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DEGENERATING FAMILIES OF LINEAR DYNAMICAL SYSTEMS I

AND

ON INVARIANTS AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS

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DEGENERATING FAMILIES OF LINEAR DYNAMICAL SYSTEMS I

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Abstract

This paper addresses itself to the question whether $M^{cr,co}_{m,n,p}(\mathbb{R})$, the space of equivalence classes of completely reachable and observable linear dynamical systems under state space equivalence, can be compactified in a system theoretically meaningful way by adding e.g. lower dimensional systems. We obtain a partial compactification $\bar{M}_{m,n,p}(\mathbb{R})$ by adding lower dimensional systems, differential operators and mixtures of these two. This partial compactification is wellbehaved with respect to the limiting input-output behaviour of (degenerating) families of linear dynamical systems. The compactification is also maximal in the sense that if the input-output behaviours of a family of systems (F_z,G_z,H_z) have a (noninfinite) limit than that limit is the input-output behaviour of one of the points of $\bar{M}_{m,n,p}(\mathbb{R})$.

1. Introduction.

Let $\dot{x} = Fx + Gu$, $y = Hx$ be a (constant) linear dynamical system of state space dimension n with m inputs and p outputs. Let $L_{m,n,p}(\mathbb{R})$ be the (affine) space ($L_{m,n,p}(\mathbb{R}) \simeq \mathbb{R}^{n^2+mn+np}$) of all such systems and let $L^{cr}_{m,n,p}(\mathbb{R})$, resp. $L^{co}_{m,n,p}(\mathbb{R})$, resp. $L^{co,cr}_{m,n,p}(\mathbb{R})$ be the open and dense subspaces of $L_{m,n,p}(\mathbb{R})$ consisting of the completely reachable, resp. completely observable, resp. completely observable and completely reachable systems. Base change in state space induces an action of $GL_n(\mathbb{R})$, the group of nxn real invertible matrices on $L_{m,n,p}(\mathbb{R})$, viz.: $(F,G,H)^S = (SFS^{-1},SG,HS^{-1})$, $S \in GL_n(\mathbb{R})$, and two systems of $L_{m,n,p}(\mathbb{R})$ which are related in this way by means of some $S \in GL_n(\mathbb{R})$ (we shall call them $GL_n(\mathbb{R})$-equivalent in that case) are indistinguishable from the point of view of their input-output behaviour. Inversely if (F,G,H) and $(\bar{F},\bar{G},\bar{H})$ are two systems of $L_{m,n,p}(\mathbb{R})$ with the same input-output behaviour and if, moreover, at least one of them is completely reachable (cr) and completely observable (co) then (F,G,H) and $(\bar{F},\bar{G},\bar{H})$ are $GL_n(\mathbb{R})$-equivalent. This makes the space $M^{co,cr}_{m,n,p}(\mathbb{R}) = L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$ of $GL_n(\mathbb{R})$ orbits in $L^{cr,co}_{m,n,p}(\mathbb{R})$ important in identification of systems theory, essentially because the input-output data of a given black-box give zero information concerning a basis for state space. More precisely suppose we have given a black-box which is to be modelled by means of a linear dynamical system (lds). Then the input-output data give us a point of $M^{co,cr}_{m,n,p}(\mathbb{R})$ and, as more and more input-output data come in, (ideally) a sequence of points of $M^{co,cr}_{m,n,p}(\mathbb{R})$ representing better and better lds approximations to the given black box. The same sort of thing happens when one is dealing with a slowly varying black box (or lds). If this sequence approaches a limit we have "identified" the black box. (In practice, of course, one also wants a concrete representation in terms of a triple of matrices; this is were the matter of continuous canonical forms comes in). Unfortunately the space $M^{co,cr}_{m,n,p}(\mathbb{R})$ is never compact, so that a sequence of points may fail to converge to anything whatever. There are "holes" in $M^{co,cr}_{m,n,p}(\mathbb{R})$.

This paper addresses itself to the question of whether $M^{co,cr}_{m,n,p}(\mathbb{R})$ can be compactified in a system theoretically meaningful way.

To illustrate what kinds of holes there are in $M^{co,cr}_{m,n,p}(\mathbb{R})$ we offer the following three 2-dimensional, one input-one output examples.

1.1. Example.

$$g_z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad h_z = (z,o)$$

The result of starting in $x_o = 0$ at time $t = 0$ with input function u(t) is $y(t) = \int_0^t ze^{t-\tau}u(\tau)d\tau$. We see (by taking e.g. $u(t) = 1$, $0 \leq t \leq n$, $u(t) = 0$, $t > n$) that the family of systems $(F_z,g_z,h_z)_z$ does not have any reasonable limiting input-output behaviour as $z \to \infty$, so that this limiting input-output behaviour can hardly model any (physical) black box.

1.2. Example.

$$g_z = \begin{pmatrix} z \\ 1 \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad h_z = (z^{-1}, 0)$$

In this case the result of input u(t), starting in $x_o = 0$ at time $t = 0$, is

$$y(t) = \int_0^t h_z e^{(t-\tau)F_z} g_z\, u(\tau)d\tau = \int_0^t e^{t-\tau} u(\tau)d\tau + \int_0^t z^{-1} e^{t-\tau}(t-\tau)u(\tau)d\tau$$

and we see that the limiting input-output behaviour of this family of systems as $z \to \infty$ is the same as the input-output behaviour of the one dimensional system $g = 1$, $F = 1$, $h = 1$. This example also illustrates that it may very well happen that the family of systems (F_z, g_z, h_z) may not converge to anything (not even a subsequence converges), while the associated family of input-output behaviours has a definite (finite) limit. (The same thing happens in example 1.3 below). Of course this kind of thing is only to be expected when one takes quotients for the action of a noncompact group.

1.3. Example.

$$g_z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F_z = \begin{pmatrix} -z & -z \\ 0 & -z \end{pmatrix}, \quad h_z = (z^2, 0)$$

In this case the limit

$$\lim_{z\to\infty} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau = \lim_{z\to\infty} \int_0^t e^{-z(t-\tau)} (z^2 - z^3(t-\tau))u(\tau)d\tau$$

does exist for all reasonable input functions u(t) (E.g. u(t) continuously differentiable suffices). But this limit is not the input-output behaviour of any linear dynamical system. The limit is in fact the linear differential operator $u(t) \mapsto \frac{du(t)}{dt}$.

Thus we see that the holes in $M^{cr,co}_{m,n,p}(\mathbb{R})$ are of very different kinds. There is little one can do about filling in the kind of holes exemplified by example 1.1, nor does this seem to be a serious matter from the point of view of identification theory. The other holes can be filled in and the result is a system theoretically meaningful partial compactification $\bar{M}_{m,n,p}(\mathbb{R})$ which is also maximal in the sense that if a family (F_z, G_z, H_z) has a finite limiting behaviour than that limiting input-output behaviour is the input-output behaviour of a "system" in $\bar{M}_{m,n,p}(\mathbb{R})$. Cf. theorems 3.4 and 3.5 and remark 5.2 for more detailed statements.

2. Differential operators of order $\leq n-1$ as limits of systems in $L^{co,cr}_{1,n,1}(\mathbb{R})$.

2.1. Definition. A differential operator of order $\leq n-1$ is (for the purposes of this paper) an input-output map of the form

$$(2.2) \quad y(t) = Du(t) = a_o u(t) + a_1 \frac{du(t)}{dt} + \dots + a_{n-1} \frac{d^{n-1}u(t)}{dt^{n-1}}$$

where the $a_o, \dots, a_{n-1}$ are real constants. (The functions u(t) are always supposed to be sufficiently differentiable, say of class C^∞).

2.3. Theorem. Let D be a differential operator of order $\leq n-1$. Then there exists a family of linear dynamical systems $(F_z, g_z, h_z)_z \subset L^{cr,co}_{1,n,1}(\mathbb{R})$ such that (F_z, g_z, h_z) converges in input-output behaviour to D as $z \to \infty$. More precisely there exists a family $(F_z, g_z, h_z)_z \subset L^{cr,co}_{1,n,1}(\mathbb{R})$ such that

$$(2.4) \quad \lim_{z\to\infty} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau = Du(t)$$

uniformly in t on every bounded t-interval in $[0,\infty)$.

2.5. To prove theorem 2.3 we need to do some preliminary exercises concerning differentiation, partial integration and determinants. To start with, here is the determinant exercise.

2.6. Lemma. Let $k \in \mathbb{N} \cup \{0\} \cup \{-1\}$, $n \in \mathbb{N}$. Let B(n,k) be the nxn matrix with the binomial coefficient entries $B(n,k)_{i,j} = \binom{i+j+k}{i+k+1}$, $i,j = 1, \dots, n$. Then $\det(B(n,k)) = 1$ for all n,k.

2.7. Lemma.

$$\int_0^t z^n e^{-z(t-\tau)} u(\tau)d\tau = z^{n-1}u(t) - z^{n-2}u'(t) + \dots + (-1)^{n-1}u^{(n-1)}(t) + O(z^{-1})$$

as $z \to \infty$, where $u^{(i)}(t)$ is the i-th derivative of u(t) and $O(z^{-1})$ is the Landau O-symbol.

Proof. Partial integration.

2.8. Lemma. Let $\phi(\tau) = (t-\tau)^m u(\tau)$. Then $\phi^{(n)}(t) = 0$ for $n < m$ and $\phi^{(n)}(t) = (-1)^m n(n-1)\dots(n-m+1)u^{(n-m)}(t)$ if $n \geq m$.

Proof. Induction with respect to m.

Combining lemma 2.7 and lemma 2.8 we find

$$(2.9)\quad \int_0^t e^{-z(t-\tau)} z^n (t-\tau)^m u(\tau)d\tau = (-1)^m m!$$

$$\sum_{i=m+1}^{n} (-1)^{i+1} z^{n-i} \binom{i-1}{m} u^{(i-1-m)}(t) + O(z^{-1})$$

2.10. Proof of theorem 2.3. Let $1 \le m \le n$. Consider the following family of n-dimensional, 1 input, 1 output, systems

$$g_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z^m \end{pmatrix}, \quad F_z = \begin{pmatrix} -z & z & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & z \\ 0 & \cdots & & 0 & -z \end{pmatrix},$$

$$h_z = (0,\dots,0,x_m,\dots,x_1)$$

where $x_1, \dots, x_m$ are still to be determined real numbers. Now

$$sF_z = \begin{pmatrix} -sz & & 0 \\ & \ddots & \\ 0 & & -sz \end{pmatrix} + \begin{pmatrix} 0 & sz & & 0 \\ & \ddots & \ddots & \\ & & \ddots & sz \\ 0 & & & 0 \end{pmatrix}$$

and these two matrices commute. It follows that

$$e^{sF_z} = e^{-sz} \begin{pmatrix} 1 & sz & \frac{s^2z^2}{2!} & \cdots & \frac{s^{n-1}z^{n-1}}{(n-1)!} \\ 0 & 1 & sz & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{s^2z^2}{2!} \\ \vdots & & & \ddots & sz \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

Hence

$$h_z e^{(t-\tau)F_z} g_z = \sum_{i=1}^{m} x_i z^{m+i} (i!)^{-1} (t-\tau)^i e^{-z(t-\tau)}$$

so that

$$\int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau = \sum_{i=1}^{m} (i!)^{-1} x_i \sum_{j=i+1}^{m+i} (-1)^i (i!)$$

$$(-1)^{j+1} \binom{j-1}{i} z^{m+i-j} u^{(j-i-1)}(t) + O(z^{-1})$$

$$= \sum_{\ell=0}^{m-1} (-1)^{m-\ell+1} z^{\ell} \Big(\sum_{i=1}^{m} x_i \binom{m+i-\ell-1}{i} \Big) u^{(m-\ell-1)}(t) +$$

$$+ O(z^{-1})$$

Now, by lemma 2.6 we know that $\det\big(\binom{m+i-\ell-1}{i}_{i,\ell}\big) = 1$

It follows that we can choose $x_1, \dots, x_m$ in such a way that

$$\int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau = x u^{(m-1)}(t) + O(z^{-1})$$

where x is any pregiven real number.

Now let D be any differential operator of order $\le n-1$, say, $D = a_0 + a\frac{d}{dt} + \dots + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}$. For each $i = 0, \dots, n-1$, let $(F_z(i), g_z(i), h_z(i))_z$ be a family of lds's, as constructed above, such that

$$\lim_{z\to\infty} \int_0^t h_z(i) e^{(t-\tau)F_z(i)} g_z(i) u(\tau)d\tau = a_i u^{(i)}(t)$$

Now let $(\hat F_z, \hat g_z, \hat h_z)$ be the n^2-dimensional system which is the direct sum of the n n-dimensional systems $(F_z(i), g_z(i), h_z(i))$, i.e.,

$$\hat F_z = \begin{pmatrix} F_z(1) & & 0 \\ & F_z(2) & \\ & & \ddots \\ 0 & & F_z(n) \end{pmatrix}, \quad \hat g_z = \begin{pmatrix} g_z(1) \\ \vdots \\ g_z(n) \end{pmatrix},$$

$$\hat h_z = (h_z(1), \dots, h_z(n))$$

Then

$$\lim_{z\to\infty} \int_0^t \hat h_z e^{(t-\tau)\hat F_z} \hat g_z u(\tau)d\tau = Du(t)$$

The transfer function of $(\hat F_z, \hat g_z, \hat h_z)$ is

$$T_z(s) = \hat h_z (s - \hat F_z)^{-1} \hat g_z = \sum_{i=1}^{n} h_z(i)(s - F_z(i))^{-1} g_z(i)$$

and because $F_z(i)$ is the same matrix for all i, it follows that the degree of the denominator of $T_z(s)$ is $\le n$. By realization or decomposition theory, cf. [4] or [5], it follows that there exists for every z an n-dimensional system $(\bar F_z, \bar g_z, \bar h_z)$ such that

$$\bar h_z e^{(t-\tau)\bar F_z} \bar g_z = \hat h_z e^{(t-\tau)\hat F_z} \hat g_z$$

giving us a family of n-dimensional systems $(\bar F_z, \bar g_z, \bar h_z)$ which in input-output behaviour converges to D. Finally because $L^{co,cr}_{1,n,1}(\mathbb{R})$ is open and dense in $L_{1,n,1}(\mathbb{R})$ we can find for every z a co and cr system (F_z, g_z, h_z) such that

$$|\bar{h}_z e^{(t-\tau)\bar{F}_z}\bar{g}_z - h_z e^{(t-\tau)F_z}g_z| \le \varepsilon_z |t-\tau| e^{|t-\tau| M_z}$$

where M_z is the maximum of the absolute values of the entries of F_z plus 1, and where ε_z can be chosen arbitrarily. Taking e.g. $\varepsilon_z = e^{-zM_z}$ we see that the families $(\bar{F}_z,\bar{g}_z,\bar{h}_z)$ and (F_z,g_z,h_z) have the same limiting input-output behaviour. This concludes the proof of theorem 2.3.

3. Limits of transfer functions.

3.1. Let (F,g,h) be a co and cr system of dimension n. Its transfer function is $T(s) = h(s-F)^{-1}g$, which is a rational function of the form

$$T(s) = \frac{b_{n-1}s^{n-1}+\ldots+b_1 s+b_o}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_o}$$

such that numerator and denominator have no factors in common. The system $(F,g,h) \in L^{cr,co}_{1,n,1}(\mathbb{R})$ is uniquely determined up-to-$GL_n(\mathbb{R})$-equivalence by T(s), so that we can (and shall) identify $M^{cr,co}_{1,n,1}(\mathbb{R})$ with the space of all such rational functions T(s). There is an obvious compactification of this space of all rational functions, viz. $\mathbb{P}^{2n}(\mathbb{R})$, real projective space of dimension 2n, which consists of all ratios $(x_o:x_1:\ldots:x_{2n})$, $x_i \in \mathbb{R}$, not all x_i equal to zero. The embedding $\psi : M^{co,cr}_{1,n,1}(\mathbb{R}) \to P^{2n}(\mathbb{R})$ is given by $\psi(T(s)) = (b_o:\ldots:b_{n-1}:a_o\ldots:a_{n-1}:1)$. The image of ψ is clearly open and dense.

In this section we relate this compactification of $M^{co,cr}_{1,n,1}(\mathbb{R})$ to the considerations of section 2 above and we construct the partial compactification $\bar{M}_{1,n,1}(\mathbb{R})$ mentioned in the introduction.

3.2. Let $\bar{M}_{1,n,1}(\mathbb{R})$ be the subspace of $\mathbb{P}^{2n}(\mathbb{R})$ consisting of those points $(x_o:\ldots:x_{2n}) \in \mathbb{P}^{2n}(\mathbb{R})$ for which at least one of the $x_n, \ldots, x_{2n}$ is non-zero. To each $x \in \bar{M}_{1,n,1}$ we associated the (generalized) transfer function

$$T_x(s) = \frac{x_{n-1}s^{n-1}+\ldots+x_1 s+x_o}{x_{2n}s^{2n}+\ldots+x_{n+1}s+x_n}$$

$$= c_{k-1}s^{k-1}+\ldots+c_1 s+c_o+\frac{b_{n-k-1}s^{n-k-1}+\ldots+b_1 s+b_o}{s^{n-k}+\ldots+a_1 s+a_o}$$

where k = 2n − m if m is the index of the last coordinate of x which is nonzero. We write $D_x(s) = c_o + c_1 s + \ldots + c_{k-1}s^{k-1}$ and $T^r_x(s)$, the reduced transfer function of x, for $T_x(s) - D_x(s)$.

3.3. <u>Lemma</u>. Let $T_z(s) = (s^n+a_{n-1}(z)s^{n-1}+ \ldots + a_1(z)s + a_o(z))^{-1}(b_{n-1}(z)s^{n-1}+ \ldots + b_1(z)s + b_o(z))$ be a family of transfer functions of systems in $L^{co,cr}_{1,n,1}(\mathbb{R})$. Then $\lim_{z\to\infty} T_z(s)$ exists pointwise for infinitely many values of s if and only if

(i) all limit points of the sequence $(x_z)_z$, $x_z = \psi(T_z(s))$, are in $\bar{M}_{1,n,1}(\mathbb{R}) \subset \mathbb{P}^{2n}(\mathbb{R})$

(ii) if x,x' are two limit points of $(x_z)_z$ then $T_x(s) = T_{x'}(s)$.

Moreover, if these conditions are fullfilled then $\lim_{z\to\infty} T_z(s) = T_x(s)$, where x is any limit point of $(x_z)_z$. (There always is one because $\mathbb{P}^{2n}(\mathbb{R})$ is compact).

The proof is elementary. First suppose we have a (sub)sequence $(x_{z'})_{z'}$ which converges to an element $x \in \bar{M}_{1,n,1}(\mathbb{R})$. Then, clearly, $\lim_{z'\to\infty} T_{z'}(s) = T_x(s)$. Now suppose $(x_{z'})_{z'}$ is a subsequence which converges to an element $x' \in \mathbb{P}^{2n}(\mathbb{R}) \setminus \bar{M}_{1,n,1}(\mathbb{R})$, then $\lim_{z'\to\infty} T_{z'}(s) = \pm\infty$ for all but finitely many values of s, where the sign depends on the parity of the index of the last coordinate of x' which is non-zero and the sign of s. Finally if $(x_z)_z$ has all its limit points in $\bar{M}_{1,n,1}(\mathbb{R})$ and there are limit points x',x such that $T_x(s) \neq T_{x'}(s)$ then $\lim_{z\to\infty} T_z(s)$ cannot exist for infinitely many values of s because then we would have two unequal rational functions which are equal for infinitely many values of the argument.

3.4. <u>Theorem.</u> Let $x \in M_{1,n,1}(\mathbb{R})$ and let (F,g,h) be any cr (n-k)-dimensional system with transferfunction equal to $T^r_x(s)$, and $\det(s-F) = s^{n-k} + x_m^{-1}x_{m-1}s^{n-k-1} + \ldots + x_m^{-1}x_{2n}$, where m = 2n − k is the index of the last coordinate of x which is unequal to zero (so that degree $(D_x(s)) \le k-1$). Then there exists a family of systems $(F_z,g_z,h_z) \subset L^{co,cr}_{1,n,1}(\mathbb{R})$ such that

(i) $\lim_{z\to\infty} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau =$

$$= D_x(\frac{d}{dt})u(t) + \int_0^t he^{(t-\tau)F} gu(\tau)d\tau$$

(ii) $\lim_{z\to\infty} \psi\pi(F_z,g_z,h_z) = x$

where $\pi : L^{co,cr}_{1,n,1}(\mathbb{R}) \to M^{co,cr}_{1,n,1}(\mathbb{R})$ is the natural projection and ψ is the embedding of 3.1 above.

(iii) $\lim_{z\to\infty} T_z(s) = T_x(s)$

Proof. Let $(\bar F_z,\bar g_z,\bar h_z)$ be a family of k dimensional systems in $L_{1,k,1}(\mathbb{R})$ whose input-output behaviour converges to the differential operator $D_x(\frac{d}{dt})$ (Theorem 2.3). Then

$$F_z = \begin{pmatrix} \bar F_z & 0 \\ 0 & F \end{pmatrix}, \quad g_z = \begin{pmatrix} g_z \\ g \end{pmatrix}, \quad h_z = (h_z,h)$$

has the desired limiting input-output behaviour. As in the proof of theorem 2.3 we can change $(F_z,g_z,h_z)_z$ to a family of cr and co systems with the same limit input-output behaviour. This proves (i). To prove (iii) apply (i) with u(t) smooth of bounded support. Then the integrals and $D_x(\frac{d}{dt})u(t)$ are all Laplace transformable and (iii) follows by the continuity of the Laplace transform. (Cf. [6], theorem 8.3.3 and theorem 4.3.1). Finally (ii) follows from (iii) because the determinant requirement prevents the family $\psi\pi(F_z,g_z,h_z)$ from having any other limit point $x' \neq x$ with $T_{x'}(s) = T_x(s)$.

3.5. <u>Theorem</u>. Let $(F_z,g_z,h_z)_z$ be a family of n-dimensional systems such that

$$\lim_{z\to\infty} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau$$

converges uniformly in t on bounded t intervals for all smooth input function u(t) of bounded support. Then there exist a $k \in \mathbb{N} \cup \{0\}$, a differential operator D of degree $\leq k-1$ and an (n-k)-dimensional system (F,g,h) such that

$$\lim_{z\to\infty} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau = Du(t) + \int_0^t he^{(t-\tau)F} gu(\tau)d\tau$$

Proof. Changing (F_z,g_z,h_z) slightly if necessary (as in the proof of theorem 2.3) we can assume that $(F_z,g_z,h_z) \in L^{co,cr}_{1,n,1}(\mathbb{R})$ for all z. Let u(t) be a given smooth bounded support input function. Let U(s) be its Laplace transform. Then

$$\mathcal{L} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau = T_z(s)U(s)$$

and the continuity of the Laplace transform ([6], theorem 8.3.3) and lemma 3.3 together imply that there is an $x \in \bar M_{1,n,1}(\mathbb{R})$ such that $\lim_{z\to\infty} T_z(s) = T_x(s)$. Take $D = D_x(\frac{d}{dt})$ and let (F,g,h) be any (n-k)-dimensional system with transfer function $T^r_x(s)$. Then the statement of the theorem follows because the Laplace transform is injective and continuous.

3.6. Theorem 3.4 says that every point of $\bar M_{1,n,1}(\mathbb{R})$ is system theoretically meaningful while theorem 3.5 that the compactification $\bar M_{1,n,1}(\mathbb{R})$ of $M^{co,cr}_{1,n,1}(\mathbb{R})$ is in a certain sense maximal.

4. <u>Compatibility of the compactification $\bar M_{1,n,1}(\mathbb{R})$ with various other (partial) compactifications.</u>

4.1. <u>Compatibility with $M^{co}_{1,n,1}(\mathbb{R})$ and $M^{cr}_{1,n,1}(\mathbb{R})$</u>. Let $M^{cr}_{1,n,1}(\mathbb{R})$ be the orbit space $L^{cr}_{1,n,1}(\mathbb{R})/GL_n(\mathbb{R})$. This is a differentiable manifold isomorphic to $\mathbb{R}^{2n}$ of which $M^{co,cr}_{1,n,1}(\mathbb{R})$ is an open submanifold. Cf. [1]. We have the following situation

$$\begin{array}{ccc} M^{co,cr}_{1,n,1}(\mathbb{R}) & \hookrightarrow & M^{cr}_{1,n,1}(\mathbb{R}) = \mathbb{R}^{2n} \\ & \searrow & \\ & & \bar M_{1,n,1}(\mathbb{R}) \end{array}$$

where the identification $M^{cr}_{1,n,1}(\mathbb{R}) \simeq \mathbb{R}^{2n}$ is given by associating to $(a_1,\dots,a_n,b_1,\dots,b_n) \in \mathbb{R}^{2n}$ the $GL_n(\mathbb{R})$-orbit of the cr system

$$g = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & 0 \\ 0 & & & \\ & & 0 & 1 \\ -b_1 & -b_2 & \cdots & -b_n \end{pmatrix}, \quad h = (a_1,\dots,a_n)$$

(This is a slightly different "canonical form" from the one used in [1], cf. e.g., also [5]). The transfer function of this cr system is $T(s) = (s^n+b_{n-1}s^{n-1}+\dots+b_2s+b_1)^{-1}(a_ns^{n-1}+\dots+a_2s+a_1)$ and we see that the embedding $M^{co,cr}_{1,n,1}(\mathbb{R}) \to \bar M_{1,n,1}(\mathbb{R})$ naturally extends to an embedding $M^{cr}_{1,n,1}(\mathbb{R}) \to \bar M_{1,n,1}(\mathbb{R})$.

Similarly one sees that the inclusion $M^{co,cr}_{1,n,1}(\mathbb{R}) \to \bar M_{1,n,1}(\mathbb{R})$ extends uniquely to an embedding $M^{co}_{1,n,1}(\mathbb{R}) \to \bar M_{1,n,1}(\mathbb{R})$.

4.2. Caveat. As it happens the images of $M^{co}_{1,n,1}(\mathbb{R})$ and $M^{cr}_{1,n,1}(\mathbb{R})$ under these natural embeddings are equal. Let this image be Y. Then the points of $Y \setminus M^{co,cr}_{1,n,1}(\mathbb{R})$ represent more than one $GL_n(\mathbb{R})$-orbit in $L_{1,n,1}(\mathbb{R})$ (but the associated differential operator is zero). It is also not true that a point of $Y \setminus M^{co,cr}_{1,n,1}(\mathbb{R})$ corresponds uniquely to a $GL_k(\mathbb{R})$-orbit of a k-dimensional system.for some $k < n$. Thus, so to speak, the same lower dimensional system occurs more than once in the edge of $M^{co,cr}_{1,n,1}(\mathbb{R})$ in Y. Similarly the "generalized systems" with transfer functions $T_x(s) = D_x(s) + T^r_x(s)$, $x \in \bar{M}_{1,n,1} \setminus Y$, $D_x(s) \neq 0$, occur more than once in $\bar{M}_{1,n,1}(\mathbb{R})$ iff (denominator degree of $T^r_x(s)$) + (degree $D_x(s)$) $< n$.

4.3. Forgetting inputs or outputs. In [2] and [3] we considered the orbit space structure of pairs of matrices (F,G) under the action $(F,G)^S = (SFS^{-1}, SG)$, $S \in GL_n(\mathbb{R})$. Let $I^{cr}_{m,n}(\mathbb{R})$ be the space of all completely reachable pairs of matrices (F,G) of sizes nxn, nxm respectively. In [2], [3] we showed that the orbit space $K^{cr}_{m,n}(\mathbb{R}) = I^{cr}_{m,n}(\mathbb{R})/GL_n(\mathbb{R})$ is a quasi projective submanifold of a Grassmann manifold $G_{n,(n+1)m}(\mathbb{R})$. This gives us a natural compactification $\bar{K}_{m,n}(\mathbb{R})$ of $K^{cr}_{m,n}(\mathbb{R})$, viz. the closure of $K^{cr}_{m,n}(\mathbb{R})$ in the compact manifold $G_{n,(n+1)m}(\mathbb{R})$.

Specializing now to the case m = 1 we have a diagram

$$
\begin{array}{ccc}
M^{co,cr}_{1,n,1}(\mathbb{R}) & \hookrightarrow & \bar{M}_{1,n,1}(\mathbb{R}) \\
\downarrow \phi & & \vdots \bar{\phi} \\
K^{cr}_{1,n}(\mathbb{R}) & \hookrightarrow & \bar{K}_{1,n}(\mathbb{R})
\end{array}
\tag{4.4}
$$

where the left-hand vertical map is induced by $(F,g,h) \mapsto (F,g)$, i.e. by forgetting outputs. A quick check shows that under the identification $M^{cr}_{1,n,1}(\mathbb{R}) \simeq \mathbb{R}^{2n}$, used in 4.1 above and the identification $K^{cr}_{1,n}(\mathbb{R}) = \mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R}) \simeq G_{n,n+1}(\mathbb{R})$ (cf. [2], [3]) the map ϕ corresponds to the projection $(a_1,\dots,a_n,b_1,\dots,b_n) \mapsto (b_1,\dots,b_n)$, restricted to $M^{co,cr}_{1,n,1}(\mathbb{R}) \subset M^{cr}_{1,n,1}(\mathbb{R})$. Thus we see that there exists a continuous (and algebraic) map $\bar{\phi}: \bar{M}_{1,n,1}(\mathbb{R}) \to \bar{K}_{1,n}(\mathbb{R})$, viz. $(x_0: x_1:\dots:x_{2n}) \to (x_n:\dots:x_{2n})$, which completes the diagram (4.4) commutatively. (I.e. $\bar{\phi}$ extends ϕ). Moreover $\bar{\phi}$ is surjective showing that the compactification $\bar{K}_{1,n}(\mathbb{R})$ of $K^{cr}_{1,n}(\mathbb{R})$ is system theoretically meaningful in a certain sense. Cf. also 4.5 below.

Similar results hold, of course, for output systems (F,H) under the $GL_n(\mathbb{R})$-action $(F,H)^S = (SFS^{-1}, HS^{-1})$; i.e. when one forgets inputs.

4.5. On the fibres of $\bar{\phi}: \bar{M}_{1,n,1}(\mathbb{R}) \to \bar{K}_{1,n}(\mathbb{R})$ and the interpretation of the points of $K_{1,n}(\mathbb{R}) \setminus K^{cr}_{1,n}(\mathbb{R})$. Let $y \in K_{1,n}(\mathbb{R})$, $y = (y_0:\dots:y_n)$, and let k be the index of the last coordinate of y which is nonzero. Then the fibre over y of $\bar{\phi}$ is equal to

$$\bar{\phi}^{-1}(y) = \{(x_0:\dots:x_{n-1}:y^{-1}_{k-1}y_0:\dots:y^{-1}_k y_{k-1}:1:0:\dots:0)\} \subset \bar{M}_{1,n,1}(\mathbb{R})$$

and these points correspond to generalized systems with transfer functions of the form

$$D + \frac{c_{k-1}s^{k-1}+\dots+c_1 s+c_0}{s^k+y^{-1}_k y_{k-1}s^{k-1}+\dots+y^{-1}_k y_0}$$

where D is a differential operator of degree $\leq n-k-1$. It follows that all points $x \in \bar{\phi}^{-1}(y) \subset M_{1,n,1}(\mathbb{R})$ for which $D_x(s) = 0$ can be seen as $GL_k(\mathbb{R})$-orbits of k-dimensional systems (F,g,h) for which the "input system" (F,g) is uniquely determined up to $GL_k(\mathbb{R})$-equivalence. Thus the points of $K_{1,n}(\mathbb{R}) \setminus K^{cr}_{1,n}(\mathbb{R})$ can be seen as lower dimensional completely reachable pairs (F,g) and we have in fact a stratification

$$\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \dots \cup \mathbb{R} \cup \{pt\}$$

$$\bar{K}_{1,n}(\mathbb{R}) = K^{cr}_{1,n}(\mathbb{R}) \cup K^{cr}_{1,n-1}(\mathbb{R}) \cup \dots \cup K^{cr}_{1,1}(\mathbb{R}) \cup K^{cr}_{1,0}(\mathbb{R})$$

where the single point space $K^{cr}_{1,0}(\mathbb{R})$ is interpreted as the "zero input-system".

5. Concluding remarks.

5.1. There are several ways in which the elements of $K_{1,n}(\mathbb{R}) \setminus K^{cr}_{1,n}(\mathbb{R})$ can be directly interpreted as limits of cr input-systems and also as lower dimensional cr systems. Some care must be exercised when doing this, however. To illustrate one of the difficulties involved we here offer the reader the following example for reflection. Consider the two families of input-systems.

$$g_z = \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}; \quad \bar{g}_z = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\bar{F}_z = \begin{pmatrix} 1 & z^{-1} \\ 0 & 2 \end{pmatrix}$$

As $z \to \infty$ both families converge (as input-systems). The first one to the non-cr "input-system" $\binom{1}{0}$, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and the second one to the cr input-system $\binom{1}{1}$, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. This in spite of the fact that (F_z, g_z) and $(\bar{F}_z, \bar{g}_z)$ are $GL_2(\mathbb{R})$-equivalent for all finite z. There is, however, a "canonical" subspace of $\mathbb{R}^2$ on which the two limit systems agree. This is a general phenomenon to which we intend to return in a subsequent paper. (Also for the more than one input case).

5.2. One cannot use realization theory directly to prove theorem 2.3. For instance the system of rational functions $(s-z)^{-1}z$ converges to -1 as $z \to \infty$, which is the Laplace transform of the operator $u(t) \mapsto y(t) = -u(t)$. The transfer function $(s-z)^{-1}z$ is realized by the one dimensional system $g = 1$, $h = z$, $f = 1$. But the limit $\lim_{z\to\infty} \int_0^t ze^{t-\tau}u(\tau)d\tau$ does not exist for almost all $u(t)$. On the other hand, the following is true. Let (F_z, g_z, h_z) be a family of systems with transfer functions $T_z(s)$. Suppose that there exists a $c \in \mathbb{R}$ such that $T_z(s)$ has no poles with real part $\geq c$ for all z. Then $\lim_{z\to\infty} T_z(s)$ exists iff $\lim_{z\to\infty} \int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)d\tau$ exists for all smooth input functions with compact support. Half of this theorem was proved in 3.5 above. The other half is proved using a continuity property of the inverse Laplace transform (in the sense of distribution theory) when applied to a converging sequence of rational functions with the extra property just mentioned.

5.3. The results of sections 2 and 3 above generalize immediately to the case of more inputs and more outputs. The proofs remain practically the same. E.g. to prove the more dimensional analogue of theorem 2.3 one first obtains all differential operators of the form

$$A\frac{d^r}{dt^r}, \quad r < n,$$ where A is an pxm matrix with at most one entry $\neq 0$. Then one takes a direct sum of nmp n-dimensional systems to realize every differential operator of the form

$$A_0 + A_1\frac{d}{dt} + \ldots + A_{n-1}\frac{d^{n-1}}{dt^{n-1}}$$ as a limit of an n^2mp-dimensional system with F-matrices consisting of nmp identical diagonal blocks. Now apply again decomposition or realization and approximation as in 2.3. The arguments and results of section 3 above (and also of 5.2 above) generalize in the same manner. The results of section 4 above do not generalize as easily. We intend to come back to this, to the problems indicated in 5.1 above, and to questions similar to those treated in this paper for discrete systems over more general fields than $\mathbb{R}$ (or $\mathbb{C}$), in subsequent papers.

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ON INVARIANTS AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS

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The following text presents no more (nor less) than an outline and possibly a guide to the principal results of [2-5] and some related material [6,7].

A constant, linear, dynamical system is a set of equations

$$(1)\qquad \begin{array}{ll} \dot{x} = Fx + Gu & x_{t+1} = Fx_t + Gu_t \\ y = Hx & y_t = Hx_t \\ \text{(continuous time)} & \text{(discrete time)} \end{array}$$

with $u \in \mathbb{R}^m$ = input space or control space, $x \in \mathbb{R}^n$ = state space, $y \in \mathbb{R}^p$ = output space. Here F, G, H are real matrices of the appropriate sizes with constant coefficients. The system is completely given by the triple of matrices (F,G,H). We use $L_{m,n,p}(\mathbb{R})$ to denote the space of all triples of matrices of sizes nxn, nxm, pxn respectively.

Of course the discrete time systems (1) also make sense for matrices (F,G,H) with coefficients in any field.

From the "black box" or "input-output" point of view the system $\Sigma = (F,G,H)$ assigns the output function

$$(2)\qquad y = f_\Sigma\, u,\ y(t) = \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau$$

to the input function u(t) if we start in x(0) at time t = 0. From this point of view there is a redundancy about the description of the system by means of a triple of matrices (F,G,H). Indeed let $GL_n(\mathbb{R})$ be the group of invertible real nxn matrices and let $GL_n(\mathbb{R})$ act on $L_{m,n,p}(\mathbb{R})$ according to

$$(3)\qquad (F,G,H)^S = (SFS^{-1},\ SG, HS^{-1})$$

Then the input-output maps of $\Sigma = (F,G,H)$ and of $\Sigma^S = (F,G,H)^S$ (both with starting state x(0) = 0 at time t=0) are exactly the same for all $S \in GL_n(\mathbb{R})$. We thus have an (internal) group of symmetrics $GL_n(\mathbb{R})$ of "basis transformations in state space". (The action just described corresponds to the state space transformation $x' = Sx$).

Several related questions now rise:

(i) What are the invariants for the action (3)? (Here an <u>invariant</u> is any continuous function $f: L_{m,n,p}(\mathbb{R}) \to \mathbb{R}$ such that $f((F,G,H)^S) = f((F,G,H))$ for all $S \in GL_n(\mathbb{R})$).

(ii) Does (3) describe all the redundancy in the description (F,G,H) of the input-output map (2); can recover "(F,G,H)-up-to-$GL_n(\mathbb{R})$-action" from the input-output data (2). How does one recognize that an input-output map comes from a (finite dimensional) system (F,G,H)?

(iii) Do there exist continuous canonical forms on suitable subspaces of $L_{m,n,p}(\mathbb{R})$? Here a <u>continuous canonical form</u> on a subspace $L' \subset L_{m,n,p}(\mathbb{R})$ is a continuous map $c: L' \to L'$ such that: (a) if $c(F,G,H) = (\bar{F},\bar{G},\bar{H})$ then there is an $S \in GL_n(\mathbb{R})$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^S$ and (b) $c(F,G,H) = c(\bar{F},\bar{G},\bar{H})$ if and only if there is an $S \in GL_n(\mathbb{R})$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^S$.

To answer these questions it is necessary to define two more concepts. The system (F,G,H) is said to be <u>completely reachable</u> (cr) if the matrix $R(F,G) = (G\ FG\ \dots\ F^nG)$ consisting of all the columns of the matrices F^iG, $i = 0, \dots, n$, has rank n; the system (F,G,H) is said to be <u>completely observable</u> if the matrix $Q(F,H)$ defined by $Q(F,H)^T = (H^T, F^TH^T, \dots, (F^T)^nH^T)$ has rank n. Here an upper "T" denotes "transposes". These two notions have the meanings suggested by their names, cf. [6]. Let $L^{cr,co}_{m,n,p}(\mathbb{R})$ be the open subspace of $L_{m,n,p}(\mathbb{R})$ consisting of all completely observable and completely reachable triples.

<u>Theorem 1</u>. Every invariant of $GL_n(\mathbb{R})$ acting on $L_{m,n,p}(\mathbb{R})$ can be written as a continuous function in the entries of the 2n-matrices $HG, HFG, \dots, HF^{2n-1}G$.

Let $\mathcal{A} = (A_0, A_1, A_2, \dots)$ be a sequence of real pxm matrices. We say that $\mathcal{A}$ is <u>realizable</u> if there exists a triple $(F,G,H) \in L_{m,n,p}(\mathbb{R})$ (for some n) such that $A_i = HF^iG$ for all $i = 1,2,\dots$. For each $r,s \in \mathbb{N}$ let $\mathcal{H}_{r,s}(\mathcal{A})$ be the block Hankel matrix

$$\mathcal{H}_{r,s}(\mathcal{A}) = \begin{pmatrix} A_0 A_1 & \dots & A_r \\ A_1 & & \vdots \\ \vdots & & \\ A_s & \dots & A_{r+s} \end{pmatrix}$$

The answer to question (ii) is now given by

<u>Theorem 2</u>. (Ho, Kalman, Meadowes, Silverman, Tissi, Youla). The sequence $\mathcal{A}$ is <u>realizable</u> by a triple $(F,G,H) \in L_{m,n,p}(\mathbb{R})$ iff there is an n_0 such that $n \geq n_0 = \operatorname{rank} \mathcal{H}_{n_0-1,n_0-1}(\mathcal{A}) = \operatorname{rank} \mathcal{H}_{r,s}(\mathcal{A})$ for all $r,s \geq n_0-1$. Moreover all realizations of dimension n_0 are co and cr and they all are in the same $GL_n(\mathbb{R})$ orbit.

It is now clear from theorem 2, that question (iii) is especially important for the subspace $L^{cr,co}_{m,n,p}(\mathbb{R})$. Before answering it let us take time out to explain why the word continuous in question (iii) is (sometimes) important. First, using delta functions as inputs we see from (2) that knowing the input-output data of a system amounts to knowing the sequence of matrices $HG, HFG, HF^2G, \dots$. Now suppose we have an unknown black box to be modelled by a linear dynamical system (1). The algorithmic proof of theorem 2 gives us a way of calculating (F,G,H) from $HG, \ldots, HF^{2n-1}G$. Because of measurement errors it would be highly desirable to have a continuous algorithm calculating (F,G,H) from $(HG,\ldots,HF^{2n-1}G)$. Now the existence of such a continuous algorithm is easily seen to be equivalent to the existence of a continuous canonical form. Cf. also [1] for some remarks in a related case.

Theorem 3. There is a continuous canonical form on $L^{cr,co}_{m,n,p}(\mathbb{R})$ if and only if $m = 1$ or $p = 1$.

The proof of this theorem goes via a detailed study of the orbit space $L^{cr,co}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$.

Theorem 4. $L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R}) = M^{co,cr}_{m,n,p}(\mathbb{R})$ is a smooth noncompact differentiable manifold (without boundary) of dimension $mn + np$. The natural projection π: $L^{cr,co}_{m,n,p}(\mathbb{R}) \to M^{co,cr}_{m,n,p}(\mathbb{R})$ is a locally-trivial principal $GL_n(\mathbb{R})$ bundle which is (globally) trivial iff $p = 1$ or $m = 1$.

From the identification of systems point of view (cf. also just above theorem 3) it is interesting to see if $M^{co,cr}_{m,n,p}(\mathbb{R})$ can be compactified in a system theoretically meaningful way.

Theorem 5. Let $D = B_o + B_1 \frac{d}{dt} + \ldots + B_{n-1} \frac{d^{n-1}}{dt^{n-1}}$ be the linear operator $u(t) \mapsto y(t) = B_o u(t) + \ldots + B_{n-1} \frac{d^{n-1}}{dt^{n-1}} u(t)$, where $B_o, \ldots, B_{n-1}$ are constant real $p\times m$ matrices. Then every such operator D arises as a converging limit of input-output maps of systems in $L^{cr,co}_{m,n,p}(\mathbb{R})$. Inversely if $\Sigma_s, s = 1,2, \ldots$ is a sequence of systems in $L^{cr,co}_{m,n,p}(\mathbb{R})$ such that $\lim_{s\to\infty} f_{\Sigma_s} u(t) = fu(t)$ uniformly on each bounded t interval, then f is the (direct) sum of an integral operator of (size $p\times m$ and) order $\leq i-1$ and the input-output function of a co and cr system of state space dimension $n-i$.

This provides a partial, but apparently system theoretically maximal, compactification of $M^{cr,co}_{m,n,p}(\mathbb{R})$.

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