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ON THE NUMBER OF EDGE-COLOURINGS OF REGULAR BIPARTITE GRAPHS

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Let $f(k)$ be the largest number such that each k-regular bipartite graph with $2n$ vertices has at least $f(k)^n$ 1-factorizations. We prove that $f(k) \leq k!^2/k^k$, and that equality holds if k contains no other prime factors than 2 and 3. We conjecture equality for each k.

Introduction

We are interested in lower bounds for the number of edge-colourings (1-factorizations) of regular bipartite graphs. Let $f(k, n)$ be the smallest possible number of k-edge-colourings of a k-regular bipartite graph with $2n$ vertices. (A *k-edge-colouring* is an ordered partition of the edge set into perfect matchings. In this paper, graphs are allowed to have multiple edges.) In particular, we are interested in the largest possible number $f(k)$ such that $f(k, n) \geq f(k)^n$ for each n, i.e., in

$$f(k) := \inf_{n \in \mathbb{N}} \sqrt[n]{f(k, n)} = \lim_{n \to \infty} \sqrt[n]{f(k, n)} \tag{1}$$

(the second equality follows from "Fekete's lemma", as $f(k, n_1 + n_2) \leq f(k, n_1) \cdot f(k, n_2)$).

We show that

$$f(k) \leq k!^2/k^k, \tag{2}$$

and that equality holds if $k = 2^a 3^b$. We conjecture that equality holds for each natural number k. This would follow from a conjecture made in [3]: the number of perfect matchings in a k-regular bipartite graph with $2n$ vertices is at least

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \tag{3}$$

As corollaries we derive some results on the number of latin squares and on permanents, namely Bang's [1] lower bound of e^{-n} for permanents of doubly

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stochastic matrices of order n. In fact the methods in this paper have been inspired to a large extent by the paper of Bang and by discussions with W.G. Valiant on a related paper of Friedland [2].

Results

Theorem 1. $f(k, n) \leq k!^{2n} n!^k/(nk)!$.

Proof. Let $X = \{1, \ldots, kn\}$, and let Π be the collection of all ordered partitions $\mathcal{A} = (A_1, \ldots, A_n)$ of X into n classes of size k. If $\mathcal{A}$ and $\mathcal{B}$ are in Π, denote by $c(\mathcal{A}, \mathcal{B})$ the number of partitions $\mathcal{C} = (C_1, \ldots, C_k)$ of X into k classes of size n such that

$$|A_i \cap C_j| = |B_i \cap C_j| = 1, \tag{4}$$

for $i = 1, \ldots, n$; $j = 1, \ldots, k$ (i.e., each C_j is a common system of distinct representatives (SDR) for $\mathcal{A}$ and $\mathcal{B}$). It is easy to see that $c(\mathcal{A}, \mathcal{B})$ is the number of k-edge-colourings of the k-regular bipartite graph with vertices, say, $a_1, \ldots, a_n, b_1, \ldots, b_n$, where a_i and b_j are connected by $|A_1 \cap B_j|$ edges, for $i, j = 1, \ldots, n$. In particular, $c(\mathcal{A}, \mathcal{B}) \geq f(k, n)$.

Now fix some $\mathcal{A}$ in Π, and consider the sum

$$\sum_{\mathcal{B} \in \Pi} c(\mathcal{A}, \mathcal{B}). \tag{5}$$

This sum may be evaluated in two ways. First

$$\sum_{\mathcal{B} \in \Pi} c(\mathcal{A}, \mathcal{B}) \geq |\Pi| \cdot f(k, n) = \frac{(nk)!}{k!^n} \cdot f(k, n). \tag{6}$$

Alternatively, there are $k!^n$ possible partitions $\mathcal{C} = (C_1, \ldots, C_k)$ of X with $|A_i \cap C_j| = 1$ for $i = 1, \ldots, n$; $j = 1, \ldots, k$. For each such partition there are $n!^k$ partitions $\mathcal{B}$ in Π such that $|B_i \cap C_j| = 1$, for $i = 1, \ldots, n$; $j = 1, \ldots, k$. So the sum (5) is equal to $k!^n \cdot n!^k$. Combining this with (6) yields the required upper bound for $f(k, n)$ $\square$

Corollary 1a. $f(k) \leq k!^2/k^k$.

Proof. Apply Stirling's asymptotical formula for $n \to \infty$ to Theorem 1. $\square$

Theorem 2

$$f(kl, n) \geq \left(\frac{(kl)!}{k!^l \cdot l!^k}\right)^{2n} \cdot f(l, kn) \cdot f(k, n)^l.$$

Proof. Let $G = (V, E)$ be a kl-regular bipartite graph with $2n$ vertices, having exactly $f(kl, n)$ kl-edge-colourings. Consider all possible graphs G' arising from G as follows. Each vertex of G is replaced by k new vertices, while each edge e of G is replaced by one new edge connecting two of the new vertices replacing the endpoints of the original edge e, in such a way that the new graph g' is l-regular. So the number of graphs G' arising from G in this way is equal to

$$\left(\frac{(kl)!}{l!^k}\right)^{2n}, \tag{7}$$

since for each vertex v of G we have to partition, arbitrarily, the edges incident with v into k classes of size l.

Let Π be the collection of all partitions $(E_1, \ldots, E_l)$ of the edge set of G into l classes such that each class E_i induces a k-regular subgraph of G. Now any l-edge-colouring $(E_1, \ldots, E_l)$ of a derived graph G' yields a partition in Π. Conversely, each partition in Π arises in this way from an l-edge-colouring of $k!^{2ln}$ graphs G'. Hence, by (7),

$$k!^{2ln} \cdot |\Pi| \geq \left(\frac{(kl)!}{l!^k}\right)^{2n} \cdot f(l, kn). \tag{8}$$

Now each class E_i of a partition $\mathscr{E}$ in Π can be refined to a k-edge-colouring of the graph (V, E_i) in at least $f(k, n)$ ways. So $\mathscr{E}$ can be refined in at least $f(k, n)^l$ ways to a kl-edge-colouring of G. Therefore, the total number $f(kl, n)$ of kl-edge-colourings of G is at least $|\Pi| \cdot f(k, n)^l$, that is, by (8), at least the required lower bound. □

Corollary 2a

$$f(kl) \geq \left(\frac{(kl)!}{k!^l \cdot l!^k}\right)^2 \cdot f(k)^l \cdot f(l)^k.$$

Proof. Directly from the definition of $f(k)$ and Theorem 2. □

A natural function seems to be

$$g(k) := \left(\frac{k!^2}{f(k)}\right)^{1/k}. \tag{9}$$

Corollary 1a states that $g(k) \geq k$, while Corollary 2a asserts that $g(kl) \leq g(k) \cdot g(l)$.

Corollary 2b. *If $f(k) = k!^2/k^k$ for $k = k_1$ and for $k = k_2$, then also for $k = k_1k_2$.*

Proof. Directly from Corollaries 1a and 2a. □

Corollary 2c. *If $k = 2^a 3^b$, then $f(k) = k!^2/k^k$.*

Proof. By Corollaries 1a and 1b it suffices to show that $f(2)\geq 1$ and $f(3)\geq\frac{4}{3}$. The former inequality is trivial, while the latter follows from the result of Voorhoeve [4] that the permanent of a nonnegative, integral matrix of order n with line sums 3 is at least $(\frac{4}{3})^n$, i.e., that the number of perfect matchings in a 3-regular bipartite graph with $2n$ vertices is at least $(\frac{4}{3})^n$. Hence $f(3, n)\geq(\frac{4}{3})^n$. □

Let $p(k, n)$ be the smallest possible permanent of a nonnegative, integral matrix of order n with line sums k, i.e., the smallest possible number of perfect matchings of a k-regular bipartite graph with $2n$ vertices. Van der Waerden's conjecture (recently proved by Falikman (added in proof)) says

$$p(k, n)\geq\left(\frac{k}{n}\right)^n\cdot n!, \tag{10}$$

while in [3] it was conjectured that

$$p(k, n)\geq\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \tag{11}$$

(Note that the first bound is asymptotical for $k\to\infty$, while the second one is asymptotical for $n\to\infty$.) Now one easily sees that

$$f(k, n)\geq p(k, n)\cdot p(k-1, n)\cdot\;\cdots\;\cdot p(1, n). \tag{12}$$

So conjecture (11) implies our conjecture that

$$f(k, n)\geq\left(\frac{k!^2}{k^k}\right)^n. \tag{13}$$

Let $L(n)$ denote the number of latin squares of order n. Since $L(n)$ is equal to the number of n-edge-colourings of the complete bipartite graph $K_{n,n}$, we know that $L(n)\geq f(n, n)\geq p(n, n)\cdot p(n-1, n)\cdot\;\cdots\;\cdot p(1, n)$. Both Van der Waerden's conjecture (10) and conjecture (11) imply that

$$L(n)\geq n!^{2n}/n^{n^2} \tag{14}$$

(cf. Wilson [5]), so both conjectured lower bounds support some evidence to each other. In fact, the lower bound (14) indeed can be proved if n has no other prime factors than 2 and 3.

Corollary 2d. *If $n=2^a3^b$, then there are at least $n!^{2n}/n^{n^2}$ latin squares of order n.*

Proof. Directly from Corollary 2c. □

Finally we derive the following result of Bang [1].

Corollary 2e. *The permanent of a doubly stochastic matrix of order n is at least* e^{-n}.

Proof. Since the dyadic doubly stochastic matrices form a dense subset of the space of all doubly stochastic matrices it suffices to prove the lower bound for dyadic matrices only. So let $M=(m_{ij})$ be a dyadic doubly stochastic matrix of order n. Let u be a natural number such that $2^u M$ is integral, and let for each $t \geq u$, G_t be the 2^t-regular bipartite graph with vertices, say, $a_1, \ldots, a_n, b_1, \ldots, b_n$, where there are $2^t m_{ij}$ edges connecting a_i and b_j, for $i, j = 1, \ldots, n$. This means that for $t \geq u$, G_t arises from G_u by replacing each edge by 2^{t-u} parallel edges. Hence the number μ of perfect matchings in G_u is equal to

$$\mu = 2^{un} \cdot \text{per } M. \tag{15}$$

Moreover, the number γ_t of 2^t-edge-colourings of G_t satisfies

$$\gamma_t \leq \mu^{2^t} \cdot (2^{t-u})!^{2^u n}, \tag{16}$$

since each colouring is determined by 2^t perfect matchings in G_u, together with an ordering of the 2^{t-u} "copies" in G_t of each of the edges of G_u. But by Corollary 2c γ_t also satisfies

$$\gamma_t \geq (2^t)!^{2n}/2^{t2^t}n. \tag{17}$$

Combining (15), (16) and (17) we obtain a lower bound for per M, which tends to e^{-n} if $t \to \infty$ by Stirling's asymptotical formula. □

Corollary 2f. *If G is a k-regular bipartite graph with $2n$ vertices, then G has at least $(k/e)^n$ perfect matchings, i.e., $p(k,n) \geq (k/e)^n$.*

Proof. Directly from Corollary 2e. □

Corollary 2g. $f(k) \geq k!/e^k$.

Proof. Directly from (12) and Corollary 2f. □

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