ECONOMETRIC INSTITUTE

THREE RESEARCH ANNOUNCEMENTS ON FORMAL A-MODULES

M. HAZEWINKEL

REPORT 7624/M

ERASMUS UNIVERSITY ROTTERDAM, P.O. BOX 1738, ROTTERDAM, THE NETHERLANDS

ECONOMETRIC INSTITUTE

Three research announcements on formal A-modules

by

Michiel Hazewinkel

Report no. 7624/M

ERASMUS UNIVERSITY ROTTERDAM, P.O. BOX 1738, ROTTERDAM, THE NETHERLANDS

Three research announcements on formal A-modules

A. Cartier-Dieudinné theory for formal A-modules

B. Twisted Lubin-Tate formal groups laws, ramified Witt vectors and Artin-Hasse exponential mappings

C. "Tapis de Cartier" for formal A-modules.

Full proofs for the finite dimensional and Witt vector like cases will appear in my forthcoming book "Formal groups and applications"; for the infinite dimensional complications cf. M. Hazewinkel, Infinite dimensional "universal" formal groups laws and formal A-modules (in preparation).

Cartier-Dieudonné theory for formal A-modules

In this note we present a classification theory for formal A-modules which generalizes Cartier's classification theory for formal groups (cf. [1],[2]).

1. Formal A-modules. Let A be a discrete valuation ring with finite residue field k of q elements, char (k) = p, $q = p^r$. Choose a fixed uniformizing element Π of A and let K be the quotient field of A. Both characteristic 0 and p for K are permitted. Let $B \in \underline{\underline{Alg}}_A$, the category of commutative A-algebras. A formal A-module over B is a formal group law F over B (possibly of infinite dimension) together with a ring homomorphism $A \longrightarrow \mathrm{End}_B(F)$ such that $\rho_F(a) \equiv aX$ mod (degree 2) for all $a \in A$.
If B is A-torsion free there exists a unique vector of power series f(X) with coefficients in $K \otimes_A B$ such that $f(X) \equiv X$ mod (degree 2), $F(X,Y) = f^{-1}(f(X)+f(Y))$, $\rho_F(a)(X) = f^{-1}(af(X)$. We shall call f(X) the A-logarithm of F.

2. The functor W^{Π} and the formal A-module $\hat{W}^{\Pi}$. Consider the polynomials $w^{\Pi}_{q,n}(Z_0,\ldots,Z_n) = Z_0^{q^n} + \Pi Z_1^{q^{n-1}} + \ldots + \Pi^n Z_n$ with coefficients in A for n = 0,1,.... Then there is a unique functor $W^{\Pi}: \underline{\underline{Alg}}_A \longrightarrow \underline{\underline{Alg}}_A$, such that as a set-valued functor we have $W^{\Pi}(B) = \{(b_0,b_1,\ldots) | b_i \in B\}$, $W^{\Pi}(\varphi)(b_0,b_1,\ldots) = (\varphi(b_0),\varphi(b_1),\ldots)$ for $\varphi \in \underline{\underline{Alg}}_A$, and such that the

$w^{\Pi}_{q,n}$ define functorial A-algebra homomorphisms $W^{\Pi}(B) \longrightarrow B$. Moreover $W^{\Pi}(-)$ has a functorial A-algebra endomorphism τ^{Π} and an additive endomorphism $\underline{\underline{v}}$ such that $\tau^{\Pi}\underline{\underline{v}} = \Pi$. If B is of characteristic p>0 then $\tau^{\Pi}(b_0,b_1,\ldots) = (b_0^q,b_1^q,\ldots)$. The "addition polynomials" of W^{Π} define a formal A-module (of infinite dimension) which we shall denote $\hat{W}^{\Pi}$. More generally one has such a functor W^F and formal A-module $\hat{W}^F$ associated to any twisted Lubin-Tate formal group law of dimension 1 over A. cf. [6]. The functor $W^{\Pi}(-)$ has also been described in [3].

3. <u>The operator $\underline{\underline{f}}_{\Pi}$ and q-typification</u>. Let $\underline{\underline{FG}}^A_B$ be the category of formal A-modules over B. Let C(-;B) be the functor which assigns to every $F \in \underline{\underline{FG}}^A_B$ its topological group of curves. The homomorphisms ρ_F make C(-;B) a topological A-module valued functor. There now exists two functor endomorphisms ε_q and $\underline{\underline{f}}_{\Pi}$ of C(-;B) for all B with the following properties

(i) ε_q and $\underline{\underline{f}}_{\Pi}$ commute with base change;

(ii) $\varepsilon_q\varepsilon_q = \varepsilon_q$ and $\varepsilon_q\underline{\underline{f}}_{\Pi} = \underline{\underline{f}}_{\Pi}\varepsilon_q$;

(iii) if B is A-torsion free and f(X) is the A-logarithm of F(X,Y) then

$$f(\gamma(t)) = \Sigma x_i t^i \Rightarrow f(\varepsilon_q\gamma(t)) = \sum_{i=0}^{\infty} x_{q^i} t^{q^i}, \quad f(\underline{\underline{f}}_{\Pi}\gamma(t)) = \sum_{n=1}^{\infty} \Pi x_{qn} t^n$$

We shall use $C_q(-;B)$ to denote the image functor of ε_q. The subfunctor $C_q(-;B)$ is stable under the operators $\underline{\underline{V}}_q : \gamma(t) \longmapsto \gamma(t^q)$, $<c> : \gamma(t) \longmapsto \gamma(ct)$, $c \in B$ and the operators $[a]$, $a \in A$ which describe the A-module structure of C(-;B). As operators on $C_q(-;B)$ the relations amoung all these operators are

$$\langle b\rangle \mathbf{V}_q = \mathbf{V}_q \langle b^q\rangle,\ \mathbf{f}_\Pi \langle b\rangle = \langle b^q\rangle \mathbf{f}_\Pi,\ \mathbf{f}_\Pi \mathbf{V}_q = [\Pi]$$

$$\text{(3.1)}\qquad [a] \text{ commutes with } \mathbf{f}_\Pi,\ \mathbf{V}_q,\ \langle b\rangle,$$

$$[1] = \mathrm{id},\ [a+b] = [a]+[b],\ [ab] = [a][b],$$

$$\langle b\rangle+\langle c\rangle = \sum_{n=0}^{\infty} \mathbf{V}_q^n \langle r_n(b,c)\rangle \mathbf{f}_\Pi^n; \qquad [a] = \sum_{n=0}^{\infty} \mathbf{V}_q \langle \Omega_n(a)\rangle \mathbf{f}_\Pi^n$$

where the $r_n(b,c)$ are the polynomials with coefficients in A determined inductively by $Z_1^{q^n}+Z_2^{q^n} = \sum_{i=0}^{n} \Pi^i r_i(Z_1,Z_2)^{q^{n-i}}$, and the $\Omega_n(a) \in A$ are determined by $w^\Pi_{q,n}(\Omega_0(a),\ldots,\Omega_n(a)) = a$.

4. Representation theorem. Let $\gamma_\Pi(t)$ be the curve in $C_q(\hat{W}^\Pi;B)$ whose first component is the power series t and whose other components are all zero. Then for every $\gamma(t) \in C_q(F;B)$, $F \in \mathbf{FG}^A_B$, there exists a unique homomorphism of formal A-modules $\alpha_\gamma : \hat{W}^\Pi \longrightarrow F$ such that $\alpha_\gamma(\gamma_\Pi(t)) = \gamma(t)$. This theorem permits us to write every functor-endomorphism of $C_q(-;B)$ as a set-valued functor, as a unique infinite sum $\Sigma \mathbf{V}_q^n \langle b_{n,m}\rangle \mathbf{f}_\Pi^m$ with for every $n \in \mathbb{N}\cup\{0\}$ only finitely many m with $b_{n,m} \neq 0$. We shall denote this A-algebra of operators with $\mathrm{Cart}_\Pi(B)$. Relations (3.1) are the calculation rules for $\mathrm{Cart}_\Pi(B)$.

5. The equivalence of categories. Let $\mathbf{E}(\Pi,B)$ be the category of all $\mathrm{Cart}_\Pi(B)$ modules C such that $\mathbf{V}_q$ is injective, $C = \varprojlim C/\mathbf{V}_q^n C$, $C/\mathbf{V}_q C$ is a free B-module. (The operators $\langle b\rangle$ induce a B-module structure on $C/\mathbf{V}_q C$). Then $F \longmapsto C_q(F;B)$ is an equivalence of categories between $\mathbf{E}(\Pi,B)$ and the category of formal A-modules over B.

6. Height and dimension. Let ℓ be a perfect extension field of k, let $F \in \underline{\underline{FG}}_{\ell}^{A}$ be finite dimensional and let $C = C_q(F;\ell)$. Then $C/[\Pi]C$ is a vector space over ℓ. We define $h(F) = \dim_\ell(C/[\Pi]C)$. If $h<\infty$ then $h = \dim(F) + \dim_\ell(C/\underline{\underline{f}}_\Pi C)$. If A is the ring of integers of a finite extension K of $\underline{\underline{Q}}_p$, $n = [K: \underline{\underline{Q}}_p]$ and H is the height of F as a formal group then $H = nh$.

7. Description of $\mathrm{Cart}_\Pi(B)$. The subset of all elements of the form $\Sigma \underline{\underline{V}}_q^n \langle b_n \rangle \underline{\underline{f}}_\Pi^n$ of $\mathrm{Cart}_\Pi(B)$ is a sub-A-algebra which can be identified with $W^\Pi(B)$, and $\mathrm{Cart}_\Pi(B)$ then consists of all expressions $x_0 + \Sigma \underline{\underline{V}}_q^i x_i + y_i \underline{\underline{f}}_\Pi^i$ with $x_i, y_i \in W^\Pi(B)$ and $\lim_{i\to\infty} y_i = 0$. If B is a perfect field then $W^\Pi(B)$ is a complete discrete valuation ring with residue field B and uniformizing element Π. In this case the calculation rules are $\underline{\underline{f}}_\Pi x = \tau^\Pi(x)\underline{\underline{f}}_\Pi$, $x\underline{\underline{V}}_q = \underline{\underline{V}}_q \tau^\Pi(x)$ and $\underline{\underline{f}}_\Pi \underline{\underline{V}}_q = \Pi = \underline{\underline{V}}_q \underline{\underline{f}}_\Pi$. (If B is an algebraic extension of k (and A is complete) then $W^\Pi(B)$ is the completion of the ring of integers of the unramified extension of K with residue field B).

8. Classification up to isogeny. If ℓ is an algebraically closed field extension of k then $\mathrm{Cart}_\Pi(\ell)$ " Localized with respect to $\underline{\underline{V}}_q$" is a ring over which the classification up to isomorphisms of torsion modules can be carried out as usual (cf. e.g. [7] ch. 2). It results that every formal A-module over ℓ is isogenous to a direct sum of formal A-modules $G_{1,0}^\Pi$, $G_{n,m}^\Pi$, $G_{n,\infty}^\Pi$ with $n,m \in \underline{\underline{N}}$, $(n,m) = 1$ whose curve modules over $\mathrm{Cart}_\Pi(\ell)$ are the quotients of $\mathrm{Cart}_\Pi(\ell)$ by the left ideals

generated respectively by $\underline{\underline{f}}_{\Pi}-1$, $\underline{\underline{f}}_{\Pi}^{n}-\underline{\underline{V}}_{q}^{m}$, $\underline{\underline{f}}_{\Pi}^{n}$. This decomposition is unique (up to isogeny); $G^{\Pi}_{1,0}$ is the formal A-module of A-height 1, $G^{\Pi}_{n,m}$ has dimension n and A-height n+m and is simple (as a formal A-module!) and $G^{\Pi}_{n,\infty}$ is the obvious n-dimensional quotient of $\hat{W}^{\Pi}$.

9. The proofs of all of the above rely heavily on the construction of universal formal A-modules as in [4] part VIII and the functional equation techniques developped in [4] , cf. also [5], and certain infinite dimensional analogues.

References

1. P. Cartier, Groupes formels associés aux anneaux de Witt généralisés. C.R. Acad. Sci. Paris 265 (1967), A50-52.

2. P. Cartier, Modules associés à un groupe formel commutatif. Courbes typiques, C.R. Acad. Sci. Paris 265 (1967), A129-132.

3. E. Ditters, Formale Gruppen, die Vermutungen von Atkin-Swinnerton Dyer und verzweigte Witt Vektoren, Lecture Notes, Göttingen, 1975.

4. M. Hazewinkel, Constructing formal groups I-VIII (I,II,III : to appear in J. pure and applied Algebra; IV : to appear Adv. Math.; VIII : submitted Compositio Math.; available in preprint form as reports 7119, 7201, 7207, 7322, 7505, 7507, 7514, 7519, Econometric Inst., Erasmus Univ. Rotterdam, 1972-1975).

5. M. Hazewinkel, A universal formal group and complex cobordism, Bull. AMS 81, 5(1975), 930-933.

6. M. Hazewinkel, Twisted Lubin-Tate formal group laws, ramified Witt vectors and Artin-Hasse exponential mappings. In preparation.

7. Yu.I. Manin, Theory of commutative formal groups over fields of finite characteristic (Russian), Usp.Mat.Nauk 18, 6 (1963), 3-90.

Twisted Lubin-Tate formal group laws, ramified Witt vectors and Artin-Hasse exponential mappings.

For any ring R let $\Lambda(R)$ denote the multiplicative group of power series of the form $1+a_1t+\ldots$ with coefficients in R. The Artin-Hasse exponential mappings are homomorphisms $W_{p^\infty}(k) \longrightarrow \Lambda(W_{p^\infty}(k))$ which satisfy certain additional properties. In this note we present a generalization with $W_{p^\infty}(k)$ replaced by rings of Witt vectors associated to a (global) twisted Lubin-Tate formal group law and Λ replaced by the curve functor of that global formal group law.

1. Twisted Lubin-Tate formal group laws. Let A be a discrete valuation ring with residue field k of characteristic $p>0$ and quotient field K. Let $\tau: K \longrightarrow K$ be an automorphism of K such that $\tau(a) \equiv a^q \bmod \pi A$ for all $a \in A$ for a certain power q of p. Choose a uniformizing element $\pi \in A$ and let $f(X) \in A[[X]]$ be a power series over A such that $f(X) \equiv X$ mod (degree 2) and $f(X)-\pi^{-1}\tau_* f(X^q) \in A[[X]]$, where τ_* means "apply τ to the coefficients". Let $F(X,Y) = f^{-1}(f(X)+f(Y))$, then $F(X,Y)$ has its coefficients in A and hence is a formal group law over A (cf. [4] part VIII (= report 7507). We shall call these formal group laws twisted Lubin-Tate formal group laws. If τ = id and k is finite with q elements they correspond bijectively with the group laws of [6]. Cf. [2] and [5], Ch. I, §8. Two such twisted formal group laws with logarithms $f(X)-\pi^{-1}\tau_*f(X^q) \in A[[X]]$ and $\hat{f}(X)-\hat{\pi}^{-1}\tau_*\hat{f}(X^q) \in A[[X]]$ are isomorphic over A if and only if there is a unit $u \in A$ such that $\pi^{-1}\hat{\pi} = u^{-1}\tau(u)$.

2. <u>The A-algebra functor</u> $C_q(F;-)$. Let $B \in \underline{\underline{Alg}}_A$, the category of commutative A-algebras, and let F be as above and C(F;B) be the module of curves of in F with coefficients in B (cf. [1]). There exists a unique functorial additive functor endomorphism ε_q of C(F;-) such that for A-torsion free B we have for all $\gamma(t) \in C(F;B)$

$$f(\gamma(t)) = \sum_{i=1}^{\infty} x_i t^i \Rightarrow f(\varepsilon_q \gamma(t)) = \sum_{i=0}^{\infty} x_{q^i} t^{q^i}$$

One has $\varepsilon_q \varepsilon_q = \varepsilon_q$ and we shall denote the image functor of ε_q with $C_q(F;-)$ and call its elements q-typical curves. (In case q=p this coincides with [1]).

There now exists a unique functorial A-algebra structure on $C_q(F;-)$ and a unique semilinear A-algebra endomorphism $\underline{\underline{f}}_\pi^{tw}$ of $C_q(F;-)$ such that for A-torsion free B we have : if $\gamma(t), \delta(t) \in C_q(F;B)$ and $f(\gamma(t)) = \Sigma x_i t^{q^i}$, $f(\delta(t)) = \Sigma y_i t^{q^i}$, then $f(\gamma(t) * \delta(t)) = \Sigma \pi\tau(\pi)\ldots\tau^{i-1}(\pi) x_i y_i t^{q^i}$, $f(\{a\}\gamma(t)) = \Sigma \tau^i(a) x_i t^{q^i}$, $f(\underline{\underline{f}}_\pi^{tw}\gamma(t)) = \Sigma \tau^i(\pi) x_{i+1} t^{q^i}$. The ring endomorphism $\underline{\underline{f}}_\pi^{tw}$ is semilinear in the sense that $\underline{\underline{f}}_\pi^{tw}(\{a\}\gamma(t)) = \{\tau(a)\}\underline{\underline{f}}_\pi^{tw}\gamma(t)$. One also has $\underline{\underline{f}}_\pi^{tw}\underline{\underline{V}}_q = \{\pi\}$ where $\underline{\underline{V}}_q$ is the operator $\underline{\underline{V}}_q\gamma(t) = \gamma(t^q)$.

3. <u>The functor</u> $W_{q,\infty}^F$. Let $f(X) = \Sigma a_i X^i$ and write $e_i = a_{q^i}$, i=0,1,2,... Consider the polynomials $w_n^F(Z) = \pi\tau(\pi)\ldots\tau^{n-1}(\pi)$ $(Z_n + e_1 Z_{n-1}^q + \ldots + e_n Z_0^{q^n})$. There exists a unique A-algebra valued functor $W_{q,\infty}^F$ on $\underline{\underline{Alg}}_A$ which as a set valued functor is $W_{q,\infty}^F(B) = \{(b_0, b_1, \ldots) | b_i \in B$ $W_{q,\infty}^F(\varphi)(b_0, b_1, \ldots) = (\varphi(b_0), \varphi(b_1), \ldots)$ for $\varphi \in \underline{\underline{Alg}}_A$ and such that the $w_n^F(Z)$ define functorial semilinear A-algebra homomorphisms $W_{q,\infty}^F(B) \longrightarrow B$ with twist τ^n for all n=0,1,2,....The semilinear

A-algebra endomorphisms $\underline{\underline{f}}_\pi^{tw}$ satisfies $w_n^F(\underline{\underline{f}}_\pi^{tw}) = w_{n+1}^F$ functorially and is characterized thereby. The functors $C_q(F;-)$ and $W_{q,\infty}^F(-)$ are isomorphic, the isomorphism is given by $(b_0,b_1,\ldots) \longrightarrow \varepsilon_q\ (\Sigma^F b_i t^{q^i})$. In case A is the ring of integers of a finite extension K of $\underline{\underline{Q}}_p$, $\tau = \mathrm{id}$, k has q elements and $f(X) = X+\pi^{-1}X^q+\pi^{-2}X^{q^2}+\ldots$ this functor has also been described in [3].

4. Ramified Witt vectors. Now suppose that A is complete and let B be the ring of integers of a finite unramified extension L of K and let $\hat{\tau}$: $L \longrightarrow L$ be the automorphism of L such that $\hat{\tau}|K = \tau$ and $\hat{\tau}(b) \equiv b^q \bmod \pi B$. Let F(X,Y) and f(X) be as before. Then we can also regard F(X,Y) as a twisted Lubin-Tate formal group law over B (with $\hat{\tau}$ instead of τ) and we have B-algebra homomorphisms $B \longrightarrow C_q(F;B) \longrightarrow C_q(F;\ell)$ where ℓ is the residue field of L. This composed map is an isomorphism of B-algebras under which $\hat{\tau}$ corresponds to $\underline{\underline{f}}_\pi^{tw}$.

5. Artin-Hasse exponentials (local case). There exists a unique A-algebra functor morphism Δ^F: $W_{q,\infty}^F(-) \longrightarrow W_{q,\infty}^F(W_{q,\infty}^F(-))$ such that $w_n^F \circ \Delta^F = (\underline{\underline{f}}_\pi^{tw})^n$ for all $n=0,1,2,\ldots$. This morphism Δ^F has the additional property that also $W_{q,\infty}^F(w_n^F) \circ \Delta^F = (\underline{\underline{f}}_\pi^{tw})^n$.

6. Artin-Hasse exponentials (global case). Now let A be the ring of integers of a global field K, and let F(X,Y) be a formal group law over A such that for each finite valuation v F(X,Y) is a twisted

Lubin-Tate formal group law over A_v, the completion of A with respect to v (for some q_v and τ_v). Such formal group laws exist; in fact one can up to strict isomorphism arbitrarily perscribe the "local" formal group laws $F_v(X,Y)$ over A_v and this then determines F(X,Y) uniquely up to isomorphism; cf. [4] report 7201 or part VII. Now $W^F_{q,\infty} \simeq C_q(F;-)$ which is a subfunctor of C(F;-). So combining 4 and 5 above we find for all finite valuations v A-module homomorphisms $A_v \longrightarrow C(F;A_v)$ such that the composed map $A_v \longrightarrow C(F;A_v) \longrightarrow C(F;k_v) \xrightarrow{\varepsilon_q} C_q(F;k_v)$ is an isomorphism.

References

1. P. Cartier, Modules associés à un groupe formel commutatif. Courbes typiques, C.R. Acad. Sci., Paris 265 (1967). A129-132.

2. P. Cartier, Groupes de Lubin-Tate généralisés, Inv. Math. 35 (1976), 273-284.

3. E.Ditters, Formale Gruppen, die Vermutungen von Atkin-Swinnerton Dyer und verzweigte Witt Vektoren, Lecture notes, Göttingen, 1975.

4. M. Hazewinkel, Constructing formal groups I-VIII (I,II,III to appear J. pure and applied Algebra; IV to appear Adv. Math.; VIII submitted Compositio Math.; all available in preprint as reports 7119,7201,7207, 7322, 7505, 7507, 7514, 7519, Econometric Inst., Erasmus Univ., Rotterdam, 1971-1975)

5. M. Hazewinkel, Formal groups and applications, in preparation.

6. J. Lubin, J. Tate, Formal complex multiplication in local fields, Ann. Math. 81(1965), 380-387.

"Tapis de Cartier" pour les A-modules formels

Resumé. On donne une généralisation pour le cas des A-modules formels de la théorie de relèvement des groupes formels commutatifs, dit "Tapis de Cartier". Les enoncés et les preuves sont des généralisations directes de ceux esquissés par Cartier dans son séminaire à l'IHES 1972, étant donnée la théorie de type Cartier-Dieudonné pour les A-modules formels, resumée en ([1]).

Soit A l'anneau des entiers d'un corps K de valuation discrète de corps résiduel k à un nombre fini d'éléments $q = p^r$, $p = car(k)$. Soit π une uniformisante de K. Le corps K peut être soit de caractéristique zero soit de $p > 0$. Soit B un A-algèbre. Dans une note précédente on a présenté une théorie de Cartier-Dieudonné des A-modules formels et les notions divers associées: q-typification; module de courbes q-typiques $C_q(F;B)$; opérateur de Frobenius $\underline{\underline{f}}_\pi$ et exponentielle de Artin-Hasse.

1. A-modules formels de Lubin-Tate généralisés. Soit B un A-algèbre local, sans A-torsion, d'idéal maximal πB, tel qu'il existe un endomorphisme $\sigma: B \to B$ tel que $\sigma(b) \equiv b^q \mod \pi B$ pour tout $b \in B$. Soit M un B-module libre de type fini et soit $\eta : M \to M$ un endomorphisme σ-semilineaire de M (i.e. $\eta(bm) = \eta(b)\eta(m)$). Choississons une base $e_1, \ldots, e_h$ de M et soit $D(\eta)$ la matrice de η par rapport à la base $e_1, \ldots, e_h$. On pose

$$(1) \qquad g_M(X) = X + \pi^{-1}D(\eta)\sigma_* g_M(X^q), \quad G_M(X,Y) = g_M^{-1}(g_M(X) + g_M(Y))$$

où $g_M(X)$ est un h-tuple de séries formelles en $X_1, \ldots, X_h$ à coefficients dans $B \otimes_A K$; $X^q = (X_1^q, \ldots, X_h^q)$ et $\sigma_* f(X)$ est la série formelle obtenue de $f(X)$ par l'application de σ à tous les coefficients de $f(X)$. Alors (d'après le lemme d'equation fonctionelle ([2])) $G_M(X,Y)$ est un A-module formel sur B de A-logarithme $g_M(X)$. Soit $\phi : (M,\eta) \to (M',\eta')$ un homomorphisme, i.e. $\phi : M \to M'$ est un homomorphisme de B-modules et $\phi\eta = \eta'\phi$. Alors $g_{M'}^{-1}(Eg_M(X))$ est un homomorphisme de A-modules formels $G_M \to G_{M'}$, où E est la matrice de ϕ par rapport aux bases choisises $\{e_1,\ldots,e_h\}$ de M et $\{e'_1,\ldots,e'_h\}$ de M'.

Supposons maintenant que σ soit un automorphisme et soit M^σ le B-module modifié $b*m = \sigma^{-1}(b)m$. Alors $\eta : (M^\sigma,\eta) \to (M,\eta)$ est un homomorphisme et on trouve un morphisme $\underline{\underline{v}}(M): \sigma_* G_M \to G_M$ de A-modules formels qui se réduit mod πB a morphisme de "Verschiebung" $Ⓥ_q : \Gamma_M^{(q)} \to \Gamma_M$, où Γ_M est la réduction mod πB de G_M. On obtient ainsi une équivalence de catégories entre la catégories des pairs (M,η) et celle des pairs $(G, \underline{\underline{v}}: \sigma_* G \to G)$ où G est un A-module formel sur B et $\underline{\underline{v}}$ un homomorphisme de A-modules formels qui se réduit en $Ⓥ_q$ modulo πB.

2. <u>Théorème des foncteurs adjointes</u>. Soit (M,η) comme ci-dessus et soit H un A-module formel sur B et $C_q(H;B)$ le $W^A_{q,\infty}(B)[\underline{\underline{f}}_\pi, \underline{\underline{V}}_q]$-module des courbes q-typiques de H. Alors il y a une correspondence bi-univoque entre homomorphismes de A-modules formels $G_M \to H$ et homomorphismes B-lineaires $\alpha : M \to C_q(H;B)$ tel que $\alpha\eta = \underline{\underline{f}}_\pi \alpha$. Ici la structure de B-module sur $C_q(H;B)$ est donnée par l'exponentielle $\Delta : B \to W^\pi_{q,\infty}(B)$, qui est caracterisé par $w^A_{q,n} \circ \Delta = \sigma^n$, $n = 0,1,2,\ldots$

Cette correspondence est donnée par un morphisme canonique $\alpha_o : M \to C_q(G_M;B)$ defini par $g_M(\alpha_o(m)) = \sum_{i=o}^{\infty} \pi^{-i}\eta^i(m)t^{q^i}$.

3. <u>Propriété universel de G_M par rapport à certain extensions.</u> Supposons que $\sigma : B \to B$ soit un automorphisme, que B est complet et Hausdorff pour la topologie πB-adique, et que M est un B-module libre de type finie avec des endomorphismes η, ζ, où η est σ-semilineaire et ζ σ^{-1}-semilineaire, tel que $\eta\zeta = \zeta\eta = \pi$ et $\zeta^r M \subset \pi M$ pour $r \in \mathbb{N}$ suffissamment grand.

Soit $o \to R^+ \to H \to H_1 \to 0$ une suite exacte de A-modules formels où R^+ est un A-module de type additif. Supposons en plus tout sous-A-module formel de type additif de dimension un se relève en un sous-A-module formel additif de dimension 1 de H. Alors pour tout homomorphisme de A-modules formels $\beta : G_M \to H_1$ il existe un relèvement unique $\tilde{\beta} : G_M \to H$, tel que $\gamma\tilde{\beta} = \beta$.

4. <u>Quotients de G_M par sous-A-module formel additif</u>. Soient M, B, η, ζ, comme ci-dessus en 3. Soit N un sous-module libre de type fini de M tel que M/N soit libre et $N + \pi M = \zeta M$. Alors on a une immersion canonique $N^+ \to G_M$ définie par $C_q(N^+;B) \to C_q(G_M;B)$, $n \mapsto \beta(n) - \underline{\underline{V}}_q \beta(\zeta^{-1}n)$ pour $n \in \mathbb{N}$.

Et il en résulte une suite exacte de A-modules formels

(2) $$0 \to N^+ \to G_M \to G \to 0$$

Soit Γ sur $\ell = B/\pi B$ la réduction modulo πB de G. Alors le morphisme composé $M \xrightarrow{\alpha_0} C_q(G_M;B) \to C_q(G;B) \to C_q(\Gamma;\ell)$ est un isomorphisme de B-modules qui identifie η avec $\underline{\underline{f}}_\pi$ et ζ avec $\underline{\underline{V}}_q$.

La suite exacte (2) est l'extension universelle de G par un A-module formel additif.

5. Relèvements. Soit Γ un A-module formel de A-hauteur finie sur un corps parfait $\ell \supset k$. On prends $B = W^A_{q,\infty}(\ell)$, $M = C_q(\Gamma;\ell)$, $\eta = \underline{\underline{f}}_\pi$, $\zeta = \underline{\underline{V}}_q$. Alors toutes les hypothèses sur B, M, η, ζ des no's 2, 3, 4 ci-dessus sont satisfaites. Soit G un A-module formel sur B qui relève Γ. (Un tel G existe toujours parce qu'il existe pour toute dimension donnée un A-module formel universel défini sur un anneau de polynômes $A[S_1, S_2, \ldots]$ (3)). Alors il existe un homomorphisme unique $\beta : G_M \to G$ tel que $M \xrightarrow{\alpha_0} C_q(G_M;B) \to C_q(G;B) \to C_q(\Gamma;\ell)$ est l'identité. Le noyau de β est un sous-A-module formel de type additif d'algèbre de Lie $N = \mathrm{Ker}(\mathrm{Lie}(\beta))$ et β est surjectif. En plus M/N est libre et $N + \pi M = \zeta M$. C'est à dire l'extension $o \to N^+ \to G_M \to G \to 0$ obtenu à partir de Γ est de la type construite en 4 ci-dessus. Comme corollaire on obtient que l'extension universelle par un A-module formel additif d'un relèvement G de Γ ne dépend pas du relèvement choisi.

(1) M. Hazewinkel, preprint

(2) M. Hazewinkel, Formal groups and applications (à paraître, Acad. Pr). §10.2

(3) M. Hazewinkel, loc. cit. §§21.4, 25.4.

Michiel Hazewinkel
Univ. Erasmus à Rotterdam
50, Burg. Oudlaan,
Rotterdam, Pays Bas