

SENSITIVITY THEOREMS IN INTEGER LINEAR PROGRAMMING

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We consider integer linear programming problems with a fixed coefficient matrix and varying objective function and right-hand-side vector. Among our results, we show that, for any optimal solution to a linear program $\max\{wx: Ax \leq b\}$, the distance to the nearest optimal solution to the corresponding integer program is at most the dimension of the problem multiplied by the largest subdeterminant of the integral matrix A . Using this, we strengthen several integer programming 'proximity' results of Blair and Jeroslow; Graver; and Wolsey. We also show that the Chvátal rank of a polyhedron $\{x: Ax \leq b\}$ can be bounded above by a function of the matrix A , independent of the vector b , a result which, as Blair observed, is equivalent to Blair and Jeroslow's theorem that 'each integer programming value function is a Gomory function.'

Key words: Integer Linear Programming, Chvátal Rank, Cutting Planes, Sensitivity Analysis.

1. Introduction

For a given integer program $\max\{wx: Ax \leq b, x \text{ integral}\}$, how does the set of optimal solutions change as the vectors w and b are varied? Early work on this topic was carried out by Gomory [14, 15], who considered the connection between optimal solutions to an integer program and its linear programming relaxation for a range of right-hand-side vectors b . His work was continued and extended by Wolsey [28]. Other studies have been made by Blair and Jeroslow [1, 2, 3].

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In this paper, we consider several different aspects of the problem. We first show (Theorem 1) that, for any optimal solution to a linear program $\max\{wx: Ax \leq b\}$, where A is an integral matrix, the distance to the nearest optimal solution to the corresponding integer program is at most the number of variables multiplied by the largest subdeterminant of A . This implies results of Blair and Jeroslow [2], von zur Gathen and Sieveking [10], and Wolsey [28]. Next (Theorem 5), we sharpen a result of Blair and Jeroslow [1], which implies that a change in the right-hand-side vector cannot produce more than an affine change in the optimal value of an integer program. We then show (Theorem 6) that for any nonoptimal integral solution to $\max\{wx: Ax \leq b, x \text{ integral}\}$ there exists an integral solution nearby (for a fixed matrix A) which has a greater objective value. This improves a result of Graver [16] and Blair and Jeroslow [3]. Finally, we show (Theorem 10) that the Chvátal rank of a polyhedron $\{x: Ax \leq b\}$ can be bounded above by a function of the matrix A , independent of the vector b .

We assume throughout the paper that all polyhedra, matrices, and vectors are rational. The l_∞ -norm of a vector $x = (x_1, \dots, x_n)$ is denoted by $\|x\|_\infty = \max\{|x_i|: i = 1, \dots, n\}$ and the l_1 -norm is denoted by $\|x\|_1 = \sum\{|x_i|: i = 1, \dots, n\}$. For a matrix A , we denote by $\Delta(A)$ the maximum of the absolute values of the determinants of the square submatrices of A . If A is an $m \times n$ matrix and b is an m -component vector, then $(A|b)$ denotes the matrix obtained by adjoining b , as an $n + 1$ st column, to A . The greatest integer less than or equal to a number β is denoted by $\lfloor \beta \rfloor$.

For basic results in the theory of polyhedra and integer linear programming, the reader is referred to Schrijver [26] and Stoer and Witzgall [27].

2. Proximity results

We begin with a theorem on the distance between optimal solutions to an integer program and its linear programming relaxation.

Theorem 1. *Let A be an integral $m \times n$ matrix and let b and w be vectors such that $Ax \leq b$ has an integral solution and $\max\{wx: Ax \leq b\}$ exists. Then*

(i) *for each optimal solution \bar{x} to $\max\{wx: Ax \leq b\}$ there exists an optimal solution z^* to $\max\{wx: Ax \leq b, x \text{ integral}\}$ with $\|\bar{x} - z^*\|_\infty \leq n\Delta(A)$*

and

(ii) *for each optimal solution \bar{z} to $\max\{wx: Ax \leq b, x \text{ integral}\}$ there exists an optimal solution x^* to $\max\{wx: Ax \leq b\}$ with $\|\bar{z} - x^*\|_\infty \leq n\Delta(A)$.*

Proof. Let \bar{x} and \bar{z} be optimal solutions to $\max\{wx: Ax \leq b\}$ and $\max\{wx: Ax \leq b, x \text{ integral}\}$ respectively. Split A into submatrices A_1, A_2 such that $A_1\bar{x} \geq A_1\bar{z}$ and $A_2\bar{x} < A_2\bar{z}$. Since $a_i\bar{x} < b_i$ for each inequality $a_i x \leq b_i$, of $Ax \leq b$, with a_i a row of A_2 , the dual variables corresponding to the rows of A_2 are equal to zero in every

optimal solution to the dual linear program of $\max\{wx: Ax \leq b\}$. So there exists a vector $y^1 \geq 0$ such that $y^1 A_1 = w$. This implies that $w\tilde{x} \geq 0$ for each vector \tilde{x} in the cone $C = \{x: A_1 x \geq 0, A_2 x \leq 0\}$. Let G be a finite set of integral vectors which generates C (so C is the set of vectors which can be written as a nonnegative linear combination of vectors in G). Using Cramer's rule, we may assume that $\|g\|_\infty \leq \Delta(A)$ for each $g \in G$. As $\bar{x} - \bar{z} \in C$, there exists, by Carathéodory's theorem, a set $\{g^1, \dots, g^t\} \subseteq G$ and numbers $\lambda_i \geq 0, i = 1, \dots, t$, such that $\bar{x} - \bar{z} = \lambda_1 g^1 + \dots + \lambda_t g^t$, where t is the dimension of C .

To verify (i), let

$$z^* = \bar{z} + [\lambda_1]g^1 + \dots + [\lambda_t]g^t = \bar{x} - (\lambda_1 - [\lambda_1])g^1 - \dots - (\lambda_t - [\lambda_t])g^t. \quad (1)$$

Since \bar{z} is integral and g^1, \dots, g^t are integral, z^* is also integral. Furthermore,

$$A_2 z^* = A_2 \bar{z} + [\lambda_1]A_2 g^1 + \dots + [\lambda_t]A_2 g^t \leq A_2 \bar{z} \quad \text{and}$$

$$A_1 z^* = A_1 \bar{x} - (\lambda_1 - [\lambda_1])A_1 g^1 - \dots - (\lambda_t - [\lambda_t])A_1 g^t \leq A_1 \bar{x}.$$

So $Az^* \leq b$. Now since $wg^i \geq 0$ for $i = 1, \dots, t$, we have that $wz^* \geq w\bar{z}$ and, hence, that z^* is an optimal solution to $\max\{wx: Ax \leq b, x \text{ integral}\}$. (This implies that $wg^i = 0$ for all $i \in \{1, \dots, t\}$ with $\lambda_i \geq 1$, a fact which is used below.) Finally,

$$\|\bar{x} - z^*\|_\infty = \|(\lambda_1 - [\lambda_1])g^1 + \dots + (\lambda_t - [\lambda_t])g^t\|_\infty \leq \|g^1\|_\infty + \dots + \|g^t\|_\infty. \quad (2)$$

So $\|\bar{x} - z^*\|_\infty \leq t\Delta(A) \leq n\Delta(A)$.

To verify (ii), let

$$x^* = \bar{x} - [\lambda_1]g^1 - \dots - [\lambda_t]g^t = \bar{z} + (\lambda_1 - [\lambda_1])g^1 + \dots + (\lambda_t - [\lambda_t])g^t. \quad (3)$$

Using the above arguments, it follows that $Ax^* \leq b$ and $\|\bar{z} - x^*\|_\infty \leq n\Delta(A)$. Since $wg^i = 0$ for all $i \in \{1, \dots, t\}$ with $\lambda_i \geq 1$, we have $wx^* = w\bar{x}$. So x^* is an optimal solution to $\max\{wx: Ax \leq b\}$. \square

This result strengthens a theorem of Blair and Jeroslow [2, Theorem 1.2], who showed that for any fixed matrix A and fixed vector w , there exists a constant T such that for any optimal solution \bar{x} to $\max\{wx: Ax \leq b\}$, there exists an optimal solution \bar{z} to $\max\{wx: Ax \leq b, x \text{ integral}\}$ such that $\|\bar{x} - \bar{z}\|_\infty \leq T$ (assuming that $Ax \leq b$ has an integral solution). (In fact, an analysis of Blair and Jeroslow's proof (see also the proof of [1, Theorem 2.1(1)]) will show that T is independent of w .) Similarly, the following consequence of Theorem 1 strengthens a result of Blair and Jeroslow [1, Theorem 2.1(2), 3, Corollary 4.7] on the difference between the optimal value of an integer program and its linear programming relaxation.

Corollary 2. *Let A be an integral $m \times n$ matrix and let b and w be vectors such that $Ax \leq b$ has an integral solution and $\max\{wx: Ax \leq b\}$ exists. Then*

$$\max\{wx: Ax \leq b\} - \max\{wx: Ax \leq b, x \text{ integral}\} \leq n\Delta(A)\|w\|_1. \quad \square \quad (4)$$

Another consequence of Theorem 1 is the following result of von zur Gathen and Sieveking [10] (which implies that integer programming (feasibility) is in the class NP).

Corollary 3. *Let A be an integral $m \times n$ matrix and b an integral m -component vector. Then if $Ax \leq b$ has an integral solution, then it has one with components at most $(n+1)\Delta((A|b))$ in absolute value.*

Proof. Suppose that $Ax \leq b$ has an integral solution. There exists a vector \bar{x} with $A\bar{x} \leq b$ such that the nonzero components of \bar{x} are given by $B^{-1}\bar{b}$ for some submatrix B of A and some part \bar{b} of b . We have $\|\bar{x}\|_\infty \leq \Delta((A|b))$. By Theorem 1, there exists an integral vector \bar{z} such that $A\bar{z} \leq b$ and $\|\bar{z} - \bar{x}\|_\infty \leq n\Delta(A)$. So

$$\|\bar{z}\|_\infty \leq \|\bar{z} - \bar{x}\|_\infty + \|\bar{x}\|_\infty \leq n\Delta(A) + \Delta((A|b)) \leq (n+1)\Delta((A|b)). \quad \square \quad (5)$$

A third consequence of Theorem 1 is a result of Wolsey [28], which shows that an integer program can be solved by first solving its linear programming relaxation and then checking a finite set of lower dimensional ‘correction vectors’. Wolsey [28] works with linear programs of the form $\max\{wx: Ax = b, x \geq 0\}$, where A is an $m \times n$ matrix of rank m . If such a linear program has an optimal solution, then it has one of the form $\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$ where B is a basis of A (that is, a $m \times m$ nonsingular submatrix of A), x_B are those variables corresponding to columns of B and x_N are those variables corresponding to columns of N (the submatrix of A formed by those columns not in B). Such a basis B is an *optimal basis*.

Corollary 4. *Let A be an integral $m \times n$ matrix of rank m . Then there exists a finite set V of nonnegative, integral $(n-m)$ -component vectors such that: For any vectors b and w for which $\max\{wx: Ax = b, x \geq 0\}$ has an optimal solution and any optimal basis B , if $Ax = b, x \geq 0$ has an integral solution then for some vector $v \in V$ an optimal solution to $\max\{wx: Ax = b, x \geq 0, x \text{ integral}\}$ is $\bar{x}_N = v$, $\bar{x}_B = B^{-1}b - B^{-1}Nv$.*

Proof. Let $V = \{v \in Z^{n-m}: \|v\|_\infty \leq n\Delta(A), v \geq 0\}$. Suppose that B is an optimal basis for $\max\{wx: Ax = b, x \geq 0\}$ and let $\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$ be the corresponding optimal solution. By Theorem 1, there exists an optimal solution \bar{z} to $\max\{wx: Ax = b, x \geq 0, x \text{ integral}\}$ with $\|\bar{x} - \bar{z}\|_\infty \leq n\Delta(A)$. Thus $\|\bar{z}_N\|_\infty \leq n\Delta(A)$, which implies that $\bar{z}_N \in V$. \square

For a fixed matrix A , Theorem 1 shows that optimal solutions to an integer program $\max\{wx: Ax \leq b, x \text{ integral}\}$ and its linear programming relaxation are near to each other. Our next theorem, which is a sharpened form of the integer programming ‘strong proximity result’ of Blair and Jeroslow [1], shows that for small changes in the right-hand-side vector b , optimal solutions to the corresponding integer programs are near to each other. Assertion (i) extends results of Hoffman [19] and Mangasarian [23].

Theorem 5. Let A be an integral $m \times n$ matrix and let b, b' , and w be vectors such that $\max\{wx: Ax \leq b\}$ and $\max\{wx: Ax \leq b'\}$ each have optimal solutions. Then

- (i) for each optimal solution \bar{x} to $\max\{wx: Ax \leq b\}$ there exists an optimal solution \bar{x}' to $\max\{wx: Ax \leq b'\}$ with $\|\bar{x} - \bar{x}'\|_\infty \leq n\Delta(A)\|b - b'\|_\infty$, and
- (ii) if $Ax \leq b$ and $Ax \leq b'$ each have integral solutions, then for each optimal solution \bar{z} to $\max\{wx: Ax \leq b, x \text{ integral}\}$ there exists an optimal solution \bar{z}' to $\max\{wx: Ax \leq b', x \text{ integral}\}$ with $\|\bar{z} - \bar{z}'\|_\infty \leq n\Delta(A)(\|b - b'\|_\infty + 2)$.

Proof. We first show part (i) in the case where w is the zero vector, that is, if \bar{x} is a solution of $Ax \leq b$ then $A\bar{x}' \leq b'$ for some \bar{x}' with $\|\bar{x} - \bar{x}'\|_\infty \leq n\Delta(A)\|b - b'\|_\infty$. Suppose such an \bar{x}' does not exist. Then the system

$$\begin{aligned} Ax &\leq b', \\ x &\leq \bar{x} + \varepsilon \mathbf{1}, \\ -x &\leq -\bar{x} + \varepsilon \mathbf{1} \end{aligned} \tag{6}$$

(where $\varepsilon = n\Delta(A)\|b - b'\|_\infty$ and $\mathbf{1} = (1, \dots, 1)^T$) has no solution. By Farkas' Lemma, we have

$$yA + u - v = 0, \quad yb' + u(\bar{x} + \varepsilon \mathbf{1}) + v(-\bar{x} + \varepsilon \mathbf{1}) < 0 \tag{7}$$

for some nonnegative vectors y, u, v . As $Ax \leq b'$ has a solution, we have $u + v \neq 0$. We may assume $\|u + v\|_1 = 1$. We may also assume, by Carathéodory's theorem, that the positive components of y correspond to linearly independent rows of A . So the positive part of y is equal to $B^{-1}(-\tilde{u} + \tilde{v})$ for some parts \tilde{u}, \tilde{v} of u, v and some submatrix B of A . Hence, $\|y\|_1 \leq n\Delta(A)\|u - v\|_1 \leq n\Delta(A)\|u + v\|_1 = n\Delta(A)$. Now we have the contradiction

$$\begin{aligned} 0 > yb' + u(\bar{x} + \varepsilon \mathbf{1}) + v(-\bar{x} + \varepsilon \mathbf{1}) &= yb' - yA\bar{x} + \varepsilon\|u + v\|_1 \\ &\geq y(b' - b) + \varepsilon \geq -\|y\|_1\|b - b'\|_\infty + \varepsilon \geq 0. \end{aligned} \tag{8}$$

We next show part (i) in general. Let \bar{x} be an optimal solution to $\max\{wx: Ax \leq b\}$ and let x^* be any optimal solution to $\max\{wx: Ax \leq b'\}$. Let $A_0x \leq b_0$ be those inequalities from $Ax \leq b$ that are satisfied by \bar{x} with equality. Then $yA_0 = w$ for some $y \geq 0$ (by the duality theorem of linear programming). Since \bar{x} satisfies:

$$A\bar{x} \leq b, \quad A_0\bar{x} \geq A_0x^* - \|b - b'\|_\infty \mathbf{1} \tag{9}$$

and since x^* is a solution of $[Ax \leq b', A_0x \geq A_0x^*]$, we have from above that $[A\bar{x}' \leq b', A_0\bar{x}' \geq A_0x^*]$ for some \bar{x}' with $\|\bar{x} - \bar{x}'\|_\infty \leq n\Delta(A)\|b - b'\|_\infty$. As $w\bar{x}' = yA_0\bar{x}' \geq yA_0x^* = wx^*$, we have that \bar{x}' is an optimal solution to $\max\{wx: Ax \leq b'\}$.

Finally, we show part (ii). Suppose that $Ax \leq b$ and $Ax \leq b'$ each have integral solutions and let \bar{z} be an optimal solution to $\max\{wx: Ax \leq b, x \text{ integral}\}$. By Theorem 1(ii), there exists an optimal solution x^* to $\max\{wx: Ax \leq b\}$ with $\|\bar{z} - x^*\|_\infty \leq n\Delta(A)$. Part (i), above, implies that there exists an optimal solution \bar{x}' to $\max\{wx: Ax \leq b'\}$ with $\|x^* - \bar{x}'\|_\infty \leq n\Delta(A)\|b - b'\|_\infty$. Now, by Theorem 1(i), there exists an optimal

solution \bar{z}' to $\max\{wx: Ax \leq b', x \text{ integral}\}$ with $\|\bar{x}' - \bar{z}'\|_\infty \leq n\Delta(A)$. Putting this together, we have

$$\begin{aligned} \|\bar{z} - \bar{z}'\|_\infty &\leq \|\bar{z} - x^*\|_\infty + \|x^* - \bar{x}'\|_\infty + \|\bar{x}' - \bar{z}'\|_\infty \\ &\leq n\Delta(A)(\|b - b'\|_\infty + 2). \quad \square \end{aligned} \tag{10}$$

A result related to Theorem 1 was proven by Graver [16] (see also Blair and Jeroslow [3, Lemma 4.3]), who showed that for every integral matrix A there exists a finite 'test set' L of integral vectors such that: For any objective vector w and right-hand-side vector b and any nonoptimal feasible solution z to the integer program $\max\{wx: Ax \leq b, x \text{ integral}\}$ there exists a vector $l \in L$ such that $z+l$ is a feasible solution with greater objective value. The following result shows that if A is an $m \times n$ integral matrix, then we can take as a 'test set' simply the set of integral vectors with components at most $n\Delta(A)$ in absolute value. (An analysis of Blair and Jeroslow's proof [3, Lemma 4.3] will give a similar result.)

Theorem 6. *Let $\max\{wx: Ax \leq b, x \text{ integral}\}$ be an integer program with A an integral $m \times n$ matrix. Then, for each integral solution z of $Ax \leq b$, either z is an optimal solution to the integer program or there exists an integral solution \bar{z} of $Ax \leq b$ with $\|z - \bar{z}\|_\infty \leq n\Delta(A)$ and $w\bar{z} > wz$.*

Proof. Let z be an integral solution of $Ax \leq b$. If z is not an optimal solution to $\max\{wx: Ax \leq b, x \text{ integral}\}$, then there exists an integral solution z^* of $Ax \leq b$ with $wz^* > wz$. Split A into submatrices A_1, A_2 such that $A_1z^* \geq A_1z$ and $A_2z^* < A_2z$. As in the proof of Theorem 1, we have $z^* - z = \lambda_1g^1 + \dots + \lambda_tg^t$ for some numbers $\lambda_i \geq 0, i = 1, \dots, t$, and integral vectors g^1, \dots, g^t contained in the cone $C = \{x: A_1x \geq 0, A_2x \leq 0\}$ with $\|g^i\|_\infty \leq \Delta(A)$ for $i = 1, \dots, t$, where t is the dimension of C .

If $\lambda_1 \geq 1$, then we have that $z + g^1 = z^* - (\lambda_1 - 1)g^1 - \lambda_2g^2 - \dots - \lambda_tg^t$ is an integral solution of $Ax \leq b$. So if $\lambda_1 \geq 1$ and $wg^1 > 0$, then we may set $\bar{z} = z + g^1$. Thus, we may assume that $wg^i \leq 0$ for all $i \in \{1, \dots, t\}$ such that $\lambda_i \geq 1$. Let

$$\bar{z} = z^* - \lfloor \lambda_1 \rfloor g^1 - \dots - \lfloor \lambda_t \rfloor g^t = z + (\lambda_1 - \lfloor \lambda_1 \rfloor)g^1 + \dots + (\lambda_t - \lfloor \lambda_t \rfloor)g^t. \tag{11}$$

We have that \bar{z} is an integral solution of $Ax \leq b$ with

$$\|z - \bar{z}\|_\infty \leq (\lambda_1 - \lfloor \lambda_1 \rfloor)\|g^1\|_\infty + \dots + (\lambda_t - \lfloor \lambda_t \rfloor)\|g^t\|_\infty \leq n\Delta(A). \tag{12}$$

Since $wg^i \leq 0$ for each $i \in \{1, \dots, t\}$ such that $\lfloor \lambda_i \rfloor > 0$, we have $w\bar{z} \geq wz^*$. Thus $w\bar{z} > wz$. \square

For a linear system $Ax \leq b$, let $\{x: Ax \leq b\}_I$ denote the convex hull of its integral solutions. The following result of Wolsey [28, Theorem 2'], on the defining systems of $\{x: Ax \leq b\}_I$ as b varies, will be used in the next section. We use the proof

technique of Blair and Jeroslow [3, Theorem 6.2], together with Theorem 6 to prove this result, as this allows us to bound the size of certain coefficients, as stated below.

Theorem 7. *For each integral $m \times n$ matrix A , there exists an integral matrix M , with entries at most $n^{2n}\Delta(A)^n$ in absolute value, such that for each vector b for which $Ax \leq b$ has an integral solution there exists a vector d_b such that $\{x: Ax \leq b\}_I = \{x: Mx \leq d_b\}$.*

Proof. We may assume that $A \neq 0$. Let K be the cone generated by the rows of A . If b is a vector such that $Ax \leq b$ has an integral solution, then, by linear programming duality, $\max\{wx: Ax \leq b, x \text{ integral}\}$ exists if and only if $w \in K$. Let $L = \{z \in Z^n: \|z\|_\infty \leq n\Delta(A)\}$ and for each $T \subseteq L$ let $G(T)$ be a finite set of integral vectors which generates the cone $C(T) = \{w: wz \leq 0 \text{ for all } z \in T\} \cap K$. Finally, let M be the matrix with rows $\bigcup \{G(T): T \subseteq L\}$.

Suppose that b is a fixed vector such that $Ax \leq b$ has an integral solution. By replacing b by $\lfloor b \rfloor$ if necessary, we may assume that b is integral. Let d_b be the minimal vector (with respect to the \leq ordering) such that $\{x: Ax \leq b\}_I \subseteq \{x: Mx \leq d_b\}$. (Since each row of M is in the cone K , each component of d_b is finite.) To prove that $\{x: Ax \leq b\}_I \supseteq \{x: Mx \leq d_b\}$, we will show that each valid inequality for $\{x: Ax \leq b\}_I$ is implied by $Mx \leq d_b$.

Let $\bar{w}x \leq t$ be an inequality such that $\bar{w} \neq 0$ and $\{x: Ax \leq b\}_I \subseteq \{x: \bar{w}x \leq t\}$. We may assume that $t = \max\{\bar{w}x: Ax \leq b, x \text{ integral}\}$. Let \bar{z} be an integral solution of $Ax \leq b$ such that $\bar{w}\bar{z} = t$. Now let $\bar{T} = \{z \in L: A(\bar{z} + z) \leq b\}$. Since \bar{z} is an optimal solution to $\max\{\bar{w}x: Ax \leq b, x \text{ integral}\}$, we have that $\bar{w} \in C(\bar{T})$. So there exist nonnegative numbers λ_g , for $g \in G(\bar{T})$, such that $\sum \{\lambda_g g: g \in G(\bar{T})\} = \bar{w}$. By Theorem 6, \bar{z} is an optimal solution to $\max\{gx: Ax \leq b, x \text{ integral}\}$ for each $g \in G(\bar{T})$. Thus, letting $g(b) = \max\{gx: Ax \leq b, x \text{ integral}\}$ for each $g \in G(\bar{T})$, we have

$$\sum \{\lambda_g g(b): g \in G(\bar{T})\} = \sum \{\lambda_g g\bar{z}: g \in G(\bar{T})\} = \bar{w}\bar{z} = t. \tag{13}$$

So $\bar{w}x \leq t$ is a nonnegative linear combination of inequalities in $Mx \leq d_b$, and, hence, is implied by $Mx \leq d_b$.

To bound the size of the entries in M , notice that Cramer's rule implies that there exists an integral matrix D such that $K = \{x: Dx \leq 0\}$ and each entry of D having absolute value at most $\Delta(A)$. Thus, for each $T \subseteq L$ the cone $C(T)$ is defined by a linear system $Fx \leq 0$ for some integral matrix F with entries at most $n\Delta(A)$ in absolute value. Using Cramer's rule, it follows that for each $T \subseteq L$ the cone $C(T)$ can be generated by a finite set of integral vectors, each of which has components which are at most $n!(n\Delta(A))^n \leq n^{2n}\Delta(A)^n$ in absolute value. \square

For an integral matrix A , the bound given above on the size of the entries in M is polynomial in n and the size of the largest entry in A , and is independent of the number of rows of A . Thus, this bound implies the result, of Karp and Papadimitriou [21], that if a polyhedron $P \subseteq Q^n$ can be defined by a system of integral linear inequalities, each of which has size at most σ , then the convex hull of the integral

points in P can be defined by a system of linear inequalities, each of which has size bounded above by a polynomial function of n and σ .

Remarks. (1) Blair and Jeroslow [1, 2] proved their results in the more general context of mixed integer programming. It is straightforward, however, to modify the proof of Theorem 1 to obtain the following mixed integer programming result:

Let A be an integral $m \times k$ matrix, B an integral $m \times (n - k)$ matrix, and w, v , and b vectors such that $\max\{wx + vy : Ax + By \leq b, x \text{ integral}\}$ has an optimal solution. Then, for each optimal solution (x^1, y^1) to $\max\{wx + vy : Ax + By \leq b\}$, there exists an optimal solution (x^2, y^2) to $\max\{wx + vy : Ax + By \leq b, x \text{ integral}\}$ such that $\|(x^1, y^1) - (x^2, y^2)\|_\infty \leq n\Delta((A|B))$. And, for each optimal solution (x^3, y^3) to $\max\{wx + vy : Ax + By \leq b, x \text{ integral}\}$, there exists an optimal solution (x^4, y^4) to $\max\{wx + vy : Ax + By \leq b\}$ such that $\|(x^3, y^3) - (x^4, y^4)\|_\infty \leq n\Delta((A|B))$.

Using this, one can prove the mixed integer programming analogues of Corollary 2, Corollary 3, and Theorem 5(ii).

In a similar way, one can obtain an analogue of Theorem 6. Note, however, that the mixed integer programming version of Theorem 6 does not imply a finite 'test set' result for mixed integer programs. This is due to the fact that, unlike the integer programming case, there may exist infinitely many vectors in $\{(x, y) : x \text{ integral}, \|(x, y)\|_\infty \leq n\Delta(A|B)\}$. Thus one cannot use the proof of Theorem 7 to obtain an analogous result for mixed integer programming. (In fact, the mixed integer programming analogue of Theorem 7 is not true, since, for example, the convex hull of $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, x_2 \leq \alpha, x_1 \text{ integral}\}$, for any α such that $0 < \alpha < 1$, has $x_1 + (1/\alpha)x_2 \leq 1$ as a facet-inducing inequality.) Similarly, the mixed integer programming version of Theorem 1 does not imply a mixed integer analogue of Corollary 4.

However, observe that if A, B and b are integral and $\max\{wx + vy : Ax + By \leq b, x \text{ integral}\}$ has an optimal solution then it has one such that the denominator of each component of the vector y is the determinant of a square submatrix of B . This observation can be used to prove a mixed integer analogue of Corollary 4 and a finite 'test set' consequence of Theorem 6 for the case where b is integral. This latter finite 'test set' result can be used to modify the proof of Theorem 7 to obtain the following:

Let A be an integral $m \times k$ matrix and let B be an integral $m \times (n - k)$ matrix. Then there exist integral matrices M and N , each with entries at most $n^{2n}\Delta(B)^{2n}\Delta((A|B))^n$ in absolute value, such that for each integral vector b for which $\{(x, y) : Ax + By \leq b, x \text{ integral}\}$ is nonempty, there exists a vector d_b such that $\{(x, y) : Mx + Ny \leq d_b\}$ is equal to the convex hull of $\{(x, y) : Ax + By \leq b, x \text{ integral}\}$.

(2) The proof of Theorem 5(i) remains valid, even when A is an irrational matrix, if we replace $\Delta(A)$ by $\Delta'(A)$, the maximum, over all invertible submatrices B of A , of the absolute values of the entries of B^{-1} .

3. The Chvátal rank of matrices

If all vectors in a polyhedron P satisfy the linear inequality $ax \leq \beta$, where a is an integral vector and β is a number, then each integral vector in P satisfies the *Chvátal cut* $ax \leq \lfloor \beta \rfloor$. Denote by P' the set of vectors which satisfy every Chvátal cut of P . It is easy to see that $P_I \subseteq P'$, where P_I denotes the convex hull of integral vectors in P . Schrijver [24], continuing the work of Chvátal [5] and Gomory [13], proved that P' is a polyhedron and that $P^{(t)} = P_I$ for some natural number t , where $P^{(0)} = P$ and $P^{(i)} = P^{(i-1)'}$ for all $i \geq 1$. The least number t such that $P^{(t)} = P_I$ is the *Chvátal rank* of P . Results on the Chvátal rank of combinatorially described polyhedra can be found in Boyd and Pulleyblank [4] and Chvátal [5, 6, 7].

We define the *Chvátal rank of matrix* A to be the supremum over all integral vectors b of the Chvátal rank of $\{x: Ax \leq b\}$. It is convenient also to define the *strong Chvátal rank* of A , which is the Chvátal rank of the matrix

$$\begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix}.$$

Then the well-known theorem of Hoffman and Kruskal [20] is that an integral matrix has strong Chvátal rank 0 if and only if it is totally unimodular. Moreover, Hoffman and Kruskal showed that an integral matrix A has Chvátal rank 0 if and only if A^T is unimodular. (A matrix C is called *unimodular* if for each submatrix B consisting of r ($= \text{rank } C$) linearly independent columns of C , the g.c.d. of the subdeterminants of B of order r is 1.) In Edmonds and Johnson [9] and Gerards and Schrijver [11], characterisations of classes of matrices having Chvátal rank 1 are given. The main result of this section is that every matrix has a finite Chvátal rank.

To prove this result we need the following theorem of Cook, Coullard, and Turán [8]. For completeness, a proof of this result is given below. In the proof, we use a lemma of Schrijver [24] which implies that if F is a face of a polyhedron P then $P^{(k)} \cap F \subseteq F^{(k)}$ for all $k \geq 0$. We also use the fact that if T is an affine transformation from Q^n to Q^n which maps Z^n onto Z^n , then for any polyhedron $P \subseteq Q^n$ we have $T(P^{(k)}) = T(P)^{(k)}$ for all $k \geq 0$.

Theorem 9. *If $P \subseteq Q^n$ is a polyhedron with $P \cap Z^n = \emptyset$, then $P^{(n^2 2^{n^3})} = \emptyset$.*

Proof. The proof is by induction on n . It is easy to see that the result is true for $n = 1$. Suppose $n \geq 2$ and that for all polyhedra $G \subseteq Q^{n-1}$ with $G \cap Z^{n-1} = \emptyset$, we have $G^{(\gamma_{n-1})} = \emptyset$, where $\gamma_{n-1} = (n-1)^{2(n-1)} 2^{(n-1)^3}$.

Let $P \subseteq Q^n$ be a polyhedron with $P \cap Z^n = \emptyset$. It follows from a result of Grötschel, Lovász, and Schrijver [18] that there exists a nonzero integral vector w such that $|w(x - x')| < n(n+1)2^{(1/2)n^2}$ for all $x, x' \in P$. (In the case where P is a bounded polyhedron, the existence of such a vector w was shown by Lenstra [22], see also

Grötschel, Lovász, and Schrijver [17].) Let

$$\gamma_n = n^{2n_2n^3} \geq n(n+1)2^{(1/2)n^2}(\gamma_{n-1} + 1) + 1. \tag{14}$$

We will complete the proof by showing that $P^{(\gamma_n)} = \emptyset$.

We may assume that the components of w are relatively prime integers. Let $\beta = \lfloor \max\{wx : x \in P\} \rfloor$. It follows that $P^{(1)} \subseteq \{x : wx \leq \beta\}$. If $P^{(1)} = \emptyset$ we are finished, so suppose not. Let $G = P^{(1)} \cap \{x : wx = \beta\}$. Since $\{x : wx = \beta\}$ contains integral vectors, there exists an affine transformation T which maps Z^n onto Z^n and $\{x : wx = \beta\}$ onto $\{x : x_n = 0\}$. Let $\bar{G} = \{(x_1, \dots, x_{n-1}) \in Q^{n-1} : (x_1, \dots, x_{n-1}, 0) \in T(G)\}$. By assumption, $\bar{G}^{(\gamma_{n-1})} = \emptyset$. But this implies that $T(G)^{(\gamma_{n-1})} = \emptyset$ and hence that $G^{(\gamma_{n-1})} = \emptyset$. Thus, by the lemma of Schrijver [24] mentioned above, we have $P^{(\gamma_{n+1}+1)} \cap G = \emptyset$. So $P^{(\gamma_{n+1}+2)} \subseteq \{x : wx \leq \beta - 1\}$: Since $P \subseteq \{x : wx > \beta - n(n+1)2^{(1/2)n^2}\}$, repeating this procedure at most $n(n+1)2^{(1/2)n^2} - 1$ times, we obtain the empty set. Hence $P^{(\gamma_n)} = \emptyset$. \square

We will also make use of the following consequence of this theorem.

Corollary 9. *Let $P \subseteq Q^n$ be a polyhedron such that $P \cap Z^n \neq \emptyset$ and let w be an integral vector such that $q = \max\{wx : x \in P_I\}$ exists. Then $P^{(r)} \subseteq \{x : wx \leq q\}$, where $r = (n^{2n}2^{n^3} + 1)(\lfloor \max\{wx : x \in P\} \rfloor - q) + 1$.*

Proof. Let $\beta = \lfloor \max\{wx : x \in P\} \rfloor$, which exists since q exists. We have $P^{(1)} \subseteq \{x : wx \leq \beta\}$. If $q = \beta$ we are finished, so suppose $q < \beta$. Let $G = P^{(1)} \cap \{x : wx = \beta\}$. Since $G \cap Z^n = \emptyset$, we have $G^{(n^{2n}2^{n^3})} = \emptyset$, by Theorem 8. Using the lemma of Schrijver [24], this implies that $P^{(n^{2n}2^{n^3}+1)} \cap \{x : wx = \beta\} = \emptyset$. So $P^{(n^{2n}2^{n^3}+2)} \subseteq \{x : wx \leq \beta - 1\}$. Repeating this operation $(\beta - q - 1)$ times, we have $P^{(r)} \subseteq \{x : wx \leq q\}$. \square

We are now ready to prove our finite Chvátal rank theorem.

Theorem 10. *The Chvátal rank of an integral $m \times n$ matrix A is at most $2^{n^3+1}n^{5n}\Delta(A)^{n+1}$.*

Proof. Let M be an integral matrix satisfying the conditions given in Theorem 7, for the matrix A . We have $\max\{\|m\|_1 : m \text{ is a row of } M\} \leq n^{2n+1}\Delta(A)^n$. Let

$$k = (n^{2n}2^{n^3} + 1)(n^{2n+2}\Delta(A)^{n+1}) + 1 \tag{15}$$

and let $P = \{x : Ax \leq b\}$ for some vector b . Since $k \leq 2^{n^3+1}n^{5n}\Delta(A)^{n+1}$, it suffices to show that $P^{(k)} = P_I$. If $P \cap Z^n = \emptyset$, then, by Theorem 8, $P^{(k)} = \emptyset = P_I$. Suppose $P \cap Z^n \neq \emptyset$. We have $P_I = \{x : Mx \leq d_b\}$ for some vector d_b . Let m be a row of M and let $q = \max\{mx : x \in P_I\}$. By Corollary 2,

$$\max\{mx : x \in P\} - q \leq n\Delta(A)\|m\|_1 \leq n^{2n+2}\Delta(A)^{n+1}. \tag{16}$$

Thus, by Corollary 9, $P^{(k)} \subseteq \{x : mx \leq q\}$. So $P^{(k)} \subseteq \{x : Mx \leq d_b\}$, which implies that $P^{(k)} = P_I$. \square

Remarks. (1) A consequence of Theorem 10 is that the number of Chvátal cuts that must be added to a linear system $Ax \leq b$ to obtain a defining system for $\{x: Ax \leq b\}_t$ can be bounded above by a function of the matrix A , independent of the right-hand-side vector b . Indeed, for each linear system $Ax \leq b$ with A an integral $m \times n$ matrix of Chvátal rank t , the polyhedron $\{x: Ax \leq b\}_t$ is the solution set of a system of inequalities

$$\begin{aligned} Ax &\leq b, \\ \alpha_i x &\leq \beta_i \quad (i = 1, \dots, N) \end{aligned} \tag{17}$$

where $N \leq 2^n t n^m S^n$ (with S denoting the maximum of the absolute values of the entries of A) and for each $k = 1, \dots, N$, the inequality $\alpha_k x \leq \beta_k$ is a Chvátal cut for the polyhedron $\{x: Ax \leq b, \alpha_i x \leq \beta_i, i = 1, \dots, k - 1\}$. To prove this, we use the result of Schrijver [24] that if $P = \{x: Mx \leq d\}$, where M is integral and $Mx \leq d$ is a totally dual integral system, then $P' = \{x: Mx \leq \lfloor d \rfloor\}$. (A linear system $Mx \leq d$ is a *totally dual integral system* if $\min\{yd: yM = w, y \geq 0\}$ can be achieved by an integral vector for each integral w for which the minimum exists.) Giles and Pulleyblank [15] proved that every polyhedron can be defined by such a totally dual integral system (see also Schrijver [25]). Their proof, together with Carathéodory's theorem, implies that if $Dx \leq f$ is a linear system with D an integral $m \times n$ matrix, each entry of which has absolute value at most T , then there exists a totally dual integral system $D'x \leq f'$ with $\{x: D'x \leq f'\} = \{x: Dx \leq f\}$ such that each entry of D' is an integer with absolute value at most nT . This implies that if $P = \{x: Ax \leq b\}$, then for each $i = 1, \dots, t$ the polyhedron $P^{(i)}$ can be defined by a linear system $M^i x \leq b^i$ where each inequality is a Chvátal cut for $P^{(i-1)}$ and where M^i is an integral matrix having at most $(2n^i S)^n$ rows. (This technique for bounding the size of M^i is also used in Boyd and Pulleyblank [4].)

(2) Professor C. Blair pointed out to us several relations of his and Jeroslow's work with our results. Especially, the fact that each matrix has a finite Chvátal rank can be seen to be equivalent to their result that 'each integer programming value function is a Gomory function'. Here we shall discuss this relation.

A function $f: \mathbb{Q}^m \rightarrow \mathbb{Q}$ is a *Gomory function* if there exist rational matrices M_1, \dots, M_t so that M_1, \dots, M_{t-1} are nonnegative, and so that, for each $b \in \mathbb{Q}^m$,

$$f(b) = \max_j (M_1 \lceil M_2 \lceil \dots \lceil M_t b \rceil \dots \rceil \rceil)_j \tag{18}$$

(here $\lceil \cdot \rceil$ denotes component-wise upper integer parts; M_i has m columns, and M_i has the same number of rows as M_{i-1} has columns ($i = 2, \dots, t$); the maximum ranges over all coordinates j of the vector $(M_i \lceil M_2 \lceil \dots \lceil M_t b \rceil \dots \rceil)_j$).

Blair and Jeroslow [3, Theorems 5.1 and 5.2] showed:

Blair–Jeroslow Theorem. For each rational $m \times n$ -matrix A and row vector $c \in \mathbb{Q}^n$ with $\min\{cx: Ax = 0, x \geq 0\}$ finite, there exist Gomory functions $f, g: \mathbb{Q}^m \rightarrow \mathbb{Q}$ so that, for

each $b \in \mathbb{Q}^m$,

- (i) $f(b) \leq 0$ if and only if $\{x \mid x \geq 0; Ax = b; x \text{ integral}\}$ is nonempty;
- (ii) $g(b) = \min\{cx \mid x \geq 0; Ax = b; x \text{ integral}\}$ iff $f(b) \leq 0$.

Proposition. *The Blair-Jeroslow theorem is equivalent to each rational matrix having finite Chvátal rank.*

Proof (sketch). I. We first show that the Blair-Jeroslow theorem implies that each rational matrix has a finite Chvátal rank. One easily checks that it suffices to derive from the Blair-Jeroslow theorem that for each rational matrix A and vector b , the Chvátal rank of the polyhedron $\{x \mid x \geq 0; Ax = b\}$ has an upper bound only depending on A .

Choose a rational matrix A . By Theorem 7, there exists a matrix C so that for each vector b there exists a vector d_b with $\{x \mid x \geq 0; Ax = b\}_I = \{x \mid x \geq 0; Cx \leq d_b\}$.

Take any row c of C . By the Blair-Jeroslow theorem there exist Gomory functions f and g with the properties described in (19). Hence there exist matrices M_1, \dots, M_t and $N_1, \dots, N_{t'}$, so that $M_1, \dots, M_{t-1}, N_1, \dots, N_{t'-1} \geq 0$ and, for each $b \in \mathbb{Q}^m$,

$$\begin{aligned} \min\{cx \mid x \geq 0; Ax = b; x \text{ integral}\} &= \max\{uM_1 \lceil M_2 \cdots \lceil M_t b \rceil \cdots \rceil \\ &+ vN_1 \lceil N_2 \cdots \lceil N_{t'} b \rceil \cdots \rceil \mid u \geq 0; v \geq 0; v\mathbf{1} = 1\}. \end{aligned} \tag{20}$$

Without loss of generality, $t = t'$ (otherwise add some matrices I). By taking for b any column of A one sees that for each $u \geq 0, v \geq 0$ with $v\mathbf{1} = 1$ one has:

$$uM_1 \lceil M_2 \cdots \lceil M_t A \rceil \cdots \rceil + vN_1 \lceil N_2 \cdots \lceil N_{t'} A \rceil \cdots \rceil \leq c. \tag{21}$$

Hence, for each b ,

$$\begin{aligned} \min\{cx \mid x \geq 0; Ax = b; x \text{ integral}\} \\ \leq \max\{w \lceil P_2 \cdots \lceil P_t b \rceil \cdots \rceil \mid w \geq 0; w \lceil P_2 \cdots \lceil P_t A \rceil \cdots \rceil \leq c\}, \end{aligned} \tag{22}$$

where

$$P_i := \begin{bmatrix} M_i & 0 \\ 0 & N_i \end{bmatrix} \quad (i = 2, \dots, t-1), \quad P_t := \begin{bmatrix} M_t \\ N_t \end{bmatrix} \tag{23}$$

((22) follows by taking $w := (uM_1 vN_1)$).

By applying LP-duality to the maximum in (22) we see that for each b :

$$\begin{aligned} \min\{cx \mid x \geq 0; Ax = b; x \text{ integral}\} \\ \leq \min\{cx \mid x \geq 0; \lceil P_2 \cdots \lceil P_t A \rceil \cdots \rceil x \leq \lceil P_2 \cdots \lceil P_t b \rceil \cdots \rceil\}. \end{aligned} \tag{24}$$

Hence equality follows.

Let t_{\max} be the maximum of the t above when c runs over all rows of C . Then we know that the Chvátal rank of $\{x \mid x \geq 0; Ax = b\}$ is at most t_{\max} , for each $b \in \mathbb{Q}^m$.

II. We next show the reverse implication. Choose matrix A and vector c . As, by assumption, the matrix

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix}$$

has finite Chvátal rank, we know that there exist matrices P_1, \dots, P_t so that $P_1, \dots, P_{t-1} \geq 0$ and so that, for each $b \in \mathbb{Q}^m$,

$$\begin{aligned} \{x \mid x \geq 0; Ax = b\}_t &= \{x \mid x \geq 0; [P_1 [\dots [P_t A] \dots] x \\ &\geq [P_1 [\dots [P_t b] \dots]]\}. \end{aligned} \tag{25}$$

Hence, with LP-duality,

$$\begin{aligned} \min\{cx \mid x \geq 0; Ax = b; x \text{ integral}\} \\ &= \min\{cx \mid x \geq 0; [P_1 \dots [P_t A] \dots] x \geq [P_1 \dots [P_t b] \dots]\} \\ &= \max\{y [P_1 \dots [P_t b] \dots] \mid y \geq 0; y [P_1 \dots [P_t A] \dots] \leq c\}. \end{aligned} \tag{26}$$

Let the rows of the matrix M , say, be the vertices of the polyhedron $\{y \geq 0 \mid y [P_1 \dots [P_t A] \dots] \leq c\}$, and let the rows of the matrix N , say, be the extremal (infinite) rays of this polyhedron. Then the maximum in (26) is equal to:

$$\max\{uN [P_1 \dots [P_t b] \dots] + vM [P_1 \dots [P_t b] \dots] \mid u \geq 0; v \geq 0; v\mathbf{1} = 1\}. \tag{27}$$

Hence defining, for $b \in \mathbb{Q}^m$,

$$\begin{aligned} f(b) &:= \max_j (N [P_1 \dots [P_t b] \dots])_j, \\ g(b) &:= \max_j (M [P_1 \dots [P_t b] \dots])_j, \end{aligned} \tag{28}$$

gives Gomory functions satisfying (19). \square

References

- [1] C.E. Blair and R.G. Jeroslow, "The value function of a mixed integer program: I", *Discrete Mathematics* 19 (1977) 121-138.
- [2] C.E. Blair and R.G. Jeroslow, "The value function of a mixed integer program: II", *Discrete Mathematics* 25 (1979) 7-19.
- [3] C.E. Blair and R.G. Jeroslow, "The value function of an integer program", *Mathematical Programming* 23 (1982) 237-273.
- [4] S.C. Boyd and W.R. Pulleyblank, "Facet generating techniques", in preparation.
- [5] V. Chvátal, "Edmonds polytopes and a hierarchy of combinatorial problems", *Discrete Mathematics* 4 (1973) 305-337.
- [6] V. Chvátal, "Edmonds polytopes and weakly hamiltonian graphs", *Mathematical Programming* 5 (1973) 29-40.
- [7] V. Chvátal, "Cutting-plane proofs and the stability number of a graph", Report No. 84326-OR, Institut für Ökonometrie und Operations Research, Universität Bonn, FR Germany, 1984.

- [8] W. Cook, C. Coullard and Gy. Turán, "On the complexity of cutting-plane proofs", in preparation.
- [9] J. Edmonds and E.L. Johnson, "Matching: a well-solved class of integer linear programs", in: R.K. Guy, et al., eds., *Combinatorial structures and their applications* (Gordon and Breach, New York, 1970) pp. 89-92.
- [10] J. von zur Gathen and M. Sieveking, "A bound on solutions of linear integer equalities and inequalities", *Proceedings of the American Mathematical Society* 72 (1978) 155-158.
- [11] A.M.H. Gerards and A. Schrijver, "Matrices with the Edmonds-Johnson property", to appear.
- [12] F.R. Giles and W.R. Pulleyblank, "Total dual integrality and integer polyhedra", *Linear Algebra and its Applications* 25 (1979) 191-196.
- [13] R.E. Gomory, "An algorithm for integer solutions to linear programs", in: R.L. Graves and P. Wolfe (eds.), *Recent advances in mathematical programming* (McGraw-Hill, New York, 1963) pp. 269-302.
- [14] R.E. Gomory, "On the relation between integer and noninteger solutions to linear programs", *Proceedings of the National Academy of Science* 53 (1965) 260-265.
- [15] R.E. Gomory, "Some polyhedra related to combinatorial problems", *Linear Algebra and its Applications* 2 (1969) 451-558.
- [16] J.E. Graver, "On the foundations of linear and integer linear programming I", *Mathematical Programming* 8 (1975) 207-226.
- [17] M. Grötschel, L. Lovász and A. Schrijver, "Geometric methods in combinatorial optimization", in: W.R. Pulleyblank, ed., *Progress in combinatorial optimization* (Academic Press, New York, 1984) pp. 167-183.
- [18] M. Grötschel, L. Lovász and A. Schrijver, *The ellipsoid method and combinatorial optimization* (Springer-Verlag, Berlin), to appear.
- [19] A.J. Hoffman, "On approximate solutions of systems of linear inequalities", *Journal of Research of the National Bureau of Standards* 49 (1952) 263-265.
- [20] A.J. Hoffman and J.B. Kruskal, "Integral boundary points of convex polyhedra", in: H.W. Kuhn and A.W. Tucker, eds., *Linear inequalities and related systems*, Annals of Mathematics Study 38 (Princeton University Press, Princeton, 1956) pp. 223-247.
- [21] R.M. Karp and C.H. Papadimitriou, "On linear characterizations of combinatorial optimization problems", *SIAM Journal on Computing* 11 (1982) 620-632.
- [22] H.W. Lenstra, "Integer programming with a fixed number of variables", *Mathematics of Operations Research* 8 (1983) 538-548.
- [23] O.L. Mangasarian, "A condition number for linear inequalities and linear programs", in: G. Bamberg and O. Opitz, eds., *Proc. of 6th Symposium über Operations Research, Universität Augsburg, September 7-9 (1981)* (Verlagsgruppe Athenäum/Hain/Scriptor/Hanstein, Königstein, 1981) pp. 3-15.
- [24] A. Schrijver, "On cutting planes", *Annals of Discrete Mathematics* 9 (1980) 291-296.
- [25] A. Schrijver, "On total dual integrality", *Linear Algebra and its Applications* 38 (1981) 27-32.
- [26] A. Schrijver, *Theory of linear and integer programming* (Wiley, Chichester, 1986).
- [27] J. Stoer and C. Witzgall, *Convexity and optimization in finite dimensions I* (Springer-Verlag, Berlin, 1970).
- [28] L.A. Wolsey, "The b -hull of an integer program", *Discrete Applied Mathematics* 3 (1981) 193-201.