). Next, the value of x is changed into $e^{\varepsilon i\pi}x$, which gives (2.7). □

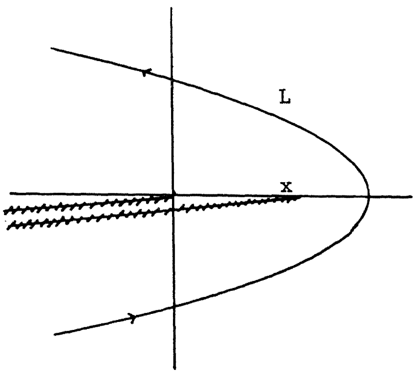

Figure 2.1

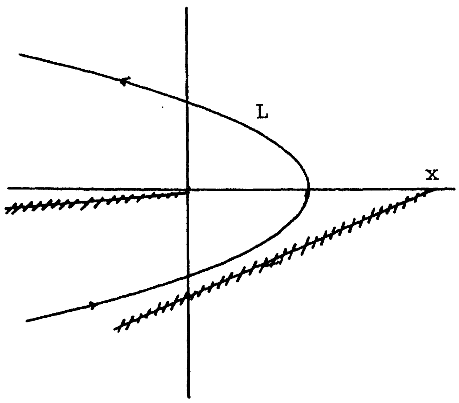

Figure 2.2

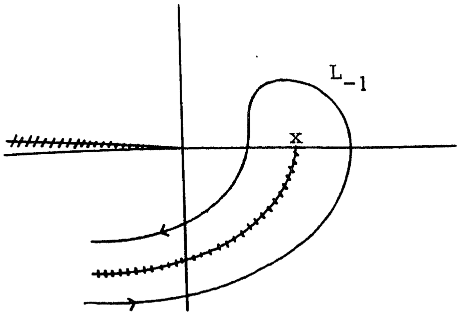

Figure 2.3

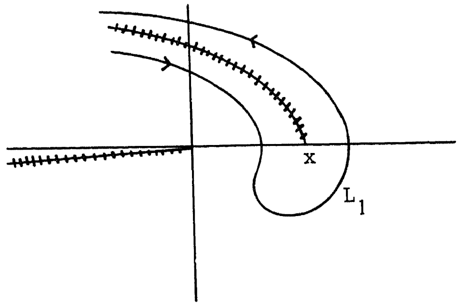

Figure 2.4

<u>REMARK 2.5.</u> The confluent hypergeometric functions in (2.5), (2.6) and (2.7) are related to each other, as follows from the connection formula

$$\text{(2.8)} \qquad M(a,b,x) = \frac{\Gamma(b)}{\Gamma(b-a)}\, e^{i\varepsilon\pi a} U(a,b,x) + \frac{\Gamma(b)}{\Gamma(a)}\, e^{i\pi\varepsilon c+x} U(b-a,b,e^{-i\varepsilon\pi}x)$$

where $\varepsilon = \pm 1$. This formula follows from our results by deforming the contour in figure 2.1 into the contours of figures 2.2 and 2.3 (for $\varepsilon = -1$) or into those of figures 2.2 and 2.4 (for $\varepsilon = +1$).

More integral representations can be derived from (2.1), but in this paper we only use the above ones. Of course, the results (2.5) through (2.7) are valid for wider ranges of the parameters.

The following function is important in the asymptotic expansions of this paper

$$\text{(2.9)} \qquad W_a(v) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} (u-v)^{-a}\, du,$$

with $a \in \mathbb{R}$, $v \in \mathbb{C}$. The contour of integration passes the singularity at $u = v$ as in the following picture.

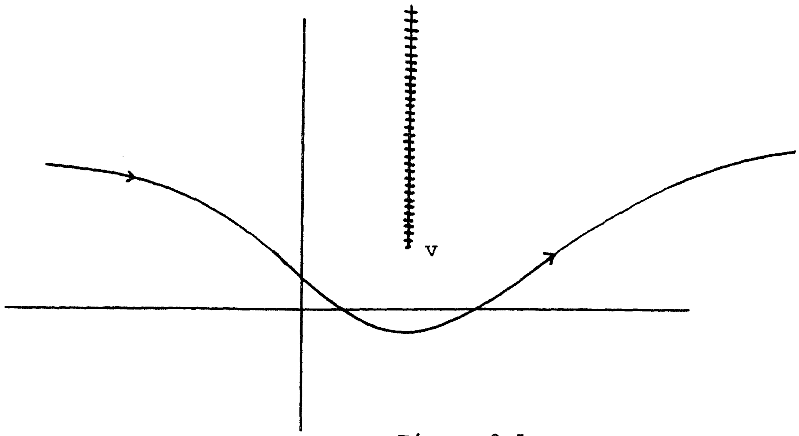

Figure 2.5

As follows from ABRAMOWITZ & STEGUN [1, p. 688], $W_a(v)$ is related to the parabolic cylinder function, this relation being given by

$$(2.10) \qquad W_a(v) = (2\pi)^{\frac{1}{2}} \exp(-\tfrac{1}{4}v^2+\tfrac{1}{2}ia\pi)\ D_{-a}(-iv).$$

Clearly, we have

$$(2.11) \qquad \frac{d}{dv} W_a(v) = a\ W_{a+1}(v).$$

Furthermore, we use the functions

$$(2.12) \qquad \begin{aligned} F_k(v) &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ u^k (u-v)^{k-a} du, \\ G_k(v) &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ u^{k+1} (u-v)^{k-a} du \end{aligned}$$

for $k = 0,1,\ldots$, where, again, the integration is as in figure 2.5. By expanding $u^k = [(u-v)+v]^k$ in a finite binomial series, $F_k(v)$ and $G_k(v)$ can be expressed as finite linear combinations of $W_{a-n}(v)$. By integration by parts, recurrence relations can be derived, for instance

$$(2.13) \qquad \begin{aligned} G_k(v) &= (2k-a)G_{k-1}(v) - vk\ F_{k-1}(v), \\ F_k(v) &= (2k-a-1)F_{k-1}(v) - v(k-1)[G_{k-2}(v) - vF_{k-2}(v)]. \end{aligned}$$

3. UNIFORM EXPANSIONS

3.1. *Saddle point contours*. Let us start with M(a,b,x). We derive an asymptotic expansion of this function for $x \to \infty$ and/or $b \to \infty$, uniformly valid with respect to λ, where

$$(3.1) \qquad \lambda = x/b.$$

From (2.5) we obtain

$$(3.2) \qquad M(a,b,x) = \frac{\Gamma(b+1)e^{b}b^{-b}}{2\pi i} \int_{-\infty}^{(\lambda^+)} e^{b\phi(t)} (1-\lambda/t)^{-a} dt,$$

where

$$(3.3) \qquad \phi(t) = t - 1 - \ln t.$$

Let us suppose temporarily $\lambda < 1$. As in [7], we choose the contour in (3.2) through the saddle point of ϕ at $t = 1$ along the steepest descent curve L given by $\operatorname{Im} \phi(t) = 0$, or explicitly

$$(3.4) \qquad \sigma = \tau \operatorname{cotg} \tau, \qquad -\pi < \tau < \pi,$$

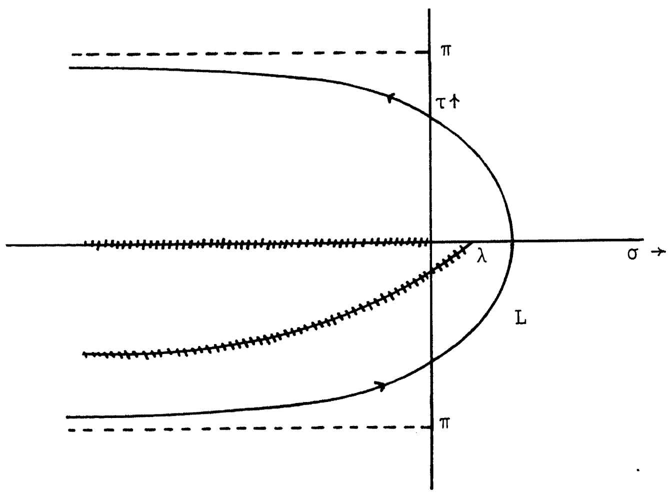

Figure 3.1

where $t = \sigma + i\,\tau$ $(\sigma,\tau \in \mathbb{R})$, see figure 3.1. On L the function ϕ is real and non-positive. Next, we define a mapping of the t-plane into the u-plane by the equation

$$(3.5) \qquad -\tfrac{1}{2}u^2 = \phi(t)$$

with the condition that $t \in L$ corresponds with $u \in \mathbb{R}$, and $u < 0$ if $\tau < 0$, $u > 0$ if $\tau > 0$. From these conditions it follows that

$$(3.6) \qquad u = i(1-t)\,[2(t-1-\ln t)/(1-t)^2]^{\frac{1}{2}},$$

where the square root is positive for positive values of its argument.

Integration with respect to u gives for (3.2)

$$(3.7) \qquad M(a,b,x) = \frac{\Gamma(b+1)e^b b^{-b}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, \frac{dt}{du}\, \frac{du}{(1-\lambda/t)^a}\,.$$

The singular points of the integrand in (3.7) are of two different types. First, we have the singularity due to the factor $(1-\lambda/t)^{-a}$ (of course, a singularity will only occur if $a \neq 0,-1,-2,\ldots$). The singular point $t = \lambda$ in the t-plane corresponds with a point $u = u(\lambda) = u_1$, say, in the u-plane explicitly given by (cf. (3.6))

$$(3.8) \qquad u_1 = i(1-\lambda)\,[2(\lambda-1-\ln \lambda)/(1-\lambda)^2]^{\frac{1}{2}},$$

and if $\lambda \to 1$, then $u_1 \to 0$. The contour in (3.7) is as in figure 2.5, with $v = u_1$. If $\operatorname{Im} u_1 > 0$, an ideal contour of integration is the steepest descent path $\operatorname{Im} u = 0$. If $\operatorname{Im} u_1 \leq 0$, the contour in (3.7) will be deformed around the branch-cut of $(1-\lambda/t)^{-a}$. Hence, we may dispose of the condition $0 \leq \lambda < 1$ and we suppose henceforth $\lambda \geq 0$.

The second type of singularities of the integrand are due to the factor dt/du, which, by using (3.3) and (3.5) can be written as

$$(3.9) \qquad \frac{dt}{du} = \frac{ut}{1-t}$$

The point $t = 1$, corresponding to $u = 0$, gives a regular point. But, on account of the many-valuedness of the logarithm in (3.3), we also must consider the points $\exp(2\pi in)$, for integer values of n, giving a sequence of singular points

$$(3.10) \qquad 2(\pi in)^{\frac{1}{2}},$$

in the u-plane. When distorting the contour in (3.7) in order to allow non-positive values of Im u_1, the singularities (3.10) must be avoided.

It is important to note that the singularities of the second type, given in (3.10), are fixed points in the u-plane, whereas u_1 given in (3.8) may be close to the origin (the saddle point). The point u_1 causes a non-uniform behaviour in (3.7) while the points in (3.10) are of a secundary interest.

The standard method for obtaining an asymptotic expansion via (3.7) is based on the substitution of the expansion

$$\frac{dt}{du}\,(1 - \lambda/t)^{-a} = \sum c_k(\lambda)\, u^k$$

in (3.7) followed by termwise integration. Owing to the singularity at u_1, a non-uniform expansion is obtained in this way. In fact, all $c_k(\lambda)$ are singular for $\lambda = 1$. In the following subsections, we give two types of uniform asymptotic expansions.

3.2. *Bleistein's method.* In the first place, we use an integration by parts procedure suggested by BLEISTEIN [2]. The integral in (3.7) is written as

$$(3.11) \qquad J(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, G(u) \frac{du}{(u-u_1)^a},$$

where

$$(3.12) \qquad G(u) = [(u-u_1)/(\lambda/t-1)]^a\, \frac{dt}{du}.$$

Except for the points given in (3.10), G is a holomorphic function of u. Especially, it is regular at $u = u_1$. Let us write

(3.13) $$G(u) = \gamma_0 + (u-u_1)\,\gamma_1 + u(u-u_1)G_1(u),$$

where γ_0, γ_1 and G_1 must be determined. Substituting $u = u_1$, $u = 0$ respectively, we obtain

(3.14) $$\gamma_0 = G(u_1), \qquad \gamma_1 = [G(u_1) - G(0)]/u_1,$$

and with γ_0 and γ_1, G_1 follows from (3.13). As G, it is regular except for the points in (3.10).

Upon inserting (3.13) into (3.11), we can rewrite $J(a,b,u_1)$ in the form

(3.15) $$J(a,b,u_1) = b^{\frac{1}{2}(a-1)}[\gamma_0\, W_a\,(u_1 b^{\frac{1}{2}}) + b^{-\frac{1}{2}}\gamma_1\, W_{a-1}(u_1 b^{\frac{1}{2}})] + J_1(a,b,u_1),$$

where W_a is defined in (2.7) and

(3.16) $$J_1(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, u(u-u_1)^{-a+1}\, G_1(u)\, du.$$

We integrate by parts in (3.15) and obtain

$$b\, J_1(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}(u-u_1)^{-a}\, [(1-a)G_1(u) + (u-u_1)G_1'(u)]\, du.$$

The procedure of (3.15) and (3.16) can now be applied to $b\, J_1(a,b,u_1)$ if we set

$$(1-a)G_1(u) + (u-u_1)G_1'(u) = \gamma_2 + (u-u_1)\gamma_3 + u(u-u_1)G_2(u).$$

It then follows that

$$J(a,b,u_1) = b^{\frac{1}{2}(a-1)}[(\gamma_0 + \gamma_2\, b^{-1})W_a(u_1 b^{\frac{1}{2}}) + b^{-\frac{1}{2}}\,(\gamma_1 + \gamma_3 b^{-1})W_{a-1}(u_1 b^{\frac{1}{2}})]$$
$$+\, b^{-1}\, J_2(a,b,u_1),$$
$$J_2(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, u(u-u_1)^{-a+1}G_2(u)\, du.$$

This process can be continued to obtain an arbitrary number of terms. The final result is the asymptotic expansion

$$(3.17)\qquad M(a,b,x) \sim \frac{\Gamma(b+1)e^{b}b^{-b+\frac{1}{2}(a-1)}}{2\pi i}\left[W_a(u_1 b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n}b^{-n} + \right.$$

$$\left. + b^{-\frac{1}{2}} W_{a-1}(u_1 b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n+1}b^{-n}\right].$$

From (3.14), (3.12) and (3.8) it follows that

$$(3.18)\qquad \begin{aligned} \gamma_0 &= \lambda\,[(1-\lambda)/u_1]^{a-1},\\ \gamma_1 &= \{\lambda\,[(1-\lambda)/u_1]^{a-1} - i\,[u_1/(\lambda-1)]^{a}\}/u_1. \end{aligned}$$

In general,

$$\gamma_{2n} = (1-a)\;G_n(u_1),$$

$$\gamma_{2n+1} = (1-a)\;[G_n(u_1) - G_n(0)]/u_1 + G_n'(0),$$

the functions G_n are determined recursively from the equations

$$(3.19)\qquad (1-a)G_n(u) + (u-u_1)G_n'(u) = \gamma_{2n} + (u-u_1)\;\gamma_{2n+1} + u(u-u_1)G_{n+1}(u),$$

$n = 0,1,\ldots$, with $G_0(u) = G(u)$ given in (3.12). By inspection $\gamma_k = 0(1)$ in λ if $\lambda \to 1$.

By using the more familiar parabolic cylinder functions we obtain, by considering (2.8) and the recurrence relation

$$D_{\nu+1}(x) = (x/2)\;D_\nu(x) - D_\nu'(x),$$

the asymptotic expansion in which all variables are real

$$(3.20)\qquad M(a,b,x) \sim (2\pi)^{-\frac{1}{2}}\Gamma(b+1)b^{-b+\frac{1}{2}(a-1)} e^{b+\frac{1}{4}\zeta^2}\Big[D_{-a}(\zeta)\sum_{n=0}^{\infty}\delta_{2n}b^{-n} + b^{-\frac{1}{2}}D'_{-a}(\zeta)\sum_{n=0}^{\infty}\delta_{2n+1}b^{-n-\frac{1}{2}}\Big]$$

where

$$(3.21)\qquad \zeta = -iu_1 b^{\frac{1}{2}},\ \delta_{2n} = -ie^{\frac{1}{2}ai\pi}(\gamma_{2n} - \tfrac{1}{2}u_1\gamma_{2n+1}),\ \delta_{2n+1} = e^{\frac{1}{2}ai\pi}\gamma_{2n+1}.$$

REMARK 3.1. For $a = 1$, the confluent hypergeometric functions can be expressed as incomplete gamma functions, see (1.2). For this case $W_0(u_1 b^{\frac{1}{2}}) = (2\pi)^{\frac{1}{2}}$, $W_1(u_1 b^{\frac{1}{2}}) = i\pi \exp(-\frac{1}{2}bu_1^{\,2})\mathrm{erfc}(\zeta/\sqrt{2})$, $\gamma_0 = \lambda$, $\gamma_{2n} = 0$ $(n > 9)$. As can be verified, expansion (3.17) reduces indeed to the result of our previous paper.

Bleistein showed that an expansion like (3.17) is uniformly valid with respect to λ in a neighbourhood of $\lambda = 1$. In the case of the incomplete gamma functions, the expansions turned out to be uniformly valid for $\lambda \geq 0$, and we might expect the expansion (3.17) to hold in the same λ-domain. In terms of ζ given in (3.21), the expansions then are expected to hold uniformly for all real ζ. For that purpose, the following properties have to be verified.

(i) The sequences $\{\gamma_{2n}b^{-n}\}$ and $\{\gamma_{2n+1}b^{-n}\}$ are uniform asymptotic sequences. That is to say, the elements of the sequences have to satisfy

$$\gamma_{2n+2}\, b^{-n-1} = o(\gamma_{2n}b^{-n}),\quad \gamma_{2n+3}\, b^{-n-1} = o(\gamma_{2n+1}b^{-n}),$$

$n = 0,1,2,\ldots$, for $b \to \infty$, uniform in $\zeta \in \mathbb{R}$.

(ii) There are sequences $\{\alpha_{2n}\}$, $\{\alpha_{2n+1}\}$, which are uniform asymptotic sequences for $b \to \infty$, uniformly in $\zeta \in \mathbb{R}$, such that for n, $m = 0,1,2,\ldots$

$$M(a,b,x) = \frac{\Gamma(b+1)e^b b^{-b+\frac{1}{2}(a-1)}}{2\pi i}\Big[W_a(u_1 b^{\frac{1}{2}})\Big\{\sum_{i=0}^{n-1}\gamma_{2i}b^{-i} + O(\alpha_{2n})\Big\} + b^{-\frac{1}{2}}W_{a-1}(u_1 b^{\frac{1}{2}})\Big\{\sum_{i=0}^{m-1}\gamma_{2i+1}b^{-i} + O(\alpha_{2m+1})\Big\}\Big]$$

for $b \to \infty$, uniform in $\zeta \in \mathbb{R}$.

A drawback of Bleistein's method is a lack of an explicit expression for the coefficients γ_n and the functions G_n, which are only given recursively. As a consequence, it is difficult to verify the properties in (i) and (ii), even in our case where the function G is given explicitly. However, inspection of the first ratio γ_2/γ_0 for $\zeta \to \pm\infty$ indicates that the uniformity with respect to ζ cannot be given for the whole domain $\mathbb{R}$. The most we can expect is uniformity with respect to an interval $[-A,B]$, where A,B depend on a,b, such that $A,B \to \infty$ for $b \to \infty$ (a fixed).

A pleasant feature of Bleistein's method is the form of the asymptotic expansion, in which only two parabolic cylinder functions occur. In the following section, we give an alternative expansion, but first we give the results for U(a,b,x) corresponding to (3.17).

The starting point is (2.6). After some transformations, we obtain

$$U(a,b,x) = \frac{b^{1-b}e^{b}\Gamma(b-a)}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}G(u)\,\frac{du}{(u_1-u)^a},$$

where G and u_1 are given in (3.12) and (3.8). The contour passes the singularity at $u = u_1$ as in figure 3.2.

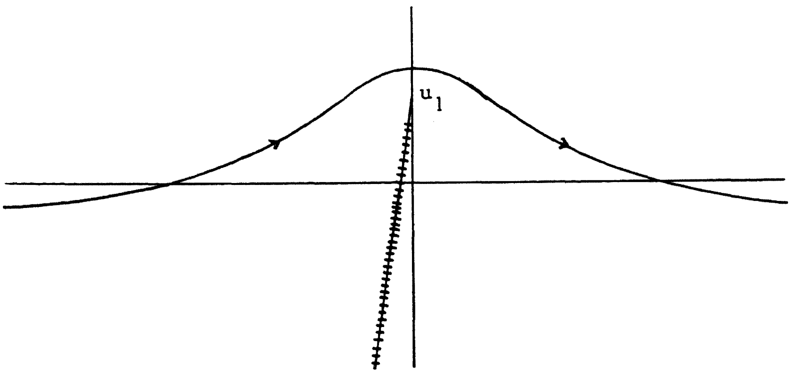

Figure 3.2

The asymptotic expansion now contains functions of the type

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}(v-u)^{-a}du.$$

If we make the change of variable $u \to -u$, it turns out that this integral equals $W_a(-v)$.

Proceeding as before, we arrive at the expansion

$$(3.22)\quad U(a,b,x) \sim \frac{b^{\frac{1}{2}(1+a)-b}e^b\Gamma(b-a)}{2\pi i}\left[W_a(-u_1b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n}b^{-n} - b^{-\frac{1}{2}}W_{a-1}(-u_1b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n+1}b^{-n}\right].$$

The coefficients γ_n are the same as those for $M(a,b,x)$ in (3.17).

The expansions for the integrals along the contours in figures 2.3 and 2.4 follow now from (2.8). By using (3.17), (3.20) and a connection formula for the parabolic cylinder functions, viz.

$$D_\nu(z) = e^{-\varepsilon i\nu\pi}\, D_\nu(-z) + (2\pi)^{\frac{1}{2}}\, \Gamma(-\nu)^{-1}\, D_{-\nu-1}(\varepsilon iz)e^{-\varepsilon i\pi(\nu+1)/2},$$

with $\varepsilon = \pm 1$, we obtain

$$(3.23)\qquad U(b-a,b,e^{-\varepsilon i\pi}x) \sim -\, (2\pi)^{-\frac{1}{2}}b^{\frac{1}{2}(a+1)-b}\, e^{-\frac{1}{2}v^2+\frac{1}{2}i\pi[2a+\varepsilon(1-a+2b)]+b-x}$$
$$\left\{W_{1-a}(-i\varepsilon u_1b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n}b^{-n} - (1-a)b^{-\frac{1}{2}}e^{\frac{1}{2}\pi i\varepsilon}W_{2-a}(-i\varepsilon u_1b^{-\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n+1}b^{-n}\right\}.$$

REMARK 3.2. The expansions in (3.17), (3.22) and (3.23) are given as series of inverse powers of b. The results are valid for $b \to \infty$, uniformly valid with respect to λ in a neighbourhood of $\lambda = 1$. By considering in (2.5) and (2.6) x as a large parameter, we can derive asymptotic expansions for $U(a,b,x)$ and $M(a,b,x)$ with series in inverse powers of x for $x \to \infty$, uniformly valid with respect to λ, again in a neighbourhood of $\lambda = 1$.

3.3. *Alternative expansions.* If we expand the function G in (3.12) in a two-points Maclaurin expansion

(3.24) $$G(u) = \sum_{k=0}^{\infty} c_k u^k (u-u_1)^k + u \sum_{k=0}^{\infty} d_k u^k (u-u_1)^k,$$

were the c_k and d_k are to be determined, we obtain by termwise integration in (3.11)

(3.25) $$J(a,b,u_1) \sim b^{\frac{1}{2}(a-1)} \left[\sum_{k=0}^{\infty} c_k F_k(u_1 b^{\frac{1}{2}}) b^{-k} + \sum_{k=0}^{\infty} d_k G_k(u_1 b^{\frac{1}{2}}) b^{-k-\frac{1}{2}} \right].$$

The functions F_k and G_k are given in (2.12) and recursion relations between them in (2.13). The coefficients c_k and d_k may be obtained by substituting the values $u = 0$ and $u = u_1$ and differentiating the series. The first few are

$$c_0 = G(0), \qquad d_0 = [G(u_1) - G(0)]/u_1,$$

$$c_1 = G'(0) + d_0/u_1, d_1 = [G'(u_1) - d_0 - c_1 u_1]/u_1^2.$$

The following lemma gives an explicit formula for c_k and d_k for general k.

LEMMA 3.3. *Let* $w(v) = (v + \frac{1}{4}u_1^2)^{\frac{1}{2}}$ *and let the functions* H_1 *and* H_2 *be given by*

$$H_1(v) = [G(\tfrac{1}{2}u_1 + w(v)) - G(\tfrac{1}{2}u_1 - w(v))]/w(v),$$

$$H_2(v) = [(w(v) - \tfrac{1}{2}u_1)\, G(\tfrac{1}{2}u_1 + w(v)) + (w(v) + \tfrac{1}{2}u_1)\, G(\tfrac{1}{2}u_1 - w(v))]/w(v),$$

where the square root in w *is real for positive arguments, then the coefficients* c_k *and* d_k *in* (3.24) *are given by*

(3.25) $$2c_k = \frac{1}{2\pi i} \int_{C_2} \frac{H_2(v)}{v^{k+1}}\, dv, \qquad 2d_k = \frac{1}{2\pi i} \int_{C_1} \frac{H_1(v)}{v^{k+1}}\, dv,$$

where C_i *are simple closed contours encircling* $v = 0$ *but not encircling any singularity of* H_i, $i = 1,2$.

PROOF. If we substitute $u = \frac{1}{2}u_1 + v$ in (3.24), we obtain

$$G(\tfrac{1}{2}u_1+v) = \sum c_k(v^2-\tfrac{1}{4}u_1^2)^k + (v+\tfrac{1}{2}u_1) \sum d_k(v^2-\tfrac{1}{4}u_1^2)^k.$$

Splitting up the right-hand side in odd and even parts (with respect to v) and using Cauchy's integral formula for the coefficients of the MacLaurin expansion of holomorphic functions we obtain the representations for c_k and d_k. Since G is holomorphic except at the points (3.10), H_1 and H_2 are holomorphic in a neighbourhood of $v = 0$. □

It is not difficult to prove that the sequences $\{c_k F_k b^{-k}\}$ and $\{d_k G_k b^{-k}\}$ are uniform asymptotic sequences for $b \to \infty$, uniform in a neighbourhood of $\lambda = 1$, and that (3.25) gives a uniform asymptotic expansion. An optimal interval of uniformity has not been obtained, but we expect that the investigations on this subject are carrid out easier with (3.25) than by using the expansion of the previous subsection.

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The Asymptotic Expansion of the Incomplete Gamma Functions

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ABSTRACT

Earlier investigations on uniform asymptotic expansions of the incomplete gamma functions are reconsidered. The new results include estimations for the remainder and the extension of the results to complex variables. Furthermore asymptotic expansions of the inverse functions are given.

KEY WORDS & PHRASES: *incomplete gamma function, asymptotic expansion, inverse incomplete gamma functions.*

1. INTRODUCTION

We consider the incomplete gamma functions ratios P and Q defined by

$$(1.1)\qquad \begin{aligned} P(a,x) &= \frac{1}{\Gamma(a)}\int_0^x t^{a-1}\, e^{-t}\, dt, \\ Q(a,x) &= \frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}\, e^{-t}\, dt. \end{aligned}$$

We suppose first that x and a are real with

$$(1.2)\qquad x \geq 0, \quad a > 0.$$

In TEMME [5] we derived asymptotic expansions of P and Q for $a \to \infty$, uniformly valid for $x \geq 0$. In this paper we reconsider these expansions. Our new results concern the representations of the remainder in the asymptotic expansion, representations for the coefficients of the expansion for numerical applications, numerical upper bounds for the remainder of the case of real variables, and extension of the asymptotic expansions to the case of complex variables. Furthermore we give an asymptotic expansion of the inverse function.

To describe the expansions given in [5] we introduce the following parameters

$$(1.3)\qquad \lambda = x/a, \quad \mu = \lambda - 1, \quad \eta = \{2[\mu - \ell n(1+\mu)]\}^{\frac{1}{2}},$$

with the convention that the square root has the sign of $\mu (\mu > -1)$. As a function of μ, η is monotone and infinitely differentiable on $(-1,\infty)$. Analytic properties of $\eta(\mu)$ for complex μ are considered in §5.

The asymptotic expansions of P and Q derived in [5] follow from the representations

$$(1.4)\qquad \begin{aligned} P(a,x) &= \tfrac{1}{2}\,\mathrm{erfc}[-\eta(a/2)^{\frac{1}{2}}] - R_a(\eta) \\ Q(a,x) &= \tfrac{1}{2}\,\mathrm{erfc}[\eta(a/2)^{\frac{1}{2}}] + R_a(\eta) \end{aligned}$$

with

$$(1.5)\qquad R_a(\eta) \sim (2\pi a)^{-\frac{1}{2}}\, e^{-\frac{1}{2}a\eta^2} \sum_{k=0}^{\infty} c_k(\eta)\, a^{-k}$$

for $a \to \infty$, uniformly valid with respect to $\eta \in \mathbb{R}$; erfc is the complementary error function defined by

$$(1.6)\qquad \operatorname{erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-t^2}\, dt.$$

The expansion (1.5) was derived by using saddle point methods. In § 2 we use a different method which yields recurrence relations for the coefficients c_k and a representation for the remainder of (1.5). In §3 we discuss representations for c_k that can be used for numerical applications. In §4 numerical error bounds are constructed for the remainder of the series in (1.5) when the first n terms in the series are used. Bounds are given up to $n = 10$. As a side result this section gives bounds for the remainder of the asymptotic expansion of the reciprocal gamma function $1/\Gamma(x)$ for real x. In § 5 the results are extended to complex values of a and x. In § 6 a new asymptotic expansion for the inverse of the incomplete gamma functions is derived. To describe this, let $q \in [0,1]$. Then the function $x(q,a)$ implicitly defined by the equation $Q(a,x) = q$ is called the inverse. We give an asymptotic expansion of the form

$$(1.7)\qquad x(q,a) \sim a(x_0 + x_1\, a^{-1} + x_2\, a^{-2} + \ldots)$$

for $a \to \infty$. This expansion is based on inversion of the uniform asymptotic expansion for Q. The analysis is formal but it appears that (1.7) is valid in $q \in [0,1]$. Some information is given about the first coefficients in (1.7).

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2. RECURRENCE RELATIONS FOR THE COEFFICIENTS AND REPRESENTATION OF THE REMAINDER.

First we remark that the asymptotic expansion for $a \to \infty$, of $dR_a(\eta)/d\eta$ may be obtained by formal differentiation of (1.5). This is not proved here, but it follows from the representation of $R_a(\eta)$ in our previous paper (formula (2.10) of TEMME [5]). The result is

$$\frac{dR_a(\eta)}{d\eta} \sim a(2\pi a)^{-\frac{1}{2}}\, e^{-\frac{1}{2}a\eta^2} \sum_{k=0}^{\infty} c_k^{(1)}(\eta)\, a^{-k} \tag{2.1}$$

with

$$\begin{aligned} c_0^{(1)}(\eta) &= -\eta c_0(\eta) \\ c_k^{(1)}(\eta) &= -\eta c_k(\eta) + \frac{d\, c_{k-1}(\eta)}{d\eta}; \quad k \geq 1. \end{aligned} \tag{2.2}$$

Secondly, we need the coefficients of the asymptotic expansion of the complete gamma function. Let us define

$$\Gamma^*(a) = (a/2\pi)^{\frac{1}{2}}\, e^a\, a^{-a}\,\Gamma(a), \quad a > 0. \tag{2.3}$$

Then Γ^* and $1/\Gamma^*$ have the well-known asymptotic expansions for $a \to \infty$

$$\begin{aligned} \Gamma^*(a) &\sim \sum_{k=0}^{\infty} (-1)^k \gamma_k\, a^{-k} \\ 1/\Gamma^*(a) &\sim \sum_{k=0}^{\infty} \gamma_k\, a^{-k}. \end{aligned} \tag{2.4}$$

The first few coefficients are

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}.$$

Further coefficients follow from SPIRA [3] and WRENCH [6]. Wrench gives $(-1)^k\gamma_k$ up to $k = 20$ in rational form, Spira the remaining up to $k = 30$. Decimal representations are also given in both references.

With these preparations we have

OREM 1. *Let* $\{\gamma_k\}$ *be defined by* (2.4). *Then the coefficients* c_k *of* (1.5) *isfy the recurrence relation*

$$c_0(\eta) = \frac{1}{\mu} - \frac{1}{\eta},$$

.5)

$$\eta c_k(\eta) = \frac{d\,c_{k-1}(\eta)}{d\eta} + \frac{\eta}{\mu}\gamma_k, \qquad k \geq 1.$$

OOF. By differentiating one of the formulas in (1.4) with respect to η d by using (2.1) it follows that

.6) $$\frac{d}{d\eta}R_a(\eta) = (a/2\pi)^{\frac{1}{2}}\left(1 - \frac{1}{\mu+1}\,\frac{1}{\Gamma^*(a)}\,\frac{d\mu}{d\eta}\right) e^{-\frac{1}{2}a\eta^2}.$$

om (1.4) we have

.7) $$\frac{d\mu}{d\eta} = \frac{(\mu+1)\eta}{\mu},$$

d substituting (2.1) and the second relation of (2.4) we obtain (2.5) collecting equal powers of a^{-1} and using (2.2). □

As follows from [5], the coefficients c_k are holomorphic in a neighourhood of $\eta = 0$. In fact the singularities of $1/\mu$ and $1/\eta$ in c_0 cancel ach other. So the limiting value of c_0 for $\eta \to 0$ is well defined.

Owing to the derivative of c_{k-1} in (2.5) this formula cannot be andled easily from a numerical point of view. Further, the above mentioned ancellation of singular parts in c_0 occurs in all c_k when working with 2.5). Therefore other representations are given for these coefficients. In he next section we discuss some aspects of the Taylor expansions for small $|\eta|$-values, while for larger $|\eta|$-values a recurrence relation is constructed rom which the coefficients can be computed directly. But first we give epresentations of the remainder in the asymptotic expansion (1.5).

From (1.4) it follows that $R_a(\infty) = R_a(-\infty) = 0$. Hence, integration of (2.6) gives

(2.8) $$R_a(\zeta) = (a/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\zeta} \left[1 - \frac{\eta}{\mu}\,\frac{1}{\Gamma^*(a)}\right] e^{-\frac{1}{2}a\eta^2}\, d\eta =$$

$$= -(a/2\pi)^{\frac{1}{2}} \int_{\zeta}^{\infty} [1 - \frac{\eta}{\mu} \frac{1}{\Gamma^*(a)}] \, e^{-\frac{1}{2}a\eta^2} \, d\eta,$$

where μ as a function of η is defined implicitly in (1.4). From these representations and the recurrence relations for c_k a simple expression for the remainder follows. For this purpose we introduce the notation

$$R_a(\eta) = (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} [\sum_{k=0}^{N-1} c_k(\eta) \, a^{-k} + a^{-N} G_N(\eta;a)], \tag{2.9}$$

$a > 0$, $\eta \in \mathbb{R}$, $N = 0,1,2,\ldots$. Furthermore, we need a notation for the remainder in the asymptotic expansion of $1/\Gamma^*(a)$, which is written as

$$1/\Gamma^*(a) = \sum_{k=0}^{N-1} \gamma_k \, a^{-k} + a^{-N} H_N(a), \qquad a > 0, \; N = 0,1,2,\ldots . \tag{2.10}$$

<u>THEOREM</u> 2. *Let* G_N *and* H_N *be defined by (2.9) and (2.10). Then*

$$e^{-\frac{1}{2}a\zeta^2} G_N(\zeta;a) = a \int_{\zeta}^{\infty} \eta \, c_N(\eta) \, e^{-\frac{1}{2}a\eta^2} \, d\eta + H_{N+1}(a) \int_{\zeta}^{\infty} \frac{\eta}{\mu} \, e^{-\frac{1}{2}a\eta^2} \, d\eta. \tag{2.11}$$

<u>PROOF</u>. Follows immediately from substitution of (2.9) and (2.10) in (2.6) (and by using (2.5) and (2.7)). □

The second integral in (2.11) can be expressed in $Q(a,x)$. From representations of c_k to be given in the following sections it follows that $|c_k(\eta)|$ is a bounded function of $\eta \in \mathbb{R}$. For numerical applications the following is important.

<u>COROLLARY</u> 1. *If* $|c_k(\eta)|$ *is bounded on* $\mathbb{R}$ *then for* $N = 0,1,2,\ldots$

$$|Q(a,x) - \tfrac{1}{2}\,\mathrm{erfc}[\eta(a/2)^{\frac{1}{2}}] - \frac{e^{-\frac{1}{2}a\eta^2}}{(2\pi a)^{\frac{1}{2}}} \sum_{k=0}^{N-1} c_k(\eta) \, a^{-k}| \le \frac{Q_N(\eta;a)}{(2\pi a)^{\frac{1}{2}}} a^{-N} \tag{2.12}$$

with

$$(2.13) \qquad Q_N(\eta;a) = \begin{cases} C_N\, e^{-\frac{1}{2}a\eta^2} \\ C_N\,(2-e^{-\frac{1}{2}a\eta^2}) \end{cases} + |H_{N+1}(a)|\; e^a a^{-a}\Gamma(a)Q(a,x)$$

where the upper term is for $\eta \ge 0$*, the lower one for* $\eta \le 0$*, and where*

$$(2.14) \qquad C_N = \sup_{\eta\in\mathbb{R}} |c_N(\eta)|.$$

In §4 we give numerical values of C_N and bounds for H_N, $0 \le N \le 10$. With these values we have strict and realistic error bounds for the remainder of the uniform asymptotic expansion of $Q(a,x)$. Similar results hold for the function $P(a,x)$. For $N = 0,1,2,\ldots$ we have

$$(2.15) \qquad |P(a,x) - \tfrac{1}{2}\,\mathrm{erfc}[-\eta(a/2)^{\frac{1}{2}}] + \frac{e^{-\frac{1}{2}a\eta^2}}{(2\pi a)^{\frac{1}{2}}} \sum_{k=0}^{N-1} c_k(\eta)\, a^{-k}| \le \frac{P_N(\eta;a)}{(2\pi a)^{\frac{1}{2}}}\, a^{-N}$$

with

$$(2.16) \qquad P_N(\eta;a) = \begin{cases} C_N(2 - e^{-\frac{1}{2}a\eta^2}) \\ C_N \end{cases} + |H_{N+1}(a)|\; e^a a^{-a}\Gamma(a)\; P(a,x),$$

where the upper term is for $\eta \ge 0$, the lower one for $\eta \le 0$.

<u>REMARK</u> 1. The functions multiplying the constants C_N in (2.13) and (2.16) have quite different behaviour for $\eta < 0$ and $\eta > 0$. This, however, is in agreement with the behaviour of the functions P and Q in the same formula. In fact, the bounds P_N and Q_N give a measure for the relative accuracy for the error in the uniform expansions.

<u>REMARK</u> 2. The asymptotic expansion (1.5) and the representation for the remainder is easily obtained by partial integration of one of the integrals in (2.8) and by using the recursions (2.5) and $H_k(a) = \gamma_k + \frac{1}{a} H_{k+1}(a)$.

3. REPRESENTATIONS OF c_k.

Using (2.5) with k = 1 we obtain

$$\eta c_1(\eta) = -\frac{1}{\mu^2}\frac{d\mu}{d\eta} + \frac{1}{\eta^2} + \frac{\eta}{\mu}\gamma_1, \tag{3.1}$$

and using (2.7) we have

$$c_1(\eta) = \frac{1}{\eta 3} - \frac{1+\mu+\mu^2/12}{\mu^3} \tag{3.2}$$

Computing higher order coefficients we notice the following structure

$$c_k(\eta) = (-1)^k\{\frac{Q_k(\mu)}{\mu^{2k+1}} - \frac{A_k}{\eta^{2k+1}}\}, \tag{3.3}$$

where Q_k is a polynomial in μ of degree 2k and $A_k = 2^k\Gamma(k+\frac{1}{2})/\Gamma(\frac{1}{2})$,

$$A_0 = 1, \quad A_k = (2k-1)A_{k-1}, \quad k \geq 1.$$

We obtain for Q_k a recurrence relation with respect to k by substituting (3.3) in (2.5). The result is

$$Q_k(\mu) = (1+\mu)[(2k-1)Q_{k-1}(\mu) - \mu\, Q'_{k-1}(\mu)] + (-1)^k\,\gamma_k\,\mu^{2k}, \tag{3.4}$$

where the derivative is with respect to μ. The first few polynomials are

$$\begin{aligned} Q_0(\mu) &= 1 \\ Q_1(\mu) &= 1 + \mu + \frac{1}{12}\mu^2 \\ Q_2(\mu) &= 3 + 5\mu + \frac{25}{12}\mu^2 + \frac{1}{12}\mu^3 + \frac{1}{288}\mu^4. \end{aligned} \tag{3.5}$$

Again, the recurrence relation (3.4) contains the derivative of Q_{k-1}, but now Q_{k-1} is a polynomial. In order to preserve accuracy for $\mu = -1$ we write

$$Q_k(\mu) = (1+\mu)\,P_k(\mu) + (-1)^k\,\gamma_k\,\mu^{2k}\,, \tag{3.6}$$

and we proceed with P_k. Writing

$$(3.7) \qquad P_k(\mu) = p_0^{(k)} + p_1^{(k)}\mu \ldots p_{2k-2}^{(k)}\,\mu^{2k-2}$$

we have the relation (which is easily obtained by substituting (3.7) and (3.6) in (3.4))

$$p_0^{(k)} = (2k-1)\,p_0^{(k-1)}$$

$$(3.8) \qquad p_j^{(k)} = (2k-1-j)[p_j^{(k-1)} + p_{j-1}^{(k-1)}], \quad j = 1,2,\ldots,2k-4$$

$$p_{2k-3}^{(k)} = 2p_{2k-4}^{(k-1)}, \; p_{2k-2}^{(k)} = (-1)^{k-1}\,\gamma_{k-1},$$

with as starting polynomial $P_1(\mu) = 1$, or $p_0^{(1)} = 1$.

In Table I we give the coefficients $p_j^{(k)}$ of (3.7) for $k = 1,2,\ldots,5$, $j = 0,1,\ldots,2k-2$

TABLE I

k	$p_j^{(k)}$
1	1
2	3, 2, 1/12
3	15, 20, 25/4, 1/6, 1/288
4	105, 210, 525/4, 77/3, 49/96, 1/144, -139/51840
5	945, 2520, 9555/4, 1883/2, 12565/96, 149/72, 221/17280, -139/25920, -571/2488320.

At this stage, it is not clear for which η-values direct computation of c_k via (3.3) is safe. This depends of course on the desired accuracy. In applications, the desired accuracy in c_k will depend on k. For, when using the asymptotic expansion, first terms (i.e., terms $c_k(\eta)a^{-k}$ with k small) are needed with higher accuracy than late terms. Since the terms in the asymptotic expansion are decreasing in absolute value (if a is large) the coefficient c_0 is needed in good relative accuracy, while for the remaining terms a criterion based on absolute accuracy can be used. In Table II we give the μ-part and the η-part of c_k (cf.(3.3)) for $k = 0,1,2$, and $\eta = \pm 1$. The μ-values corresponding with $\eta = \pm 1$ are $\mu(-1) = -0.698...$, $\mu(1) = 1.35...$.

TABLE II

k	η	η-part	μ-part
0	1	-1	0.74
	-1	1	-1.43
1	1	1	-1.0034
	-1	-1	1.0054
2	1	-3	3.0022
	-1	3	2.9926

It appears that $\eta = \pm 1$ are safe values for summing the asymptotic series as far as it concerns coefficients c_k up to and including $k = 2$. To give an indication for the c_k with $k \geq 2$, we notice that absolute accuracy in subtracting the η-part from the μ-part in (3.3) is preserved if both parts are in absolute value not larger than 1. From Stirling's approximation for A_k it follows that the η-part is in absolute value approximately $(2k/e\eta^2)^k$. This expression is smaller than 1 if $|\eta| > (2k/e)^{\frac{1}{2}}$. For $k = 10$ the righthand side is 2.71.. .

If $|\eta|$ is small it is preferred to use expansions either in terms of η or in terms of μ. We advise expansions in η, since it gives better convergence properties. When expanding c_k in powers of μ we need (among others) the expansion of η in powers of μ. Due to the singularity of the logarithm in (1.3), the radius of convergence of this series is 1. Other singularities for η are zeros of $\mu - \ell n(1+\mu)$, but they are outside the domain $|\mu| \leq 1$. This follows from straightforward analysis. The reader may also consult an interesting note of DIEKMANN [2]. The expansion of μ in powers of η has radius of convergence $2\sqrt{\pi} \simeq 3.54$. This follows from the analysis of §5. From the recurrence relation (2.5) it is easily seen that the radius of convergence of the power series for c_k either in μ or in η is the same for all k.

We conclude this section with information on the construction of the coefficients for the expansion of c_k in powers of η.

It is convenient to start with the computation of the α_k in

$$\mu(\eta) = \alpha_1\eta + \alpha_2\eta^2 + \dots , \tag{3.9}$$

where μ is defined implicitly in (1.3). Substitution of (3.9) in (2.7) yields the recurrence relation

$$(k+1)\,\alpha_k = \alpha_{k-1} - \sum_{j=2}^{k-1} j\alpha_j\,\alpha_{k-j+1}, \qquad k \geq 2.$$

The first few are

$$\alpha_1 = 1, \quad \alpha_2 = 1/3, \quad \alpha_3 = 1/36, \quad \alpha_4 = -1/270, \quad \alpha_5 = 1/4320.$$

With α_k we also have available the γ_k of (2.4), which are also needed in (2.5). The relation between α_k and γ_μ is

$$\gamma_k = (-1)^k\, 1.3.5. \dots (2k+1)\,\alpha_{2k+1}, \qquad k = 0,1,2,\dots .$$

In fact, the expansion (3.9) is of importance for the derivation of the expansion in (2.4) (see also §4), By using (2.7) and

$$\tfrac{1}{2}\eta^2 = \mu - \ell n(1+\mu), \tag{3.10}$$

it follows that the expansion of $\eta/\mu(\eta)$ occurring in (2.5) is given by

$$\eta/\mu(\eta) = 1 + 2(\alpha_2-\tfrac{1}{2})\eta + 3\alpha_3\,\eta^2 + 4\alpha_4\,\eta^3 + \dots . \tag{3.11}$$

This gives the coefficients of c_0 of (2.5)

$$c_0(\eta) = c_0^{(0)} + c_1^{(0)}\,\eta + c_2^{(0)}\,\eta^2 + \ldots \tag{3.12}$$

with

$$c_0^{(0)} = -\frac{1}{3}, \quad c_k^{(0)} = (k+2)\alpha_{k+2}, \qquad k \geq 1.$$

By repeated use of (2.5) we obtain the recursion for the coefficients in

$$c_k(\eta) = c_0^{(k)} + c_1^{(k)}\eta + c_2^{(k)}\eta^2 + \ldots$$

$$c_n^{(k)} = \gamma_k c_n^{(0)} + (n+2)\; c_{n+2}^{(k-1)} \qquad n \geq 0, \quad k \geq 1. \tag{3.13}$$

Of course, each $c_n^{(k)}$ can be expressed in terms of $c_n^{(0)}$. The relation is for $n \geq 0$ and $k \geq 1$

$$c_n^{(k)} = \gamma_k\; c_n^{(0)} + \gamma_{k-1}(n+2)\; c_{n+2}^{(0)} + \ldots\; \gamma_0(n+2)\ldots(n+2k)\; c_{n+2k}^{(0)}. \tag{3.14}$$

Other functions may be used for expanding the coefficients c_k, for instance Chebyshev polynomials. In that case recurrence relations for corresponding coefficients can be constructed again. But the Taylor case gives simple relations and the coefficients can also be used for complex values of the parameters.

On account of the convergence properties of (3.12) (with radius $2\sqrt{\pi}$) successive terms in (3.14) are decreasing in absolute value. Hence no instability problems arise when using (3.14) for the computation of $c_n^{(k)}$.

4. BOUNDS FOR THE REMAINDER IN THE ASYMPTOTIC EXPANSION

In Table III we give the numbers C_k defined in (2.14). These bounds were obtained numerically by using representations of c_k given in the foregoing section. From (3.3) it follows that

$$\lim_{\eta\to+\infty} c_k(\eta) = 0, \quad \lim_{\eta\to-\infty} c_k(\eta) = (-)^{k+1} Q_k(-1) = -\gamma_k .$$

TABLE III

k	C_{2k}	C_{2k+1}
0	1	$3.5_{10}{-3}$
1	$9.2_{10}{-3}$	$6.9_{10}{-4}$
2	$2.1_{10}{-3}$	$3.5_{10}{-4}$
3	$1.3_{10}{-3}$	$3.5_{10}{-4}$
4	$1.7_{10}{-3}$	$6.0_{10}{-4}$
5	$3.4_{10}{-3}$	

Next we give details for computing the bounds H_k (defined in (2.10)) for $k = 0,1,\ldots,10$. It is convenient to start with details for obtaining the asymptotic expansion of $1/\Gamma(a)$. Again, a is a positive number. Starting point is Hankel's integral

$$(4.11) \qquad \frac{1}{\Gamma(a)} = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} e^t \, t^{-a} \, dt.$$

As in our previous paper [5] this can be written as

$$(4.2) \qquad \frac{1}{\Gamma(a)} = \frac{a^{1-a} e^a}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} f(u) \, du$$

with

$$(4.3) \qquad f(u) = \frac{ut}{1-t} , \quad -\tfrac{1}{2}u^2 = t - 1 - \ln t.$$

The relation between u and t must be specified in more detail. Let L be the saddle point contour for (4.1) in the t-plane. That is

$$(4.4) \qquad L = \{t \mid t = \frac{\phi}{\sin\phi} e^{i\phi}, \quad -\pi < \phi < \pi\} .$$

Then u defined in (4.3) is real if $t \in L$, and sign(u) = sign [Im(t)]. The asymptotic expansion of $1/\Gamma(a)$ is obtained by expanding

$$(4.5) \qquad g(u) = \frac{1}{2i} [f(u) + f(-u)]$$

in powers of u and termwise integration. Let us define the function g_N by writing

$$(4.6) \qquad g(u) = \sum_{k=0}^{N-1} a_k u^{2k} + a_N u^{2N} g_N(u), \qquad N = 0,1,2,\ldots,$$

with

$$(4.7) \qquad a_k = \frac{1}{(2k)!} g^{(2k)}(0) ;$$

all a_k are different from zero. Then the function H_N of (2.10) is given by

$$(4.8) \qquad H_N(a) = (a/2\pi)^{\frac{1}{2}} a_N a^N \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} u^{2N} g_N(u)\, du.$$

Suppose that we have bounds

$$(4.9) \qquad G_N = \sup_{u \in \mathbb{R}} |g_N(u)|,$$

then a bound for H_N is given by

$$(4.10) \qquad |H_N(a)| \le |\gamma_N|\, G_N, \qquad a > 0,$$

where γ_k are the coefficients in (2.10).

As yet it is not clear that g_N is bounded on $\mathbb{R}$. But it follows from (4.6) that g_N is bounded if g is bounded. The function f of (4.3) is not bounded on $\mathbb{R}$, but its even part g is. This follows from using the representation of $t \in L$ as given in (4.4). In terms of u and ϕ, g is given by

$$(4.11)\qquad g(u) = \frac{u\phi\sin^2\phi}{\phi^2 + \sin^2\phi - \phi\sin 2\phi}\,, \qquad -\pi < \phi < \pi$$

with

$$\tfrac{1}{2}u^2 = 1 - \phi\,\mathrm{ctg}\,\phi + \ln\frac{\phi}{\sin\phi}\,, \qquad \mathrm{sign}(u) = \mathrm{sign}(\phi)\,,$$

from which it follows that g is bounded if $u \to \pm\infty$ or $\phi \to \pm\pi$. Table IV gives the number G_N for $N = 0,1,\ldots,11$.

Table IV

N	G_{2N}	G_{2N+1}
0	1	1
1	1.95	1
2	3.33	1
3	5.05	1
4	6.95	1
5	8.90	1

For $N = 0,1,3,5,7,9,11$ the maximal function values of $|g_N(u)|$ occur at $u = 0$; for $N = 2,4,6,8,10$ the maxima occur in the neighbourhood of $u = \pm 2\sqrt{\pi}$. These latter points are the points on the real axis marking the domain of convergence of the Taylor series of g.

With the data of Table III and Table IV and relation (4.10) the bounds Q_N and P_N defined in (2.13) and (2.16) are easily computed.

5. EXTENSION TO COMPLEX VARIABLES

In this section we will show that the asymptotic expansion for P and Q given by (1.4) and (1.5) are valid for $a \to \infty$ uniformly in $|\arg a| \leq \pi - \varepsilon_1$, $|\arg x/a| \leq 2\pi - \varepsilon_2$ where ε_1 and ε_2 are positive numbers, $0 < \varepsilon_1 < \pi$, $0 < \varepsilon_2 < 2\pi$.

The condition on the argument of a follows from the validity of the expansions in (2.4), which are known to be uniformly valid when $|\arg a| \leq \pi - \varepsilon_1$. As noticed in Remark 2 of §2 the asymptotic expansion of $R_a(\eta)$ can be obtained by partial integration of one of (2.8). If we consider the second integral, one of the assumptions by partial integration will be that $\exp(-\frac{1}{2}a\eta^2)$ vanishes at infinity in a certain direction of the η-plane. If $|\arg a| < \pi$ and if it is allowed to use η-values at infinity with $\arg(a\eta^2) < \frac{\pi}{2}$ then the convergence of the integral is established for $|\arg a| \leq \pi - \varepsilon_1$. From these inequalities it follows that it is sufficient to show that for large $|\eta|$ we can take $\arg \eta$ in $(-\frac{3\pi}{4}, \frac{3\pi}{4})$. A second aspect of using the second integral of (2.8) is the possibility of joining the point ζ with ∞ such that the function $\mu(\eta)$ of the integral is holomorphic along this path and such that the point ζ can be associated unequivocally with a point in the μ-plane. In order to settle this we discuss the relation between η and the parameter μ (or λ) for complex values.

It is convenient to consider

$$(5.1)\qquad \eta = [2(\lambda - 1 - \ell n\, \lambda)]^{\frac{1}{2}}.$$

For $\lambda > 0$ the function η is to be interpreted as drawn in Figure 1. This implies a choice of the square root.

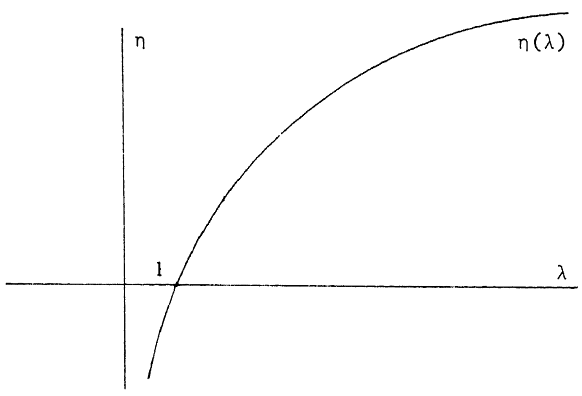

Figure 1.

We obtain a clear insight in the mapping $\lambda \to \eta(\lambda)$ and its inverse if we draw images of the half-lines ℓ_ϕ defined by

$$\ell_\phi = \{\lambda \mid \lambda = \rho e^{i\phi},\ \rho > 0\}$$

where ϕ is real, $|\phi| \le 2\pi$. Writing $\eta = \alpha + i\beta$ the image of ℓ_ϕ in the η-plane is governed by the equations

$$\tfrac{1}{2}(\alpha^2 - \beta^2) = \rho \cos\phi - 1 - \ell n\, \rho$$

$$\alpha\beta = \rho \sin\phi - \phi .$$

Taking into account the convention about the choice of the square root in (5.1) we obtain Figure 2, which contains images of ℓ_ϕ for $0 \le \phi \le 2\pi$. The complete picture for $-2\pi \le \phi \le 2\pi$ is symmetric with respect to the α-axis.

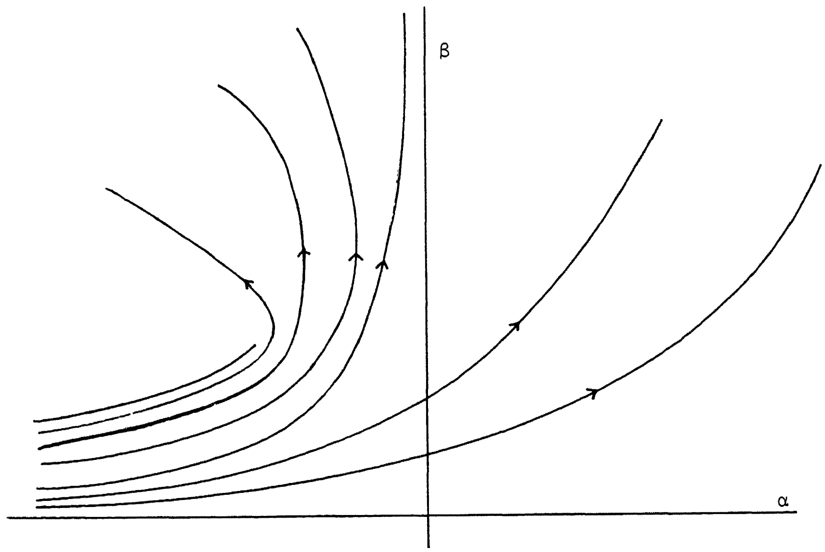

Figure 2.

The shown directions correspond to increasing values of ρ on ℓ_ϕ. The half-lines $\ell_{\pm 2\pi}$ are mapped on part of the hyperbolae $\alpha\beta = \mp 2\pi$. The points $\eta^\pm = e^{\pm 3\pi i/4}\, 2\sqrt{\pi}$ are singular points of the mapping. Other singular points are located in other Riemann sheets of the η-plane. Convenient branch-cuts for the function $\lambda(\eta)$ are the parts of the hyperbolae $\alpha\beta = \pm 2\pi$ with $\alpha \le -\sqrt{2\pi}$. With the η-plane cut along these curves, lines ℓ_ϕ with the values of ϕ outside the interval $[-2\pi, 2\pi]$ can be traced, but for our problem this is superfluous.

It is concluded that any point in the finite η-plane (not on the branch-cuts), corresponds to a point in the λ-plane with $|\arg \lambda| < 2\pi$. Consequently, if we integrate the second integral of (2.8) along a path that avoids the branch-cuts in the η-plane, the function $\mu(\eta) = \lambda(\eta) - 1$ is holomorphic. The conditions for allowing values of $\arg a$ in $(-\pi, \pi)$ are amply satisfied, since admissable directions in the η-plane can be found in the sector $-\pi < \arg \eta < \pi$.

REMARK. Singular points of the mapping $\eta \to \lambda(\eta)$ can also be found by considering the derivative $d\lambda/d\eta = \lambda\eta/(\lambda-1)$; $\lambda = 1$ gives a regular point but $\lambda = e^{2\pi i n}$ $(n = \pm 1, \pm 2, \ldots)$ gives (due to the many-valuedness of the logarithm in (5.1)) singular points η_n satisfying $\frac{1}{2}\eta_n^2 = -2\pi i n$, $n = \pm 1, \pm 2, \ldots$.

The integration by parts procedure leads eventually to (2.9) and (2.11). From the properties of the coefficients c_k and by taking appropriate contours in (2.11) it follows that for $N = 0, 1, 2, \ldots$

$$e^{-\frac{1}{2}a\eta^2} G_N(\eta; a) = O(1), \qquad a \to \infty$$

uniformly in $|\arg a| \le \pi - \varepsilon_1$, $|\arg \lambda| \le 2\pi - \varepsilon_2$.

6. ASYMPTOTIC EXPANSION OF THE INVERSE FUNCTION

The incomplete gamma functions are basic for the chi-square probability function and the Poisson distribution. Applications in this field lead us to the investigations of reliable and accurate algorithms for the computation of the functions discussed in this paper. Existing methods are not efficient if both parameters x and a are large. In statistics the inverses of the probability functions are very important. In practice the inversion is usually carried out by Newton-like methods. However, if large parameters must be considered, they are not reliable and not efficient. Our numerical experiments with the inversion of the incomplete gamma functions by using the expansions of this paper are promising, especially if the parameters are large. Since our method is based on uniform expansions, the range of application is satisfactorily large. Owing to the uniform character of our results, the coefficients of the expansion are rather complicated. But for implementation in software packages this aspect is not very important.

In [5] we also derived asymptotic expansions for the incomplete beta function. This function can be inverted by the same methods as those for the incomplete gammma function described in this section.

6.1 We consider real values of x and a satisfying (1.2). Let $q \in [0,1]$. We describe a procedure for obtaining the asymptotic expansion of the function $x(q,a)$ implicitly defined by the equation

$$Q(a,x) = q. \tag{6.1}$$

We use the representation of Q given in (1.4). If we have inverted Q then P is also inverted. The solution of $P(a,x) = p$, $0 \leq p \leq 1$, is simply $x(1-p,a)$, where $x(q,a)$ is the solution of (6.1), with $p + q = 1$.

First we describe the inversion in terms of the parameter η. Suppose we have available the value of η_0, which solves the equation $\frac{1}{2}\,\mathrm{erfc}\,[\eta_0(a/2)^{\frac{1}{2}}] = q$, where q is the same as in (6.1). This requires an inversion of the error function, but this problem is solved satisfactorily in the literature. See for instance BLAIR et.al.[1] or STRECOK [4].

The value for η implicitly defined by the equation

$$\tfrac{1}{2}\,\mathrm{erfc}\,[\eta(a/2)^{\frac{1}{2}}] + R_a(\eta) = q \tag{6.2}$$

is for large values of a approximated by η_0. Hence we write

$$\eta = \eta_0 + \varepsilon(\eta_0, a) \tag{6.3}$$

and try to determine ε. From the previous section it follows that $R_a(\eta)$ is analytic for every $\eta \in \mathbb{R}$. Substituting (6.3), we find by expansion

$$\tfrac{1}{2} \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \frac{d^k}{d\eta^k} \operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] + \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \frac{d^k}{d\eta^k} R_a(\eta) = 0, \tag{6.4}$$

where the derivatives are evaluated at $\eta = \eta_0$. In this formula we substitute for the derivatives of $R_a(\eta)$ the derivatives of the asymptotic expansion (1.5). As remarked in §2, the series can be differentiated, giving (2.1). But it can be differentiated any number of times, giving

$$\frac{d^k}{d\eta^k} R_a(\eta) \sim a^k (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} \sum_{n=0}^{\infty} c_n^{(k)}(\eta)\, a^{-n}, \tag{6.5}$$

with $c_n^{(0)}(\eta) = c_n(\eta)$, and for $k \geq 1$ (compare (2.2))

$$c_0^{(k)}(\eta) = -\eta\, c_0^{(k-1)}(\eta), \quad c_n^{(k)}(\eta) = -\eta\, c_n^{(k-1)}(\eta) + \frac{d}{d\eta} c_{n-1}^{(k-1)}(\eta), \tag{6.6}$$
$$n \geq 1.$$

The derivatives of the error function in (6.4) can be replaced by Hermite polynomials, viz.

$$\tfrac{1}{2} \frac{d^k}{d\eta^k} \operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] = (-1)^k (a/2)^{k/2} \pi^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} H_{k-1}[\eta(a/2)^{\frac{1}{2}}].$$

For a similar series as in (6.5) let us write

$$\tfrac{1}{2} \frac{d^k}{d\eta^k} \operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] = a^k (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} \sum_{n=0}^{[(k-1)/2]} h_n^{(k)}(\eta) a^{-n}. \tag{6.7}$$

This series contains as many terms as the Hermite polynomials $H_{k-1}(x)$ when expanding it in powers of x. From well-known properties of these polynomials we derive for $k = 1,2,\ldots,\quad n = 0,1,\ldots,[(k-1)/2]$

$$h_n^{(k)}(\eta) = (-1)^{k+n}\,\eta^{k-1-2n}\,2^{-n}(k-1)!/[n!(k-1-2n)!]. \tag{6.8}$$

Substituting (6.5) and (6.7) we obtain the asymptotic equality

$$\sum_{n=0}^{\infty} c_n(\eta)a^{-n} + \frac{(\varepsilon a)}{1!}\sum_{n=0}^{\infty} e_n^{(1)}(\eta)a^{-n} + \frac{(\varepsilon a)^2}{2!}\sum_{n=0}^{\infty} e_n^{(2)}(\eta)a^{-n}\ldots \sim 0, \tag{6.9}$$

with $\eta = \eta_0$ and

$$e_n^{(k)}(\eta) = c_n^{(k)}(\eta) + h_n^{(k)}(\eta), \qquad n \ge 0, \quad k \ge 1. \tag{6.10}$$

From this point the analysis is continued formally. We assume that ε in (6.3) can be developed in an asymptotic expansion. Let us make the "Ansatz"

$$\varepsilon(\eta,a) \sim \frac{\alpha(\eta)}{a}\,[1 + \alpha_1(\eta)a^{-1} + \alpha_2(\eta)a^{-2} + \ldots], \tag{6.11}$$

where α and α_i are to be determined. This will be done in §6.2. With (6.11) an expansion for x defined in (6.1) can be obtained as follows. We have

$$\begin{aligned} x(q,a) &= a\lambda(\eta) = a[1+\mu(\eta)] = a[1+\mu(\eta_0)+\varepsilon\mu'(\eta_0)+\ldots] \\ &\sim a[x_0(\eta_0) + x_1(\eta_0)a^{-1} + x_2(\eta_0)a^{-2} + \ldots], \qquad a \to \infty, \end{aligned} \tag{6.12}$$

the first coefficients x_i being given by

$$\begin{aligned} x_0(\eta) &= 1 + \mu(\eta), \\ x_1(\eta) &= \mu'(\eta)\alpha(\eta) \\ x_2(\eta) &= \tfrac{1}{2}\alpha^2(\eta)\mu''(\eta) + \alpha_1(\eta)\alpha(\eta)\mu'(\eta) \\ x_3(\eta) &= \tfrac{1}{6}\alpha^3(\eta)\mu'''(\eta) + \alpha_1(\eta)\alpha^2(\eta)\mu''(\eta) + \alpha_2(\eta)\alpha(\eta)\mu'(\eta). \end{aligned} \tag{6.13}$$

The primes denote differentiation with respect to η. The value of $\mu(\eta_0)$ can be obtained by the inversion of the relation between μ and η given in (1.3).Derivatives of $\mu(\eta)$ can be obtained via (2.7), but they also follow from the coefficients c_k and (2.5). For instance, $\mu'(\eta) = [1+\mu(\eta)][1+\eta c_0(\eta)]$. As will be seen in §6.2, the coefficients c_k are also needed in $\alpha(\eta)$, $\alpha_i(\eta)$. From the representations of α, α_1 and α_2 to be given in §6.2, the coefficients $x_0,\ldots,x_2$ of (6.13) can be determined.

6.2 The coefficients α and α_i of (6.11) are computed by substitution of (6.11) in (6.9) and by collecting equal powers of the large parameter a. By considering coefficients multiplying a^0 we obtain

$$(6.14)\qquad c_0(\eta) + \sum_{k=1}^{\infty} \frac{\alpha^k(\eta)}{k!}\, e_0^{(k)}(\eta) = 0.$$

From (6.6) and (6.8) it follows that for $k \geq 1$

$$(6.15)\qquad c_0^{(k)}(\eta) = (-\eta)^k c_0(\eta),\quad h_0^{(k)}(\eta) = -(-\eta)^{k-1}.$$

So, $e_0^{(k)}$ of (6.14) defined in (6.10) is known and α is obtained by summation. The result is

$$(6.16)\qquad \alpha(\eta) = \frac{1}{\eta}\,\ell n[1+\eta\, c_0(\eta)] = \frac{1}{\eta}\,\ell n(\eta/\mu).$$

From this representation we conclude that α is a well-defined bounded function of $\eta \in \mathbb{R}$ with $\alpha(\eta) \to 0$ if $\eta \to \pm\infty$.

For higher order coefficients α_i we need representations of $c_n^{(k)}$ in terms of c_k and their derivatives. For $n = 0$ this relation is given by the first of (6.15), for $n = 1$ it is given by

$$(6.17)\qquad c_1^{(k)}(\eta) = (-1)^k[\eta^k\, c_1(\eta) - k\eta^{k-1}\frac{d}{d\eta}\, c_0(\eta) - \tfrac{1}{2}k(k-1)\eta^{k-2}c_0(\eta)]$$

and the general formula is

$$(6.18)\qquad c_n^{(k)}(\eta) = (-1)^k\, k! \sum_{\mu=0}^{n}\sum_{\nu=0}^{\mu} \frac{(-1)^{\mu}2^{-\nu}\eta^{k-\mu-\nu}}{(k-\mu-\nu)!\,(\mu-\nu)!}\, \frac{d^{\mu-\nu}}{d\eta^{\mu-\nu}}\, c_{n-\mu}(\eta).$$

These relations follow from induction. In the last formula the summations are carried out for those μ and ν such that $k-\mu-\nu \geq 0$.

Collecting in (6.9) coefficients of a^{-1} we obtain for α_1 the equation

$$c_1(\eta) + \sum_{k=1}^{\infty} \frac{\alpha^k(\eta)}{k!} [e_1^{(k)}(\eta) + k\alpha_1(\eta)\, e_0^{(k)}(\eta)] = 0,$$

which gives after summation

$$\alpha_1(\eta) = \frac{c_1(\eta) + \alpha(\eta)c_0'(\eta) - \tfrac{1}{2}\alpha^2(\eta)c_0(\eta) + \alpha^3(\eta)r_2[\eta\alpha(\eta)]}{\alpha(\eta)[1+\eta\, c_0(\eta)]} . \tag{6.19}$$

The function r_n is for $n = 0,1,2,\ldots$ defined by

$$r_n(x) = [e^x - (1 + x + \frac{1}{2!}x^2 + \ldots \frac{x^n}{n!})]/x^{n+1}.$$

The result for α_2 is

$$\alpha_2(\eta) = \{c_2 + \alpha c_1' + [c_0'' - c_1 + \eta(1+\eta c_0)\alpha_1^2 - \tfrac{1}{2}c_0'\alpha^3 + \tfrac{1}{4}\alpha^4 c_0 - 3\alpha^5 r_4\}/ [(1+\eta c_0)\alpha], \tag{6.20}$$

where r_4 has the argument $\eta\alpha(\eta)$ and the remaining functions have argument η. In (6.19) and (6.20) the primes denote differentiation with respect to η. Derivatives of c_k can be replaced by combinations of c_k by using (2.5). The first few relations are

$$c_0'(\eta) = \eta[c_1(\eta) + \frac{1}{12}c_0(\eta)] + \frac{1}{12},$$

$$c_1'(\eta) = \eta[c_2(\eta) - \frac{1}{288}c_0(\eta)] - \frac{1}{288},$$

$$c_0''(\eta) = \eta^2[c_2(\eta) - \frac{1}{288}c_0(\eta)] - \frac{1}{288}\eta + \frac{1}{12}[c_0(\eta) + 12\,c_1(\eta) + \frac{1}{12} + \eta[c_1(\eta) + \frac{1}{12}c_0(\eta)]\}.$$

6.3 Example for $q = \frac{1}{2}$.

If $q = \frac{1}{2}$ the relations are quite simple. In that case $\eta_0 = 0$ and x_i can be determined by computing limiting values of α and α_i for $\eta \to 0$. The following expression gives the values of x_0, x_1, x_2 of (6.12), viz.

$$x(\tfrac{1}{2},a) \sim a(1 - \frac{1}{3}a^{-1} + \frac{8}{405}a^{-2} \ldots) . \tag{6.21}$$

Table V shows some results of numerical experiments. For the values of a indicated in the table we computed $x(\frac{1}{2},a)$ from (6.21). Then we computed $y = Q[a,x(\frac{1}{2},a)]$ with accuracy of about 12 significant digits. The table gives the difference $|y - \frac{1}{2}|$.

Table V

a	$\|y - \frac{1}{2}\|$
10	$0.93_{10}{-5}$
50	$0.16_{10}{-6}$
100	$0.29_{10}{-7}$
250	$0.29_{10}{-8}$
500	$0.51_{10}{-9}$
1000	$0.91_{10}{-10}$

Further experiments showed that for other q-values the results are of the same kind. In fact they show the uniform character of our expansion (6.12) with respect to $q \in [0,1]$.

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SAMENVATTING

Dit proefschrift bestaat uit acht artikelen, waarvan er zes in wetenschappelijke tijdschriften zijn verschenen terwijl de overige twee artikelen geaccepteerd zijn voor publikatie. In de Introduction worden enige algemene kenmerken van het onderzoek besproken, in de Summaries worden samenvattingen van de artikelen gegeven.

In de Introduction worden de artikelen vermeld waarbij ze aangeduid worden met [1],...,[8]. De artikelen [1], [3], [6], [7] en [8] handelen voornamelijk over asymptotische problemen, [2], [4] en [5] voornamelijk over de berekening van speciale functies. Het eerste artikel staat nogal los van de overige zeven aangezien er een asymptotisch probleem op het gebied van de partiële differentiaalvergelijkingen in wordt behandeld, terwijl in de andere artikelen vooral speciale functies aan de orde komen.

Bij de behandeling van asymptotische problemen wordt in dit proefschrift vooral de nadruk gelegd op asymptotische ontwikkelingen die uniform geldig zijn t.o.v. bepaalde parameters. Een belangrijk voorbeeld is de ontwikkeling van de incomplete gammafuncties in [3] en [8]. In [7] wordt uiteengezet hoe analoge ontwikkelingen kunnen worden afgeleid voor de confluente hypergeometrische functies, waarvan de incomplete gammafuncties speciale gevallen zijn. In [4] en [5] worden algoritmen voor de berekening van de Besselfuncties $K_\nu(z)$ en $Y_\nu(z)$ gegeven. Speciale aandacht krijgt de berekening van deze functies voor kleine waarden van $|z|$. De berekeningen zijn in dat geval gebaseerd op combinaties van de Taylorreeksen voor de Besselfuncties $J_\nu(z)$ en $J_{-\nu}(z)$. Om de berekening voor de waarden $\nu = 0,1,\ldots$ net zoals die voor andere (complexe) ν-waarden te laten verlopen wordt een uniforme methode t.o.v. ν gegeven. Voor de overige z-waarden wordt een recurrente betrekking gebruikt die, via de algoritme van Miller, een efficiënte numerieke methode oplevert. Deze methode wordt ook in [2] gebruikt om functies te berekenen die in wezen speciale confluente hypergeometrische functies zijn.

In [6] worden voor zekere integralen uniforme asymptotische ontwikkelingen gegeven, waarin als bouwstenen incomplete gammafuncties optreden.

In een van de stellingen behorende bij dit proefschrift wordt voor de incomplete betafunctie een toepassing gegeven van de resultaten uit [6].

In [8] worden de eerder verkregen resultaten voor de incomplete gammafuncties nader bestudeerd. Voor de coëfficiënten in de asymptotische ontwikkeling wordt een recurrente betrekking afgeleid en voor de restterm in de ontwikkeling wordt een numerieke schatting gegeven. Tevens worden de resultaten bestudeerd voor complexe waarden van de parameters.

STELLINGEN

BIJ HET PROEFSCHRIFT

SOME ASPECTS OF APPLIED ANALYSIS:

ASYMPTOTICS, SPECIAL FUNCTIONS AND THEIR NUMERICAL COMPUTATION

VAN

N.M. TEMME

14 JUNI 1978

I

Laat de incomplete betafunctie gedefinieerd zijn door

$$I_x(p,q) = \frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1}dt, \qquad p > 0, \quad q > 0, \qquad 0 \le x \le 1,$$

waarin $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. Zij voor $k = 0,1,\ldots$ gegeven

$$F_k = \int_a^\infty e^{-(p-a)t} t^{q-1}(t-a)^k dt, \qquad c_k = \frac{d^k}{dt^k}\left(\frac{1-e^{-t}}{t}\right)^{q-1}\Bigg|_{t=a}, \qquad a = -\ln x,$$

(dus $c_0 = \left[\frac{1-x}{-\ln x}\right]^{q-1}$ en $F_0 = p^{-q}(1-x)^{-p}\Gamma(q,-p\ln x)$, incomplete gammafunctie).

Dan geldt voor $p \to \infty$, q vast en uniform in x, $x \in [0,1]$, de asymptotische ontwikkeling

$$I_x(p,q) = \frac{x^p}{B(p,q)}\left[\sum_{k=0}^{n-1} c_k F_k + R_n\right], \qquad n = 1,2,\ldots;$$

als $q \ge 1$, dan bestaan er getallen a_n (onafhankelijk van p en x) zodat

$$|R_n| \le a_n\, p^{-n-q}\, \Gamma(n+q)/n!, \qquad n = 1,2,\ldots .$$

II

a. Voor de incomplete betafunctie (zie Stelling I) geldt de integraalrepresentatie

$$(*) \quad I_x(p,q) = \frac{\Gamma(p+q)}{(2\pi i)^2} \int_{L_s} \int_{L_\sigma} \frac{e^{\sigma + x(s-\sigma)}}{s^p \sigma^q (s-\sigma)}\, ds\, d\sigma;$$

hierin stellen L_s en L_σ verticalen voor in het complexe s-, respectievelijk σ-vlak, te weten

$$L_s = \{c_s + it \mid t \in \mathbb{R}\}, \qquad L_\sigma = \{c_\sigma + it \mid t \in \mathbb{R}\}$$

met $c_\sigma < c_s$, of contouren die op grond van de stelling van Cauchy door vervorming van L_σ en L_s kunnen worden verkregen. Indien $c_\sigma > c_s$ dan is de integraal gelijk aan $-I_{1-x}(q,p)$.

b. Bovenstaande integraal kan beschouwd worden als een meer-dimensionale variant op de integraalvoorstelling voor de incomplete gammafunctie, namelijk

$$\gamma(a,x) = \frac{\Gamma(a)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{x(s-1)}}{s^a(s-1)}\, ds, \qquad c > 0,$$

waarmee in dit proefschrift de uniforme ontwikkeling van deze laatste functie is verkregen.

Verwacht wordt dat (*) een geschikt uitgangspunt zal zijn voor de constructie van de asymptotische ontwikkeling van $I_x(p,q)$ voor $p \to \infty$ (of $q \to \infty$), welke ontwikkeling uniform geldig is ten opzichte van $x \in [0,1]$ en ten opzichte van $q \geq \delta > 0$ (of $p \geq \delta > 0$).

III

Laat q en v niet-negatieve getallen zijn en n een niet-negatief geheel getal. Laat de functie $T_n(v,q)$ gedefinieerd zijn door

$$T_n(v,q) = e^{-q}(2q)^{-n} \int_0^{2\sqrt{vq}} x^n\, e^{-x^2/(4q)}\, I_{n-1}(x)\, dx,$$

waarin $I_\nu(x)$ de gemodificeerde Besselfunctie is. Dan geldt voor $n \to \infty$, uniform ten opzichte van $q \geq 0$ en $v \geq 0$

$$T_n(v,q) = \tfrac{1}{2}\,\mathrm{erfc}(\zeta) + \mathcal{O}(e^{-\zeta^2} n^{-\frac{1}{2}}),$$

waarin

$$\zeta = \mathrm{sign}(q+n-v)\left[q+v+n\ln\left\{\frac{n+(n^2+4qv)^{\frac{1}{2}}}{2v}\right\} - (n^2+4qv)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

en $\mathrm{erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty e^{-t^2}\, dt$, de complementaire errorfunctie.

($T_n(v,q)$ is een cumulatieve distributie van n variaties van signaal plus ruis en komt voor in radar- en sonarproblemen bij het detecteren van vals alarm.

J. de Vries, Physisch Laboratorium TNO, Den Haag).

IV

De studie van het asymptotisch gedrag van de nulpunten van de incomplete gammafunctie, zie bijv. Kölbig, verdient een nieuwe aanpak met de resultaten uit dit proefschrift.

K.S. KÖLBIG, *On the zeros of the incomplete gamma function*, Math. Comp. 26, 751-755, 1972.

V

De schattingen voor de startwaarde van de algoritme van Miller voor de berekening van Besselfuncties zijn in Gautschi (1967) gebaseerd op niet-uniforme asymptotische formules voor de Besselfuncties. Een realistischer schatting wordt verkregen door uniforme benaderingen (bijv. die van Debye) te gebruiken.

W. GAUTSCHI, *Computational aspects of three-term recurrence relations*, SIAM Rev. 9, 24-82, 1967.

吉田年雄 浅野道雄 梅野正義 三木七郎 漸化式を用いる複素変数のベッセル関数 $I_n(z)$ の数値計算 社団法人 情報処理学会 Vol. 14, 23-29, 1973.

NUMAL, Hoofdstuk 6 (Revisie 1978), Mathematisch Centrum, Amsterdam.

VI

Bij het kiezen van een numerieke methode voor de berekening van integralen wordt de trapeziumregel nogal eens over het hoofd gezien.

N.M. TEMME, *The numerical computation of special functions by use of quadrature rules for saddle point integrals* I. *Trapezoidal integration rules*, Rapport TW 164, Mathematisch Centrum, Amsterdam, 1977.

VII

Voor de berekening van een functie f, die gerepresenteerd wordt door

$$f(x) = \sum_{k=0}^{n} a_k \, \phi_k(x)$$

waarbij ϕ_k aan een homogene recursierelatie voldoet waarvan de coëfficiënten niet van k afhangen, wordt vaak een algoritme gebruikt die ten onrechte naar Clenshaw is vernoemd; de algoritme is een Horner-schema voor een polynoom waarvan het argument een matrix is.

C.G. VAN DER LAAN, *Approximatie van functies en data* in *Colloquium numerieke programmatuur, deel 2* (H.J.J. te Riele (red.)), MC Syllabus 29.2, Mathematisch Centrum, Amsterdam, 1977.

VIII

Zij gegeven een L^2-functie g: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ met de eigenschap $g(E,T) = 0$ als $T > \gamma E$, met $0 < \gamma < 1$. Laat de rij functies $\{g_n\}$ gegeven zijn door

$$g_1(E,T) = g(E,T),$$

$$g_n(E,T) = \int_0^T g_{n-1}(E,\tau)g(E-\tau,T-\tau)d\tau \qquad (n = 2,3,\ldots).$$

Dan is $g_n(E,T) = 0$ voor $T > [1-(1-\gamma)^n]E$.

(Deze functies treden op bij de beschrijving van de energieoverdracht bij botsingen van ionen van twee verschillende metalen; g is dan een kansdichtheidsfunctie. J.B. Sanders, FOM-instituut, Amsterdam).

IX

Luke's sceptische opmerkingen over de mogelijkheid om met groepentheoretische methoden nieuwe resultaten op het gebied van speciale functies te verkrijgen zijn door het werk van o.a. Koornwinder achterhaald.

Y.L. LUKE, Math. Comp., 24, p.231, 1970, bespreking van

J.D. TALMAN, *Special Functions, A Group Theoretic Approach*, New York, 1969.

X

Pirsig's boek *Zen and the art of motorcycle maintenance* is een geslaagde poging inzichten op het gebied van de filosofie, natuurwetenschappen en hedendaagse maatschappelijke dilemma's in romanvorm onder de aandacht te brengen.

XI

Om te voorkomen dat een schooladviesdienst een verwijsbureau naar het buitengewoon onderwijs wordt, dienen bij het gewoon lager onderwijs mogelijkheden aanwezig te zijn om kinderen met leermoeilijkheden te begeleiden.

XII

De opzet van consumentenbladen, waarin produkten besproken worden die voor een groot aantal lezers niet van belang zijn, dient zo herzien te worden dat het initiatief van de lezer uit moet gaan om aan te geven over welk specifiek produkt hij informatie wenst te ontvangen.