ECONOMETRIC INSTITUTE

INVARIANTS, CANONICAL FORMS AND MODULI FOR TIME VARYING LINEAR DYNAMICAL SYSTEMS

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REPORT 7725/M

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ABSTRACT.

We consider time variable linear dynamical systems $\dot{x} = Fx + Gu$, $y = Hx$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ where the F,G and H are matrices of the appropriate sizes with time variable coefficients. A state space basis change changes the triple of matrices (F,G,H) into $(SFS^{-1}+\dot{S}S^{-1}, SG, HS^{-1})$. Now assume that the coefficients of F,G,H and S all belong to some field like e.g. the field of rational functions over $\mathbb{R}$ or $\mathbb{C}$ or the field of complex or real meromorphic functions. Then most of the results concerning invariants, canonical forms and moduli of our previous papers "Moduli and canonical forms for linear dynamical systems II, III" go through in these time variable cases. The proper setting for studying these questions appears to be differential algebraic geometry. And in fact the results referred to will be established for equations $\delta x = Fx + Gu$, $y = Hx$, where the F,G,H are matrices with coefficients in some arbitrary (ordinary) differential field with differentiation operator δ.

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October, 27 1977

1. INTRODUCTION.

Consider a linear, time varying, dynamical system

$$x = Fx + Gu, \quad y = Hx \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and where F,G,H are matrices of the appropriate sizes with coefficients which may depend on t. To fix the ideas suppose for example that the coefficients of F,G,H all belong to the field of rational functions over $\mathbb{R}$. Then it makes perfect sense to consider base changes of the type $\hat{x} = Sx$ where S is an n x n matrix also with coefficients in $\mathbb{R}(t)$ with nonzero determinant. Such a base change transforms the equations (1.1) into

$$\hat{x}^{\cdot} = (SFS^{-1}+\dot{S}S^{-1})\hat{x} + SGu, \quad y = HS^{-1}\hat{x} \tag{1.2}$$

and at least in the algebraic sense one can ask about invariants, moduli and canonical forms just as in the case of non time varying systems ([3-6]).

Solutions to equations like (1.1) with $u(t) \in \mathbb{R}(t)$ given, certainly exist as vectors with coefficients in some differential extension field (cf [11], [9] or [12]). They also exist as "functions" albeit as multiple valued functions with poles and branching points if F,G or u(t) have poles, cf e.g [7].

The main purpose of the present note is to point out that the results of [5,6] also go through in a time variable setting like the one discussed just above. In fact more generally these results go through for systems

$$\delta x = Fx + Gu, \quad y = Hx \tag{1.3}$$

where the F,G,H are matrices with coefficients in any differential field k with differentiation operator δ (for a definition cf 2.1 below). Examples of such differential fields are

(a) $k = \mathbb{R}(t)$ or $\mathbb{C}(t)$, $\delta = \frac{d}{dt}$

(b) k = real meromorphic functions or complex meromorphic functions, $\delta = \frac{d}{dt}$

(c) k one of the fields of (a) or (b), $\delta f(t) = f(t) - f(t-1)$

Thus when one specializes the results for abstract differential fields obtained below to one of these cases one obtains results for "real life" dynamical systems with time variable coefficients.

The techniques used to obtain the results below are basically the same as in [5,6]. Most of the (minor) difficulties are caused by the fact that differential algebraic geometry is more difficult and certainly for less developped than ordinary algebraic geometry.

2. PRELIMINARIES CONCERNING DIFFERENTIAL ALGEBRA AND DIFFERENTIAL ALGEBRAIC GEOMETRY.

2.1. Differential rings, fields, ... Let R be a commutative ring with unit element. A derivation on R is an additive operator $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a,b \in R$. A differential ring is a ring R together with a derivation operator δ. A differential field is a differential ring whose underlying ring is a field. Examples of differential fields were mentioned in the introduction.

Let (k,δ) be a differential field. Let $X_1, \ldots, X_n$; $X_1^{(1)}, \ldots, X_n^{(1)}$; $X_1^{(2)}, \ldots, X_n^{(2)}$; ... be indeterminates over k. Consider the ring of polynomials $R = k[X_1,\ldots,X_n;X_1^{(1)},\ldots,X_n^{(1)};\ldots]$. Define $\delta X_j^{(i)} = X_j^{(i+1)}$, $i = 0,1,2,\ldots$; $j = 1, \ldots, n$, where $X_j^{(o)} = X_j$, $j = 1, \ldots, n$. There is precisely one derivation δ on R which extends δ on k and which behaves on the $X_j^{(i)}$ as defined (cf [1], Ch 5, §9, prop. 4). The ring R with this derivation is called the ring of differential polynomials in $X_1, \ldots, X_n$ over k and it is denoted $k\{X_1,\ldots,X_n\}$. Roughly a differential polynomial is therefore a polynomial in the $X_1, \ldots, X_n$ and their derivatives. The quotient field of $k\{X_1,\ldots,X_n\}$ is denoted $k<X_1,\ldots,X_n>$. There is a unique derivation on $k<X_1,\ldots,X_n>$ extending the one on $k\{X_1,\ldots,X_n\}$, viz. the obvious one, $\delta(f/g) = g^{-1}(\delta f) - f(\delta g)g^{-2}$.

A differential ideal I in a differential ring (R,δ) is an ideal I of R such that $\delta I \subset I$. If $A \subset R$ is a subset then $[A] \subset R$ denotes the differential ideal generated by A. I.e. [A] is the ordinary ideal generated by the $\delta^i f$, $f \in A$, $i = 0,1,2,\ldots$.

2.2. Affine differential algebraic varieties. Let k be a differential field. Let K be a universal differential field extension of k. Cf [9], Ch. III, §7 for this notion; roughly K is a large enough field to contain all finitely generated extensions of k and finitely generated separable extensions of these. If char(k) = 0 we can take K to be differentiably algebraically closed. (I.e. such that every differential polynomial over K has a solution in K, cf [12]; if char(k) > 0 there are difficulties concerning the existence of algebraic closures).

Let I be a differential ideal in $K\{X_1,\dots,X_n\}$. We define $V(I) = \{(x_1,\dots,x_n) \in K^n | f(x) = 0 \text{ for all } f \in I\}$. Inversely, given a subset $Y \subset K^n$ we define $I(Y) = \{f \in K\{X_1,\dots,X_n\} | f(y) = 0 \text{ all } y \in Y\}$. The subsets of the form $V(I) \subset K^n$ are said to be differentiably closed. This defines a topology on K^n and K^n with this topology is called affine differential space of dimension n. The closed sets V(I) with the induced topology are the affine differential algebraic varieties. We shall from now on use the abbreviation d.a. for differential algebraic. The affine d.a. variety V is defined over k if it is of the form V = V([A]) where A is a set of elements of $k\{X_1,\dots,X_n\}$ and [A] is the differential ideal in $K\{X_1,\dots,X_n\}$ generayed by A. A differential open subset U of an affine d.a variety V over k is defined over k if $V \setminus U$ is defined over k.

The mappings $I \mapsto V(I)$, $Y \mapsto I(Y)$ set up a bijective correspondence between perfect ideals of $K\{X_1,\dots,X_n\}$ and differential closed sets in K^n. By the Ritt-Raudenbusch basis theorem (cf [9], Ch. 3, §4) every perfect ideal in $K\{X_1,\dots,X_n\}$ is generated (differentially) by finitely many elements if char(K) = 0 or more generally if K is perfect.

2.3. Morphisms between affine d.a. varieties. Let V be an affine d.a. variety and let I = I(V) be its ideal of differential polynomials with are zero on V. We write K{V} for the differential quotient ring $K\{X_1,\dots,X_n\}/I(V)$. (There is a unique derivation δ on K{V} compatible with $K\{X_1,\dots,X_n\} \to K\{V\}$ because I(V) is closed under δ). The ring K{V} may have zero divisors. We write K<V> for its full quotient ring. The elements of K{V} are called the differential polynomial functions on V and the elements of K<V> the differential rational functions on V. Let $f \in K\{V\}$, then f indeed defines a function $V \to K$ as follows. Let $x \in V$, choose a lift $\tilde{f}(X_1,\dots,X_n) \in K\{X_1,\dots,X_n\}$ of f for $K\{X_1,\dots,X_n\} \to K\{V\}$; now define $f(x) = \tilde{f}(x_1,\ldots,x_n)$. This is welldefined. Now let $x \in V$ and $f \in K\langle V\rangle$. We say that f is defined at $x \in V$ if there exist $g,h \in K\{V\}$ such that $f = g/h$ in $K\langle V\rangle$ and $h(x) \neq 0$. If f is defined at x, then $f(x) = g(x)/h(x)$ is welldefined. Let dom(f) be the set of $x \in V$ such that f is defined at x. Then dom(f) is a differential open subset of V and f defines a function dom(f) $\to K$, which in turn determines f uniquely. NB, as in ordinary algebraic geometry, given $f \in K\langle V\rangle$ it may not be possible to find a representation $f = g/h, g,h \in K\{V\}$, such that $h(x) \neq 0$ for all $x \in$ dom(f); as a rule g,h may have to depend on x.

Let V be defined over k. A differential polynomial $f \in K\{V\}$ is defined over k if it is in the image of $k\{X_1,\ldots,X_n\}$ in $K\{V\}$ under $K\{X_1,\ldots,X_n\} \to K\{V\}$. The ring of differential polynomial functions over k is denoted $k\{V\}$. Its full quotient ring is denoted $k\langle V\rangle$ and the elements of $k\langle V\rangle \subset K\langle V\rangle$ are the differential rational functions on V defined over k.

Now let $V_1 \subset K^n$ and $V_2 \subset K^m$ be two affine d.a. varieties. Let $U_i \subset V_i$, $i = 1,2$, be differential open subsets. A <u>morphism</u> $\phi : U_1 \to U_2$ is a map $x \mapsto \phi(x) = (\phi_1(x), \ldots, \phi_m(x))$ such that $\phi(x) \in U_2$ for all $x \in U_1$ and such that the ϕ_i are rational differential functions with $\mathrm{dom}(\phi_i) \supset U_1$ for all $i = 1, \ldots, m$. The morphism ϕ is an <u>isomorphism</u> if there is a morphism $\psi : U_2 \to U_1$ such that $\psi\phi = \mathrm{id}$, $\phi\psi = \mathrm{id}$. Let V_1,V_2,U_1,U_2 be defined over k, then ϕ is said to be defined over k if all the ϕ_j are defined over k.

<u>Warning</u>: Every element of $K\{V\}$ defines a morphism $V \to K$. But in general the set of morphisms $V \to K$ is larger than $K\{V\}$ (in contrast to the situation in ordinary algebraic geometry). E.g., if $V = V([\delta x - x]) \subset K$, then $f = (x-1)^{-1}$ defines a morphism $V \to K$ but this morphism is not equal to any of the morphisms defined by the elements of $K\{V\}$.

More material concerning affine d.a. varieties can be found in [2] and [9].

2.4. <u>d.a. varieties</u>. A <u>d.a. variety</u> V is a T_1-topological space V for which there exists an open covering $\{U_i \mid i \in J\}$ together with embeddings $\phi_i : U_i \to K^{n(i)}$ such that

(i) $\phi_i(U_i) \subset K^{n(i)}$ is an affine d.a. variety in $K^{n(i)}$

(ii) $\phi_i\phi_j^{-1}: \phi_j(U_i \cap U_j) \to U_i \cap U_j \to \phi_i(U_i \cap U_j)$ is an isomorphism of open differential subsets (in the sense of 2.3 above).

A d.a. variety is defined over k if all the affine d.a. varieties, differential open subsets and morphisms involved in its definition are defined over k.

For example let V be an affine d.a. variety and U an open differential subset of V. Then U is a union of open differential subsets $V_f = \{x \in V | f(x) \neq 0\}$ where f runs through the elements of K{V} which are zero on $V \setminus U$. Let $I = I(V) \subset K\{X_1,\ldots,X_n\}$ and let $V_f' = \{(x_1,\ldots,x_{n+1}) \in K^{n+1} | g(x_1,\ldots,x_n) = 0$ for all $g \in I$ and $f(x_1,\ldots,x_n)x_{n+1}=1\}$. Then V_f' is an affine d.a. variety and $(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_n,f(x_1,\ldots,x_n))$ is an isomorphism $V_f \simeq V_f'$. Thus we see that a differential open subset of an affine d.a. variety is a d.a. variety in the sense of the definition above. More generally an open subset of a d.a. variety is a d.a. variety. A d.a. variety V has a basis of open affine d.a. subvarieties (by the argument given above).

Let V,W be d.a. varieties. A morphism $\phi: V \to W$ is a continuous map such that for every two affine d.a. subvarieties $U \subset V$, $U' \subset W$, $\phi : U \cap \phi^{-1}(U') \to U'$ is a morphism of differential open subsets of affine d.a. varieties in the sense of 2.3 above.

2.5. d.a. groups and differential invariants. The category of d.a. varieties defined just above has finite products and a final object (the one point d.a. variety). A d.a. group is now a group object in this category. I.e. it is a d.a. variety G equipped with a multiplication morphism $m: G \times G \to G$, an inverses morphism $i: G \to G$ and an element $e \in G$ such that m,i and e make G a group in the usual sense of the word. Some results on affine d.a. groups can be found in [2].

An action of a d.a. group G on a d.a. variety V is a morphism $G \times V \to V$ such that (with the obvious notations) $(g_1g_2)x = g_1(g_2x)$, $ex = x$ for all $g_1,g_2 \in G$, $x \in V$. A differential invariant of an action of G on V is a differential rational function f on V such that $f(gx) = f(x)$ for all $x \in V$, $g \in G$ such that f is defined for both x and gx. This definition agrees of course with the one of S.Lie in [10], modulo the change caused by the algebraic geometric setting of the present note.

2.6. d.a. vectorbundles. An n-dimensional d.a. vectorbundle over a d.a. variety V is a morphism of d.a. varieties $\pi : E \to V$ such that there exists an open covering $\{U_i | i \in J\}$ of V by affine open d.a. subvarieties and isomorphisms $\phi_i : \pi^{-1}(U_i) \overset{\sim}{\to} U_i \times K^n$ such that

(i) $pr_1 \circ \phi_i = \pi$ for all i (where $pr_1 : U_i \times K^n \to U_i$ is the canonical projection into the first factor

(ii) $\phi_i \phi_j^{-1} : (U_i \cap U_j) \times K^n \to \pi^{-1}(U_i \cap U_j) \to (U_i \cap U_j) \times K^n$ is a morphism of the form $(x,v) \mapsto (x, \psi_{ij}(x)v)$, where ψ_{ij} is a d.a. morphism $U_i \cap U_j \to GL_n$ into the d.a. group of invertible n x n matrices.

The d.a. vectorbundle $\pi : E \to V$ is defined over k if all d.a. varieties and morphisms involved in its definition are defined over k.

2.7. Rational points of a d.a. variety. Let $V \subset K^n$ be an affine d.a. variety defined over k, then V(k), the set of k-rational points of V, is defined as $V(k) = \{(x_1, \ldots, x_n) \in V \subset K^n | x_i \in k \text{ all } i\}$. For an arbitrary d.a. variety V over k, V(k) is defined as $V(k) = \cup U_i(k)$, where $\{U_i, i \in J\}$ is an open covering of V by affine d.a. subvarieties defined over k.

3. The d.a. quotient variety $M^{ar}_{m,n,p} = L^{ar}_{m,n,p}/GL_n$. Invariants.

3.1. The setting. Let k_o be any differential field with universal extension K. For example k_o may be the field of rational or meromorphic functions over $\mathbb{R}$ or $\mathbb{C}$, with $\delta = \frac{d}{dt}$ or $\delta f(t) = f(t) - f(t-1)$. We consider equations

$$\delta x = Fx + Gu, \quad y = Hx \tag{3.1.1}$$

with $x(t) \in k^n$, $u(t) \in k^m$, $y(t) \in k^p$ and F,G,H matrices of the appropriate sizes with coefficients in k, where k is any intermediate differential field between k_o and K. As a rule we shall write $\dot{x}$ instead of δx.

Let $L_{m,n,p}$ be the d.a. variety of all triples of matrices (F,G,H) of sizes nxn, nxm, pxn respectively. Let GL_n be the d.a. group of all nxn invertible matrices. We define a d.a. action of GL_n on $L_{m,n,p}$ by

(3.1.2) $$GL_n \times L_{m,n,p} \to L_{m,n,p},$$
$$(S,(F,G,H)) \mapsto (F,G,H)^S = (SFS^{-1}+\dot{S}S^{-1}, SG, HS^{-1})$$

(Note that this is indeed a GL_n-action in that $(F,G,H)^I = (F,G,H)$ and $((F,G,H)^S)^T = (SFS^{-1}+\dot{S}S^{-1}, SG, HS^{-1})^T =$ $(TSFS^{-1}T^{-1}+T\dot{S}S^{-1}T^{-1}+\dot{T}T^{-1}, TSG, HS^{-1}T^{-1}) = (F,G,H)^{TS}$ because $(TS)^{\bullet}(TS)^{-1} = (\dot{T}S+T\dot{S})S^{-1}T^{-1} = \dot{T}T^{-1}+T\dot{S}S^{-1}T^{-1}$). Of course this action of GL_n on $L_{m,n,p}$ corresponds to the transformation $x \mapsto Sx$ in state space in (3.1.1).

3.2. Algebraically reachable and algebraically observable systems.

Let $(F,G,H) \in L_{m,n,p}$.

We define the $n \times (n+1)m$ matrix $R(F,G)$ by

(3.2.1) $$R(F,G) = (G(0) \; G(1) \; \dots \; G(n))$$

where $G(i)$ is inductively defined by

(3.2.2) $$G(0) = G,\; G(i) = FG(i-1) - \dot{G}(i-1),\; i = 1,2, \dots, n$$

More or less dually the matrix $Q(F,H)$ is defined as

(3.2.3) $$Q(F,H)^{\tau} = (H(0)^{\tau} \; H(1)^{\tau} \; \dots \; H(n)^{\tau})$$

with

(3.2.4) $$H(0) = H,\; H(i) = H(i-1)F + \dot{H}(i-1),\; i = 1,2, \dots, n$$

where the symbol τ denotes "transposes". (Note the sign difference).

The triple (F,G,H) is said to be algebraically reachable (abbreviated "ar") if rank $(R(F,G)) = n$; the triple (F,G,H) is said to be algebraically observable (abbreviated "ao") if $\text{rank}(Q(F,H)) = n$. These two conditions define open d.a. subvarieties of $L_{m,n,p}$ which we denote $L^{ar}_{m,n,p}$, $L^{ao}_{m,n,p}$. In addition we define $L^{ar,ao}_{m,n,p} = L^{ar}_{m,n,p} \cap L^{ao}_{m,n,p}$.

Of course the notions "algebraically reachable" and "algebraically observable" as defined above correspond to the usual geometric notions of reachability and observability in the cases where k is a field of rational or meromorphic function over $\mathbb{R}$ or $\mathbb{C}$. Indeed the system (F,G,H) is ao iff Q(F,H) has rank n. Because of the nature of the functions involved this happens iff Q(F(t),H(t)) has rank n pointwise in t for all t except possibly a set of measure zero and this in turn means that (F,G,H) is completely observable in the usual geometric sense (cf [14], corollary 8.8). Dually one has that algebraically reachable corresponds to completely reachable in the geometric sense for such differentiable fields (NB in [14] "determinable" is used for "observable").

3.3. Nice selections. Let $J_{n,m} = \{(0,1), \ldots, (0,m); (1,1), \ldots, (1,m); \ldots ; (n,1), \ldots, (n,m)\}$, lexicographically ordered. We use $J_{n,m}$ to label the columns of the matrices R(F,G) by assigning the label (i,j) to the j-th column of G(i). A subset $\alpha \subset J_{n,m}$ is nice if $(i,j) \in \alpha \Rightarrow (i-1,j) \in \alpha$ for all i,j. A nice subset of size n is called a nice selection. Given a nice selection α a successor index of α is an element $(i,j) \in J_{n,m} \setminus \alpha$ such that $\alpha \cup \{(i,j)\}$ is nice. For every $j \in \{1,\ldots,m\}$ and nice selection α there is precisely one successor index (i,j') of α such that $j' = j$. This successor index will be denoted $s(\alpha,j)$.

3.4. Nice selection lemma. Let $(F,G,H) \in L^{ar}_{m,n,p}$. Then there is a nice selection $\alpha \subset J_{n,m}$ such that $\det(R(F,G)_\alpha) \neq 0$. (Here $R(F,G)_\alpha$ is the square n x n matrix obtained from R(F,G) by removing all columns whose index is not in α).

Proof. Let β be a nice subset of $J_{n,m}$, which is maximal with respect to the property that all the columns of $R(F,G)_\beta$ are linearly independent. We shall show that β then has n elements which proves the lemma.

Renumbering the columns of G if necessary we can assume that β is of the form

$$\beta = \{(0,1), \ldots, (n_1,1)\} \cup \{(0,2), \ldots, (n_2,2)\} \cup \ldots$$
$$\cup \{(0,s), \ldots, (n_s,s)\}$$

We shall now show that every column of R(F,G) can be written as a linear combination of the columns of $R(F,G)_\beta$. By the maximality of β this holds for the columns with indices $(n_1+1,1), \ldots, (n_s+1,s)$, $(0,s+1), \ldots, (0,m)$. Assume with induction that the statement has been proved for all columns with indices (i,j) with $i \leq n_j+k$ where we take $n_j = 0$ for $j = s+1, \ldots, m$. Let $(i,j) \in J_{n,m}$ be such that $i = n_j + k + 1$. Now $R(F,G)_{(i,j)}$ is the j-th column of G(i). Hence

$$R(F,G)_{(i,j)} = FR(F,G)_{(i-1,j)} - \dot{R}(F,G)_{(i-1,j)}$$

Now by induction $R(F,G)_{(i-1,j)}$ is a linear combination of the columns of $R(F,G)_\beta$. Say

$$R(F,G)_{(i-1,j)} = \sum_{(u,v)\in\beta} a_{(u,v)} R(F,G)_{(u,v)}$$

Then

$$R(F,G)_{(i,j)} = \sum_{(u,v)\in\beta} a_{(u,v)} R(F,G)_{(u+1,v)} - \sum_{(u,v)\in\beta} \dot{a}_{(u,v)} R(F,G)_{u,v}$$

As we have seen that the $R(F,G)_{(u+1,v)}$ for $(u,v) \in \beta$ are linear combinations of the columns of $R(F,G)_\beta$ it follows that also $R(F,G)_{(i,j)}$ is a linear combination of the columns of $R(F,G)_\beta$. This finishes the induction. Hence $\text{rank}(R(F,G)_\beta) = \text{rank}(R(F,G)) = n$, which proves that β has n elements.

3.5. <u>The partial quotients</u> U_α/GL_n. We now proceed as in [5,6]. First note that

$$(3.5.1) \qquad R(SFS^{-1}+\dot{S}S^{-1},G) = SR(F,G)$$

(because $(SFS^{-1}+\dot{S}S^{-1})(SG(i)) - (SG(i))\dot{} = SFG(i) + \dot{S}G(i) - \dot{S}G(i) - S\dot{G}(i) = S(FG(i)-\dot{G}(i))$). Let α be a nice selection and let $x = (x_1,\ldots,x_m) \in K^{nm} = K^n \times \ldots \times K^n$. Using (3.5.1) one now shows as in [5,6] that there exists precisely one triple $(F,G,H) \in L^{ar}_{m,n,p}$ such that $R(F,G)_\alpha = I_n$, $R(F,G)_{s(\alpha,j)} = x_j$ for $j = 1, \ldots, m$. It follows that if $U_\alpha = \{(F,G,H) \in L_{m,n,p} \mid \det(R(F,G)_\alpha) \neq 0\}$ then

$$(3.5.2) \qquad U_\alpha \simeq GL_n \times K^{mn+np}, \quad U_\alpha/GL_n \simeq K^{mn+np}$$

For each nice selection α and $x = (y,z) \in K^{mn+np}$ let $\psi_\alpha(x) = (F_\alpha(x), G_\alpha(x), H_\alpha(x))$ be the unique triple such that $R(F_\alpha(x),G_\alpha(x))_\alpha = I_n$, $R(F_\alpha(x),G_\alpha(x))_{s(\alpha,j)}$ is the j-th component of $y = (y_1,\ldots,y_m) \in (K^n)^m$, and such that $H_\alpha(x) = z$.

3.6. The d.a. quotient variety $M^{ar}_{m,n,p}$. We now construct a d.a. variety $M^{ar}_{m,n,p}$ as follows; again as in [6]. For each nice selection α let $V_\alpha = K^{nm} \times K^{np}$ and let

$$V_{\alpha\beta} = \{x \in V_\alpha \mid \det(R(F_\alpha(x),G_\alpha(x))_\beta) \neq 0\}$$

We now glue the V_α together by means of the isomorphisms $\psi_{\alpha\beta}: V_{\alpha\beta} \to V_{\beta\alpha}$, which are defined by

(3.6.1) $$\psi_{\alpha\beta}(x) = y \iff (F_\beta(y),G_\beta(y),H_\beta(y)) = (F_\alpha(x),G_\alpha(x),H_\alpha(x))^S$$

where $S = R(F_\alpha(x),G_\alpha(x))_\beta^{-1}$. This defines us a d.a. variety provided we can show that $M^{ar}_{m,n,p}$ is T_1. Note that by construction $M^{ar}_{m,n,p} = L^{ar}_{m,n,p}/GL_n$, in any case as sets.

Now let $G_{n,(n+1)m}$ be the d.a. Grassmann variety of n-planes in $(n+1)m$ space. Then by (3.5.1), R induces a map

(3.6.2) $$g: M^{ar}_{m,n,p} \to G_{n,(n+1)m}$$

One now also defines $\tilde{h}: L^{ar}_{m,n,p} \to K^{(n+1)^2mp}$ by $\tilde{h}(F,G,H) = Q(F,H)R(F,G)$. Now

(3.6.3) $$Q(SFS^{-1}+\dot{S}S^{-1},HS^{-1}) = Q(F,H)S^{-1}$$

(because $(H(i)S^{-1})(SFS^{-1}+\dot{S}S^{-1}) + (H(i)S^{-1})^{\cdot} = H(i)FS^{-1} + H(i)S^{-1}\dot{S}S^{-1} + \dot{H}(i)S^{-1} - H(i)S^{-1}\dot{S}S^{-1} = H(i+1)S^{-1}$). Combining this with (3.5.1) we see that $\tilde{h}((F,G,H)^S) = \tilde{h}(F,G,H)$, so that $\tilde{h}$ induces a map

(3.6.4) $$h: M^{ar}_{m,n,p} \to K^{(n+1)^2mp}$$

One now shows as in [6] that $(g,h): M^{ar}_{m,n,p} \to G_{n,(n+1)m} \times K^{(n+1)^2mp}$ is injective which proves that $M^{ar}_{m,n,p}$ is T_1 and hence a d.a. variety. The maps g and h are d.a. morphisms (defined over k_o).

3.7. Corollary. $M^{ar}_{m,n,p}$ is an irreducible quasi projective d.a. variety. It is the quotient of $L^{ar}_{m,n,p}$ by GL_n in the category of d.a. varieties.

One also verifies with no trouble that $M^{ar}_{m,n,p}$ in addition enjoys the pleasant quotient property that $M^{ar}_{m,n,p}(k) = L^{ar}_{m,n,p}(k)/GL_n(k)$ for all intermediate differential fields $k_o \subset k \subset K$.

3.8. <u>The subvariety</u> $M^{ar,ao}_{m,n,p}$. Let $\pi : L^{ar}_{m,n,p} \to M^{ar}_{m,n,p}$ be the natural projection. Then $M^{ar,ao}_{m,n,p}$, the image of $L^{ar,ao}_{m,n,p}$, is an open d.a. subvariety of $M^{ar}_{m,n,p}$ and one shows as in [6] that the morphism h of (3.6.4) above is injective on $M^{ar,ao}_{m,n,p}$. Its image is readily described. An (n+1) x (n+1) block matrix with blocks of size p x m

$$\mathcal{A} = \begin{pmatrix} A_{o,o} & A_{o,1} & & A_{o,n} \\ A_{1,o} & & & \\ \cdot & & & \\ & & & \\ A_{n,o} & & \cdots & A_{n,n} \end{pmatrix}$$

is of the form h(F,G,H) for some triple $(F,G,H) \in L^{ar,ao}_{m,n,p}$ if and only if the following two conditions (3.8.1) - (3.8.2) hold.

(3.8.1) $\qquad \mathrm{rank}(\mathcal{A}) = n = \mathrm{rank}(\mathcal{A}')$.

where $\mathcal{A}'$ is the matrix obtained from $\mathcal{A}$ by removing the last column and row of blocks.

(3.8.2) $\qquad A_{i+1,j} - A_{i,j+1} = \dot{A}_{i,j}$ for all $i,j \in \{0,1, \ldots, n-1\}$

3.9. <u>Corollary</u>. $M^{ar,ao}_{m,n,p}$ is a quasi-affine d.a. variety.

3.10. <u>Corollary</u>. Every differential invariant of GL_n acting on $L_{m,n,p}$ is a rational function in the entries of the matrix $h(F,G,H) = Q(F,H)R(F,G)$ and their derivatives.

3.11. <u>Remarks</u>. Note that $L_{m,n,p}$, $M^{ar}_{m,n,p}$, $M^{ar,ao}_{m,n,p}$ are defined by polynomials involving no derivatives, and hence are ordinary algebraic varieties reinterpreted within the context of d.a. varieties.

On the other hand the definitions of $L^{ar}_{m,n,p}$, $L^{ar,ao}_{m,n,p}$ do involve derivatives and so do the projection map $\pi : L^{ar}_{m,n,p} \to M^{ar}_{m,n,p}$, the embedding $h: M^{ar,ao}_{m,n,p} \to K^{(n+1)^2 mp}$

and hence the description of $M^{ar,ao}_{m,n,p}$ as a quasi affine d.a. subvariety of $K^{(n+1)^2 mp}$.

All the d.a. varieties and morphisms of (3.1) - (3.10) above are defined over k_o.

4. CANONICAL FORMS

We can be brief about the matter of existence or nonexistence of global continuous canonical forms. On the one hand there exist of course the local canonical forms $c_{\mathcal{K}\alpha}: U_\alpha \to U_\alpha$ for every nice selection α defined by

$$(4.1) \qquad c_{\mathcal{K}\alpha}(F,G,H) = (F,G,H)^S, \quad S = R(F,G)_\alpha^{-1}$$

On the other hand the same examples and constructions as used in [5,6] show that global continuous canonical forms on $L^{ar,ao}_{m,n,p}$ exist if and only if $m = 1$ or $p = 1$. This is not immediate from the corresponding result in the non-time-varying case, because, a priori, the canonical form of a non-time-varying linear system could be time-varying in the present setting.

There are similar analogues of all the other results of [5,6] pertaining to canonical forms. E.g., there is a continuous canonical form on $L^{ar}_{m,n,p}$ (resp. $L^{ao}_{m,n,p}$) if and only if $m = 1$ (resp. $p = 1$).

Let us also note that $L^{ar}_{m,n,p} \to M^{ar}_{m,n,p}$ is a locally trivial principal d.a. GL_n fibre bundle over $M^{ar}_{m,n,p}$, in complete analogy with the situation in the non-time-varying case.

5. A UNIVERSAL FAMILY OF LINEAR TIME-VARYING SYSTEMS.

As in the non-time-varying case there is a natural universal family of linear dynamical systems. Here, however, the definitions of [5,6] must be recast, simply because the transformation rule $F \mapsto SFS^{-1} + \dot{S}S^{-1}$ does not correspond to the kind of transformations one encounters for an endomorphism of a vectorbundle in terms of varying local trivializations of that vector bundle.

5.1. Definition. A family Σ of linear dynamical systems parametrized by a d.a. variety V consists of

(i) a d.a. vectorbundle $\pi : E \to V$

(ii) for every open $U \subset V$ over which E is trivial and every isomorphism of d.a. vectorbundles $\psi : \pi^{-1}(E) \overset{\sim}{\to} U \times K^n$ (trivialization) a morphism $f(\psi,U): U \to L_{m,n,p}$, such that the following condition holds

(iii) let $\psi_1\psi_2^{-1}: (U_1 \cap U_2) \times K^n \overset{\sim}{\to} \pi^{-1}(U_1 \cap U_2) \to (U_1 \cap U_2) \times K^n$ be given by $(x,v) \mapsto (x,\chi_{12}(x)v)$ for $\chi_{12}: U_1 \cap U_2 \to GL_n$, then

$$f(\psi_2,U_2)(x)^{\chi_{12}(x)} = f(\psi_1,U_1)(x) \quad \text{for all } x \in U_1 \cap U_2$$

It obviously suffices to specify the $f(\psi,U)$ for all U_i of some open covering $\{U_i | i \in \mathcal{J}\}$ of V and for one particular trivialization ψ_i for each U_i. The family is said to be <u>defined over</u> k_o if all the morphisms and varieties involved are defined over k_o. The family Σ is said to be ar if all the $f(\psi,U)$ map U into $L^{ar}_{m,n,p} \subset L_{m,n,p}$.

In case k_o is a differential field of rational or meromorphic functions over $\mathbb{R}$ or $\mathbb{C}$ one can more generally define a rather hybrid sort of object: families of rational or meromophic dynamical systems parametrized by a topological space V. The definition is obvious.

5.2. <u>The universal example</u>. As in [5,6] we now construct a canonical n-vectorbundle over $M^{ar}_{m,n,p}$. It consists of the trivial pieces $V_\alpha \times K^n$ for each nice selection α glued together by the identifications

$$V_\alpha \times K^n \ni (x,v) \longleftrightarrow (y,w) \in V_\beta \times K^n$$

iff "$x = y$ in $M^{ar}_{m,n,p}$", i.e. $(F_\alpha(x),G_\alpha(x),H_\alpha(x))^S = (F_\beta(y),G_\beta(y),H_\beta(y)$ with $S = R(F_\alpha(x),G_\alpha(x)_\beta^{-1}$, and $Sv = w$. The morphisms

$$\psi_\alpha : V_\alpha \to L^{ar}_{m,n,p}, \quad x \mapsto (F_\alpha(x),G_\alpha(x),H_\alpha(x))$$

now define the required family of linear dynamical systems in the sense of 5.1 above. We denote the family just constructed by Σ^u.

5.3. <u>Universality properties of Σ^u</u>. There is an obvious notion of pull back, i.e. an obvious way of associating a family $\phi^!\Sigma$ over V' to a d.a. morphism $\phi : V' \to V$ and a family Σ over V, cf [5,6]. There is an equally obvious notion of isomorphism of families over V. This defines a contravariant functor $\mathcal{F}$: d.a. varieties → Set. As expected this functor is representable by $(M^{ar}_{m,n,p}, \Sigma^u)$. I.e., for every ar family Σ of linear dynamical systems over a d.a. variety V there is a unique d.a. morphism $\phi : V \to M^{ar}_{m,n,p}$ such that $\phi^! \Sigma^u \simeq \Sigma$ over V.

In case k_o a field of rational or meromorphic functions over $\mathbb{R}$ or $\mathbb{C}$ the family Σ^u over $M^{ar}_{m,n,p}(k_o)$ is also universal for the hybrid families briefly mentioned at the end of 5.1 above (provided one gives $M^{ar}_{m,n,p}(k_o)$ the appropriate topology of a function space of $\mathbb{R}$- or $\mathbb{C}$-valued functions). The proofs of all these facts do not really differ from those given in [5,6] for the non-time-varying case.

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