On Total Dual Integrality

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ABSTRACT

We prove that each (rational) polyhedron of full dimension is determined by a unique minimal total dual integral system of linear inequalities, with integral left hand sides (thus extending a result of Giles and Pulleyblank), and we give a characterization of total dual integrality.

1. INTRODUCTION

Let $Ax \leqslant b$ be a system of linear inequalities (in this paper, all spaces, matrices and vectors are assumed to be rational; when using expressions like $Ax \leqslant b$ we implicitly assume compatibility of sizes of matrices and vectors; we refer to Grünbaum [4], Rockafellar [7], Stoer and Witzgall [9] for the theory of polyhedra, cones, and linear programming). The system is called *total dual integral* or *t.d.i.* if the right hand side of the linear programming duality equation

$$\max\{wx \mid Ax \leqslant b\} = \min\{yb \mid y \geqslant 0,\ yA = w\} \tag{1}$$

is achieved by an integral vector y, for each integral vector w for which the minimum exists. Hoffman [6] and Edmonds and Giles [2] showed that if $Ax \leqslant b$ is total dual integral and if b is integral, then the polyhedron

$$P = \{x \mid Ax \leqslant b\} \tag{2}$$

is *integral*, i.e., each face of P contains integral vectors, that is, also the left hand side of (1) is achieved by integral vectors.

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Giles and Pulleyblank [3] showed that any integral polyhedron is determined by a t.d.i. system $Ax \leqslant b$ with b integral. In this paper we show that any polyhedron of full dimension is determined by a *unique* minimal t.d.i. system $Ax \leqslant b$ with integral A (minimal with respect to inclusion of sets of linear inequalities). The polyhedron is integral if and only if the right hand side b of this minimal system is integral. Note that a polyhedron is full-dimensional if and only if it is determined by a unique minimal system of linear inequalities, apart from the multiplication of linear inequalities by positive scalars.

We also give the following characterization of total dual integrality. A system $Ax \leqslant b$ is total dual integral if and only if: (1) for each integral vector w, if $y \geqslant 0$, $yA = w$ has a solution y, then it has a solution y each component of which is dyadic (i.e., has denominators a power of 2), and (2) for each nonnegative half-integral y such that yA is integral, there exists a nonnegative integral y' such that $y'A = yA$ and $y'b \leqslant yb$.

2. UNIQUE TOTAL DUAL INTEGRAL SYSTEMS

We show that each polyhedron of full dimension is defined by a unique minimal t.d.i. system of linear inequalities with integral left hand sides. Giles and Pulleyblank [3] cite and prove the following theorem of Hilbert [5]: if K is a polyhedral cone (i.e., if $K = \{x \mid Ax \leqslant 0\}$ for some matrix A), then there exists a finite set Z of integral vectors in K such that every integral vector in K is a nonnegative integral linear combination of vectors in Z. We shall call such a set Z of vectors a *basis* for K. We first show that if K is *pointed* (i.e., if for no nonzero vector x in K is $-x$ also in K), then K has a unique minimal basis.

PROPOSITION 1. *A pointed polyhedral cone has a unique minimal basis.*

Proof. Let K be a pointed polyhedral cone, and let Z be the set of all integral vectors in K which are not a nonnegative integral linear combination of other integral vectors in K. Clearly, each basis of K must contain Z as a subset. On the other hand, Z itself is a basis. To see this, let w be a vector such that $wx > 0$ for each nonzero x in K (such a vector w exists, since K is pointed). Suppose there is an integral vector x in K which is not a nonnegative integral linear combination of vectors in Z. We take x such that wx is as small as possible. Since x is not in Z, there are integers $\lambda_1, \dots, \lambda_k > 0$ and nonzero integral vectors $x_1, \dots, x_k$ $(\neq x)$ in K such that $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$. It follows that $wx_i < wx$ for $i = 1, \dots, k$; hence $x_1, \dots, x_k$ are nonnegative integral linear combinations of vectors in Z. But this implies that x also is a nonnegative integral linear combination of vectors in Z. ∎

It may be verified easily that if K is not pointed there are several minimal bases.

We use Proposition 1 to prove Theorem 2.

THEOREM 2. *A full-dimensional polyhedron P is determined by a unique minimal total dual integral system $Ax \leqslant b$ of linear inequalities with A integral.*

Proof. Let $F_1, \ldots, F_t$ be the minimal (nonempty) faces of P (minimal with respect to inclusion). Let K_i denote the set of vectors w such that

$$\max\{wx \mid x \in P\} \tag{3}$$

is attained by one, and hence any, vector x in F_i. It is easy and well known that each K_i is a polyhedral cone. Since P has full dimension, each K_i is pointed. Let Z_i be the unique minimal basis for K_i (Proposition 1). Let

$$Z_1 \cup \cdots \cup Z_t = \{a_1, \ldots, a_k\}, \tag{4}$$

and define

$$b_j = \max\{a_j x \mid x \in P\} \tag{5}$$

for $j = 1, \ldots, k$. So if $a_j \in Z_i$ and $x_0 \in F_i$, then $b_j = a_j x_0$. Let A be the matrix with rows $a_1, \ldots, a_k$, and let b be the column vector with components $b_1, \ldots, b_k$. We show that $Ax \leqslant b$ is the unique minimal t.d.i. system as required.

First of all, $P = \{x \mid Ax \leqslant b\}$, since if $wx \leqslant d$ is a linear inequality necessary to define P, then the face $\{x \in P \mid wx = d\}$ of P contains a minimal face, say F_i. Since w is an extreme ray of K_i, we know that a scalar multiple of w occurs in Z_i. This implies that a scalar multiple of the linear inequality $wx \leqslant d$ occurs among $Ax \leqslant b$. Since clearly P is contained in $\{x \mid Ax \leqslant b\}$, the two polyhedra are equal.

Secondly, the system $Ax \leqslant b$ is t.d.i. For suppose that w is an integral vector such that the linear programming duality equation (1) is solvable. Then w belongs to some K_i, and w is a nonnegative integral linear combination of vectors in Z_i. So there exists an integral vector $y_0 \geqslant 0$ such that $y_0 A = w$ and y_0 has zero components in positions corresponding to vectors a_j

not occurring in Z_i. Let x_0 be an arbitrary vector in F_i. Then

$$\min\{yb \mid y \geqslant 0,\ yA = w\} = \max\{wx \mid Ax \leqslant b\} = wx_0 = (y_0A)x_0 = y_0 b. \quad (6)$$

The first equality follows from linear programming duality, the second one from the fact that $w \in K_i$ and $x_0 \in F_i$, the third one from $w = y_0 A$. The last equality is implied by the fact that if $(y_0)_j \neq 0$, then $a_j \in Z_i$ and hence $a_j x_0 = b_j$. So (6) shows that the minimum is attained by an integral vector y, which proves total dual integrality.

Finally we show that each inequality in $Ax \leqslant b$ must occur in any t.d.i. system defining P with integral left hand sides. For suppose the linear inequality $a_j x \leqslant b_j$ does not occur in the t.d.i. system $A'x \leqslant b'$, where A' is integral and $P = \{x \mid A'x \leqslant b'\}$. Let $a_j \in Z_i$, and let $x_0 \in F_i$. Now

$$b_j = a_j x_0 = \max\{a_j x \mid x \in P\} = \min\{yb' \mid y \geqslant 0,\ yA' = a_j\} = y_0 b', \quad (7)$$

where y_0 is a nonnegative integral vector with $y_0 A' = a_j$, which exists because the system $A'x \leqslant b'$ is t.d.i.

If a component $(y_0)_h$ of y_0 is nonzero, then the corresponding row vector a'_h of A' belongs to K_i. For if $a'_h \notin K_i$ then, by definition of K_i,

$$a'_h x_0 < \max\{a'_h x \mid x \in P\} \leqslant b'_h. \quad (8)$$

This implies that

$$b_j = a_j x_0 = y_0 A' x_0 < y_0 b', \quad (9)$$

contradicting (7). So a_j is a nonnegative integral linear combination of other integral vectors in K_i, contradicting the fact that a_j is in the minimal basis Z_i. ∎

Since each vector in a minimal basis has relatively prime integers as components, also each row of the matrix A consists of relatively prime integers. Furthermore, P is integral if and only if the vector b is integral (necessity is trivial, while sufficiency follows from Edmonds and Giles [2]). It would be useful therefore to have an algorithmic way to find the unique minimal t.d.i. system defining P. This is useful also for the following. For any polyhedron $P = \{x \mid Ax \leqslant b\}$, where $Ax \leqslant b$ is the minimal t.d.i. system with A integral, let $P' = \{x \mid Ax \leqslant \lfloor b \rfloor\}$, where $\lfloor b \rfloor$ arises from b by taking componentwise lower integer parts (note that $Ax \leqslant \lfloor b \rfloor$ is not necessarily t.d.i. again). In [8] it is shown that the sequence of polyhedra $P, P', P'', P''', \ldots$ arrives, after a finite number of polyhedra, at the polyhedron P_I, the convex hull of the integral vertices in P. In fact this is the essence of the cutting plane method of Gomory for solving integer linear programming problems, as described by Chvátal [1]. So an algorithmic way to find the t.d.i. system gives us an algorithm to find P_I. An important special case of such an algorithm for finding t.d.i. systems is: given a finite set of vectors, find (as efficiently as possible) a basis for the cone generated by these vectors.

If P has not full dimension let H be the affine hull of P. Let L be the linear space parallel to H, i.e., $L=\{x-y \mid x, y \in H\}$. It can be shown following the same lines that there exists a unique minimal t.d.i. system $Ax \leqslant b$ such that A is integral, each row of A is in L, and $P=\{x \in H \mid Ax \leqslant b\}$.

3. A CHARACTERIZATION OF TOTAL DUAL INTEGRALITY

We prove the following characterization.

THEOREM 3. *Let $Ax \leqslant b$ be a system of linear inequalities with at least one solution. Then $Ax \leqslant b$ is total dual integral if and only if:*

(i) *for each vector $y \geqslant 0$ with yA integral, there exists a vector $y' \geqslant 0$ with dyadic components such that $y'A = yA$;*

(ii) *for each vector $y \geqslant 0$ with yA integral and y half-integral, there exists a vector $y' \geqslant 0$ with integral components such that $y'A = yA$ and $y'b \leqslant yb$.*

Proof. It is easy to see that conditions (i) and (ii) are necessary, since if $Ax \leqslant b$ is t.d.i., then for each vector $y \geqslant 0$ with yA integral there exists an integral vector $y' \geqslant 0$ such that $y'A = yA$ and $y'b \leqslant yb$.

Conversely, suppose $Ax \leqslant b$ satisfies conditions (i) and (ii). We first show that for each integer $k \geqslant 0$

$$\min\{yb \mid y \geqslant 0,\ yA = w,\ 2^k y \text{ is integral}\} \tag{10}$$

is achieved by an integral vector y, for any integral w for which this minimum exists. This is proved by induction on k. For $k=0$ the assertion is trivial. For $k \geqslant 0$ we have that

$$\begin{aligned}
&\min\{yb \mid y \geqslant 0,\ yA = w,\ 2^{k+1} y \text{ integral}\} \\
&\quad = 2^{-k} \min\{yb \mid y \geqslant 0,\ yA = 2^k w,\ 2y \text{ integral}\} \\
&\quad = 2^{-k} \min\{yb \mid y \geqslant 0,\ yA = 2^k w,\ y \text{ integral}\} \\
&\quad = \min\{yb \mid y \geqslant 0,\ yA = w,\ 2^k y \text{ integral}\} \\
&\quad = \min\{yb \mid y \geqslant 0,\ yA = w,\ y \text{ integral}\}.
\end{aligned} \tag{11}$$

The first and the third equality are straightforward, the second equality follows from condition (ii), and the last one follows from the induction hypothesis. This implies that

$$\inf\{yb \mid y \geqslant 0,\ yA = w,\ y \text{ dyadic}\} = \min\{yb \mid y \geqslant 0,\ yA = w,\ y \text{ integral}\} \qquad (12)$$

for each integral vector w for which the object set is nonempty. Now (12) is equal to

$$\min\{yb \mid y \geqslant 0,\ yA = w\}, \qquad (13)$$

since the dyadic vectors form a dense subset of the polytope $Q = \{y \mid y \geqslant 0, yA = w\}$, which can be derived from condition (i) as follows. If Q is nonempty, it contains by (i) at least one dyadic vector, say y_0. Let H be the affine hull of Q. Then the dyadic vectors in H form a dense subset of H, since the dyadic vectors in the linear space $H - y_0$ form a dense subset of this linear space (note that we are working in rational spaces). Since Q is a polyhedron in L having the same dimension as L, it follows that the dyadic vectors in Q are dense in Q. ∎

It is easily seen that we may replace condition (i) by the condition that the system $Ax \leqslant 0$ is t.d.i. Also, if A is integral, we may replace condition (ii) by: for each $\{0,1\}$-vector y such that yA has even integers as components, there exists an integral vector $y' \geqslant 0$ such that $y'A = \frac{1}{2}yA$ and $y'b \leqslant \frac{1}{2}yb$.

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