# REPRESENTATION OF QUIVERS AND MODULI OF LINEAR DYNAMICAL SYSTEMS and

# MODULI AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS, III: THE ALGEBRAIC - GEOMETRIC CASE

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REPRINT SERIES no. 211

These articles appeared in "Geometric Control Theory", Vol. VII (1976).

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# REPRESENTATIONS OF QUIVERS AND MODULI OF LINEAR DYNAMICAL SYSTEMS

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## 1. PREFACE

This note is the written version of the part which is not covered by [15] and [16] (cf. also [12], [13], [14], and [3] of the talks I gave at the Ames conference in June/July 1976. The main purpose of this part of the talks was to acquaint engineers and applied mathematicians with the fact that some of the problems they have been studying in (algebraic) system theory and identification theory are identical (or at least very similar to) a certain set of problems studied by algebraists belonging to representation theory or linear algebra (depending on one's taste and judgement) viz. the theory of representations of "quivers." Inversely it may be of interest to the algebraists that the two quivers for which results have been obtained in algebraic system theory are both of wild type.

## 2. QUIVERS AND THEIR REPRESENTATIONS

### 2.1 Definition

A quiver is a finite connected directed graph. i.e. a quiver Q consists of a finite set $P_Q$ of points and a finite set $A_Q$ of arrows between points of $P_Q$. Loops are allowed and also multiple arrows between the same points.

2.2 Some examples of quivers are

(a) (b) (c) (d)

(e) (f) (g) (h)

### 2.3 Definitions

A representation V over a field K of a quiver Q assigns to each $P \in P_Q$ a vector space V(P) and to each arrow $a \in A_Q$ a vector space homomorphism $V(a):V(s(a)) \to V(r(a))$ where a is an arrow from $s(a) \in P_Q$ to $r(a) \in P_Q$. The zero representation assigns to each $P \in P_Q$ the zero vector space. Given two representations $V_1$ and $V_2$ their direct sum $V_1 \oplus V_2$ assigns to each $P \varepsilon P_Q$ the vector space $V_1(P) \oplus V_2(P)$ and to each arrow $a \varepsilon A_Q$ the direct sum homomorphism $V_1(a) \oplus V_2(a)$. A representation V is called <u>indecomposable</u> if it cannot be written as a direct sum $V = V_1 \oplus V_2$ with $V_1$ and $V_2$ both unequal to the zero representation. Given a representation V a <u>subrepresentation</u> W consists of subspaces $W(P) \subset V(P)$ for all $P \varepsilon P_Q$ such that $V(a)(W)(s(a)) \subset W(r(a))$ for all $a \varepsilon A_Q$. A representation V is called <u>irreducible</u> if it has no other subrepresentations than itself and the zero representation. Finally two representations V, W are said to be isomorphic if there are isomorphisms $\psi(P) : V(P) \to W(P)$ for every $P \varepsilon P_Q$ such that the following diagram commutes for all $a \varepsilon A_Q$.

$$\begin{array}{ccc} V(s(a)) & \xrightarrow{V(a)} & V(r(a)) \\ \Big\downarrow \psi(s(a)) & & \Big\downarrow \psi(r(a)) \\ W(s(a)) & \xrightarrow{W(a)} & W(r(a)) \end{array}$$

2.4 The general problem is now: given a quiver, describe all isomorphism classes of (indecomposable) representations.

In the case of the quiver 2.2(a) above this is the familiar linear algebra problem of classifying square matrices up to similarity. The indecomposable representations are precisely those which have one Jordan block.

In the case of example 2.2(b) a representation consists of two matrices (A,B), and a second representation (C,D) is isomorphic to (A,B) if there are invertible matrices S, T such that C = SAT, D = SBT. Writing A + sB and C + sD for (A,B) and (C,D), where s is an indeterminate we see that the study of isomorphism classes of representations of the quiver 2.2(b) is the same as the study of pencils of matrices in the sense of Kronecker, who also solved this problem.

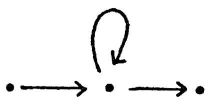
Similarly quiver 2.2(g) concerns the study of two dimensional pencils A + sB + tC. (These turn up when one studies control systems with delays.)

To conclude this section let us remark that quiver 2.2(c) is the study of pairs of matrices under simultaneous similarity a problem which has been around for some 150 years (and is still unsolved).

### 2.5 A special quiver from system theory

A linear dynamical system $\dot{x} = Fx + Gu, y = Hx$ or $x_{t+I} = Fx_t + Gu_t$, $y_t = Hx_t$ (discrete case) gives rise to a triple of matrices (F,G,H) with coefficients in $\underline{\underline{R}}$, $\underline{\underline{C}}$ in the continuous case or in any field (or ring for that matter) in the discrete case. Base change in all three of the spaces involved (input space, state space, output space) changes the triple (F,G,H) into $(T_2FT_2^{-1}, T_2GT_1^{-1}, T_3HT_2^{-1})$ where the $T_i$, $i = 1,2,3$, are invertible matrices of the appropriate sizes.

In other words the study of linear dynamical systems under base change in input space, state space and output space is the same as the study of the representations up to isomorphism of the quiver



which is the quiver 2.2(c). If one neglects outputs one obtains instead the quiver 2.2(f).

For a description of some of the results obtained recently for these quivers cf. section 4 below.

## 3. GABRIEL'S THEOREM AND ITS RELATIVES

One of the really beautiful results in the theory of representations of quivers (and also the result which started the business) is Gabriel's theorem which describes all quivers which have--up to isomorphism--only finitely many indecomposables representations. First a definition.

### 3.1 Definitions

A quiver Q is of finite type if there exist up to isomorphism only finitely many indecomposable representations; the quiver Q is tame if there are infinitely many isomorphism classes of indecomposable representations but these classes can be parametrized by a finite set of integers together with an irreducible polynomial (over the field k one happens to work over); the quiver Q is wild if given a finite dimensional k-algebra E there are infinitely many pair-wise non-isomorphic representations of Q with endomorphism algebra isomorphic to E.

These classes of quivers are clearly exclusive. They are also, as it turns out, exhaustive.

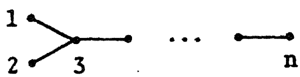
### 3.2 Gabriel's theorem

The quivers of finite type are those whose underlying undirected graph is of one of the following types:

$A_n$:


$n \geq 1$

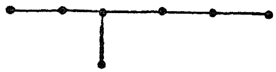
$D_n$:



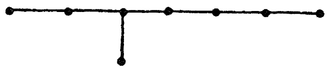

$n \geq 4$

$E_6$:

$E_7$:



$E_8$:



It is not an accident that the graphs above are Dynkin diagrams. For deatils cf. [7] and [1] and also [4] for where and how the other Dynkin diagrams fit.

3.3 Nazarova [18] has similarly described all quivers which are tame. These have as underlying undirected graphs one of the following extended Dynkin diagrams.

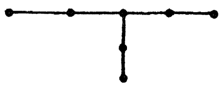
$\hat{A}_n$:


$n \geq 0$

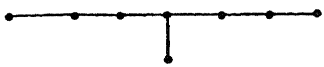
$\hat{D}_n$:


$n \geq 4$

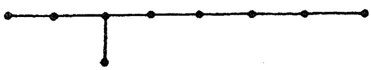
$\hat{E}_6$:



$\hat{E}_7$:



$\hat{E}_8$:



3.4 All other quivers are wild. So that in particular the quivers of algebraic system theory 2.2(c) and 2.2(f) are wild. Also wild are the quivers 2.2(g) and 2.2(e). The quivers 2.2(a), 2.2(b), 2.2(d) and 2.2(h) are all tame.

3.5 The quadratic form of a quiver

Let Q be a quiver. We attach to Q a quadratic form in as many variables $X_P$ as there are elements in $P_Q$. The quadratic form is

$$K(\ldots,X_P,\ldots) = \sum_{P \in P_Q} X_P^2 - \sum_{a \in A_Q} X_{s(a)}X_{r(a)}$$

Thus e.g. if Q is of type $A_4$ we find a form

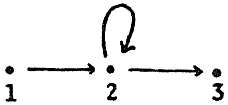
$$X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_1X_2 - X_2X_3 - X_3X_4 =$$

$$\tfrac{1}{2}X_1^2 + \tfrac{1}{2}(X_3-X_2)^2 + \tfrac{1}{2}(X_2-X_3)^2 + \tfrac{1}{2}(X_3-X_4)^{\frac{1}{2}} + \tfrac{1}{2}X_4^2$$

It now turns out that a quiver is respectively of finite type, tame or wild if this quadratic form $K_Q$ is respectively positive definite, positive semidefinite, indefinite.

## 4. ON THE QUIVERS OF (ALGEBRAIC) LINEAR SYSTEM THEORY

We now return to the quiver 2.2(c) of linear system theory. Cf. also 2.5 above. The quiver in question is

(QL) 

4.1 A representation of this quiver with dim $V(1) = m$, $\dim V(2) = n$, $\dim V(3) = P$ is a linear dynamical system with m inputs, p outputs and state space dimension n. Let $L_{m,n,p}(k)$ be the space of all representations over the field k with these dimensions. The group $G(k) = GL_m(k) \times GL_n(k) \times GL_p(k)$ acts on $L_{m,m,p}(k)$ as $((T_1,T_2,T_3),(F,G,H)) \to (T_2FT_2^{-1},T_2GT_2\ ,T_3HT_2^{-1})$ and the isomorphism classes of representations correspond bijectively to the elements of the quotient set $L_{m,n,p}(k)/G(k)$.

Now most of the results which have been obtained recently are not about $L_{m,n,p}(k)/G(k)$ but the equally interesting related quotient $L_{m,n,p}(k)/GL_n(k)$ where $GL_n(k)$ is the subgroup $1 \times GL_n(k) \times 1$ of $G(k)$. This corresponds to a finer notion of isomorphism (more isomorphism classes); viz. two representations V, W of (QL) are isomorphic in the fine sense if there is an isomorphism $\psi : V \to W$ such that $\psi(1) = id$, $\psi(3) = id$. For later purposes we define the corresponding notions: a <u>fine subrepresentation</u> of a representation V of (QL) is a subrepresentation W such that $W(1) = V(1)$ and $V(3) = W(3)$

and we say that V is <u>finely irreducible</u> if the only fine subrepresentation of V is V itself.

4.2 Complete reachability

Recall that a triple $(F,G,H) \in L_{m,n,p}(k)$ is completely reachable if and only if the space spanned by the columns of the matrices $G, FG, \ldots F^n G$ is all of $k^n$ = state space. Thus we see that a representation $V = (F,G,H)$ of (QL) is completely reachable if and only if it is finely irreducible.

4.3 Some results on $L^{cr}_{m,n,p}(k)/GL_n(k)$

Let $L^{cr}_{m,n,p}(k)$ be the subspace of all completely reachable triples (F,G,H). First suppose that k is an algebraically closed field. Then one has:

4.3.1 $L^{cr}_{m,n,p}(k)/GL_n(k)$ is a connected nonsingular algebraic variety over k of dimension np + mn.

Let us write $M^{cr}_{m,n,p}(k)$ for this variety.

4.3.2 $M^{cr}_{1,n,p}(k) = \underline{\underline{A}}_k^{n+np}$ affine space over k of dimension n + np.

4.3.3 If $m \geq 2$ then $M^{cr}_{m,n,p}(k)$ is cohomologically nontrivial.

(For these and many related results cf. [12], [13], [14], [15], [16] and [3].)

In the special case $k = \underline{\underline{R}}$ one has that $M^{cr}_{m,n,p}(\underline{\underline{R}})$ is a smooth noncompact differentiable manifold diffeomorphic to $\underline{\underline{R}}^{n+np}$ if $m = 1$ and cohomologically nontrivial if $m \geq 2$.

In terms of representations of the quiver (QL) 4.3.1 says c.q. that every fine class of finely irreducible representations of dimensions (m,n,p) can be continuously deformed into any other fine class. In particular there are mn + np "moduli" for these classes of representations, which, of course, is not unexpected given that (QL) is of wild type.

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# MODULI AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS, III: THE ALGEBRAIC-GEOMETRIC CASE

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## 1. INTRODUCTION

In this paper we treat the algebraic-geometric version of the topological theory developed in [3]. That is we study linear dynamical systems over an algebraically closed field k

$$x_{t+1} = Fx_t + Gu_t \ , \ x_t \in k^n, \ u_t \in k^m$$
$$y_t = Hx_t \ , \qquad y_t \in k^p \tag{1.1}$$

where F,G,H are matrices with coefficients in k of the appropriate sizes. A change of basis in state space changes the triple of matrices (F,G,H) into $(TFT^{-1}, TG, HT^{-1})$ and as in [3] we are interested in such questions as the following.

Does the set of orbits under this action have a (natural) structure of an algebraic variety? Do there exist continuous canonical forms? Similar questions for the case of two matrices were studied and answered in [1], cf. also [2].

Essentially the answers are as in [3]. This paper used a moderate amount of algebraic geometry (nothing much beyond definitions). Appendices 1, 2 and 3 of [1] provide sufficient background information for this paper. (Related results, usually couched in more sophisticated algebraic-geometric language can be found in [7].) All schemes in this paper will be reduced and of finite type over k, and we shall identify them with their associated algebraic varieties of closed points. We use $\underline{\underline{A}}^r$ to denote affine space of dimension r over k, and we give the space of all triples of matrices (F,G,H) of dimension $n \times n$, $n \times m$, $p \times n$ respectively, the algebraic variety structure of $\underline{\underline{A}}^{n(n+m+p)}$. Let $L_{m,n,p}$ denote this algebraic variety.

Then the assignment

$$(T,(F,G,H)) \rightarrow (TFT^{-1},TG,HT^{-1}) = (F,G,H)^T \qquad (1.2)$$

defines an action of the algebraic group $GL_n$ of invertible $n \times n$ matrices with coefficients in k on $L_{m,n,p}$. Cf. [1] Appendix 2. We can now define what a continuous algebraic canonical form on a subvariety $L' \subset L_{m,n,p}$ would be.

## 1.3. Definition

A continuous algebraic canonical form on L' is an algebraic morphism c: L' ∩ L' such that

$$\text{for every } (F,G,H) \in L' \text{ there is a } T \in GL_n \text{ such that } (F,G,H)^T = c(F,G,H) \quad (1.3.1)$$

$$c(F,G,H) = c(\overline{F},\overline{G},\overline{H}) \text{ iff there is a } T \in GL_n \text{ such that } (F,G,H)^T = (\overline{F},\overline{G},\overline{H}) \quad (1.3.2)$$

Again, as in [3], we have that continuous algebraic canonical forms on all of $L_{m,n,p}$ cannot exist for trivial reasons. ("Jump phenomena"). The conditions "completely reachable," "completely observable," "rank of G maximal and rank of H maximal and completely reachable and completely observable" all define open subvarieties of $L_{m,n,p}$ which we shall denote with $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{\rho}_{m,n,p}$ respectively. In addition, we consider the condition "F is diagonalizable (i.e. semi-simple) with distinct eigenvalues all different from zero" which defines a (non-open) subvariety $L^{\mu}_{m,n,p}$ of $L_{m,n,p}$. Combining different attributes we have the following list of (possibly interesting) subvarieties of $L_{m,n,p}$.

## 1.4. List of subvarieties

$L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{cr,co}_{m,n,p} = L^{cr}_{m,n,p} \cap L^{co}_{m,n,p}$, $L^{\mu}_{m,n,p}$

$L^{cr,co,\mu}_{m,n,p} = L^{\mu}_{m,n,p} \cap L^{cr,co}_{m,n,p}$, $L^{\rho}_{m,n,p}$, $L^{\rho,\mu}_{m,n,p} = L^{\rho}_{m,n,p} \cap L^{\mu}_{m,n,p}$

All these subvarieties of $L_{m,n,p}$ are $GL_n$-invariant. We now have the following theorem.

1.5. Theorem

The following table gives necessary and sufficient conditions for the existence of continuous algebraic canonical forms on various subvarieties of $L_{m,n,p}$.

| | variety L' | necessary and sufficient condition for the existence of an algebraic continuous canonical form |
|---|---|---|
| (i) | $L' = L^{cr}_{m,n,p}$ | m=1 |
| (ii) | $L' = L^{co}_{m,n,p}$ | p=1 |
| (iii) | $L' = L^{cr,co}_{m,n,p}$ | m=1 or p=1 |
| (iv) | $L' = L^{cr,co,\mu}_{m,n,p}$ | m=1 or p=1 |
| (v) | $L' = L^{\rho}_{m,n,p}$ | m=1 or p=1 or m=n or p=n |
| (vi) | $L' = L^{\rho,\mu}_{m,n,p}$ | m=1 or p=1 or m=n or p=n |

This theorem is "identical" with theorem 1.7 of [3]. The proof is similar in spirit but different in details.

There is of course also a corollary similar to corollary 1.8 of [3]. We shall see that the "orbit space"

$L^{cr}_{m,n,p}/GL_n$ has the structure of a quasi-projective algebraic variety and its open subvariety $L^{cr,co}_{m,n,p}/GL_n$ is in fact a quasi-affine algebraic variety. Let $M^{cr}_{m,n,p}$ denote this algebraic variety. Then we shall also see that $M^{cr}_{m,n,p}$ is a fine moduli variety for a suitable definition of (algebraic) families of linear dynamical systems.

As we said the field k we work over is supposed to be algebraically closed. This is mainly a matter of convenience: the varieties $L^{cr,co}_{m,n,p}$, $L^{cr}_{m,n,p}$, $L_{m,n,p}$, $M^{cr,co}_{m,n,p}$, $M^{cr}_{m,n,p}$, $M^{\rho}_{m,n,p}$ are all defined over any field k; in fact they are even defined over $\mathbb{Z}$. This also explains our notation $M^{cr}_{m,n,p}(\mathbb{R})$, etc. of [3]: the underlying sets of these real manifolds are simply the real points of the variety $M^{cr}_{m,n,p}$, etc. However, some care must be taken in interpreting the results of e.g. part (iii) of theorem 1.5 in this context.

Consider e.g. the following situation: let k be a finite field; let $L^{cr,co}_{m,n,p}(k)$ be the set of all k rational points of $L^{cr,co}_{m,n,p}$, i.e. the set of all completely reachable and completely controllable triples of matrices with coefficients in k; let $GL_n(k)$ be the group of $n \times n$ matrices with coefficients in k acting on $L^{cr,co}_{m,n,p}(k)$ in the abvious way. Then part (iii) of theorem 1.5 does not say that there is no map $L^{cr,co}_{m,n,p}(k) \to L^{cr,co}_{m,n,p}(k)$ (locally) given by polynomials such that the analogues of (1.3.1) and (1.3.2) hold. E.g. such a map always exists when k is $\mathbb{F}_2$, the field of two elements. But part (iii) of theorem 1.5 does say that the map $L^{cr,co}_{m,n,p}(\bar{k}) \to L^{cr,co}_{m,n,p}(\bar{k})$ defined by the same polynomials, does not satisfy the analogues of (1.3.1) and (1.3.2). Here $\bar{k}$ is the algebraic closure of k.

A large part of the proofs and constructions of [3] can be carried through unchanged in the algebraic geometric case. In these cases we shall as a rule simply refer to the appropriate section of [3].

The contents of the paper are:

## 2. THE QUOTIENT VARIETY $M^{cr}_{m,n,p}$

### 2.1. Nice Selections

Let $(F,G,H) \in L_{m,n,p}$. The matrices R(F,G) and Q(F,H) are defined as in [3], 2.2. The conditions "R(F,G) has rank n" i.e. "complete observability" define open subvarieties of $L_{m,n,p}$ which we denote by $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$ respectively.

In addition we put $L^{cr,co}_{m,n,p} = L^{cr}_{m,n,p} \quad L^{co}_{m,n,p}$ which is also an open subvariety of $L_{m,n,p}$. As in [3], 2.3 we let $J_{n,m}$ denote the set of column indices of R(F,G). Nice selections $\alpha$(from $J_{n,m}$) and the successor indices s(a,j), j = 1,...,m of the nice selection $\alpha$ are defined as in [3], 2.3. We again have (c.f. [1] 2.4.1 for a proof).

2.1.1. Lemma

If $(F,G,H) \in L^{cr}_{m,n,p}$, then there is a nice selection such that $\det(R(F,G)_\alpha \neq 0$.

2.2. The Local Quotients $U_\alpha/GL_n$.

Let $\alpha$ be a nice selection. One defines the subvarieties of $L^{cr}_{m,n,p}$

$$U_\alpha =' \{(F,G,H) \in L_{m,n;p} | \det(R(F,G)_\alpha) \neq 0\} \qquad (2.2.1)$$

$$W_\alpha =' \{(F,G,H) \in L_{m,n,p} | R(F,G)_\alpha = I_n\} \qquad (2.2.2)$$

The map $\psi_\alpha$ of [3], 2.4.5 now defines an isomorphism of algebraic varieties

$$\psi_\alpha : \underline{A}^{nm+np} \xrightarrow{\sim} W \qquad (2.2.3)$$

We define a morphism $t_\alpha : U_\alpha \to GL_n x W_\alpha$

$$t_\alpha : (F,G,H) \to (T^{-1},(F,G,H)^T), \text{ where } T = R(F,G)_\alpha^{-1} \qquad (2.2.4)$$

2.2.5. Lemma

$t_\alpha$ is a $GL_n$-invariant isomorphism of algebraic varieties (where $GL_n$ acts on $GL_n \times W_\alpha$ by left multiplication on the left hand factor.)

2.2.6. Corollary

The (categorial) quotients $U_\alpha/GL_n$ exist (as algebraic varieties) and are isomorphic to the affine space $\underline{\underline{A}}^{nm+np}$.

This follows from 2.5.5 and the isomorphism $\psi_\alpha$. For the notion of categorical quotient cf. [1] A.2.7. As a matter of fact $U_\alpha/GL_n$ is also a geometric quotient in the sense of [6]; we shall not need this fact.

## 2.3. The Quotient Variety $M^{cr}_{m,n,p}$

We are now going to define a quotient prevariety $M^{cr}_{m,n,p}$ by gluing the local quotients $U_\alpha/GL_n$ together in suitable way. For each nice selection $\alpha$ let $V_\alpha = \underline{\underline{A}}^{mn+np}$ and for each second nice selection $\beta$ let $V_{\alpha\beta}$ be the open subvariety $V_{\alpha\beta} = \psi_\alpha^{-1}(W_\alpha \cap U_\beta)$. We define $\phi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ by the formula (identical of [3], (2.5.4)).

$$\phi_{\alpha\beta}(x) = y \leftrightarrow (F_\alpha(x),G_\alpha(x),H_\alpha(x))^T = (F_\beta(y),G_\beta(y),H_\beta(y))$$

$$\text{with } T = R(F_\alpha(x),G_\alpha(x))_\beta^{-1} \qquad (2.3.1)$$

where we have written $\psi_\alpha(x) = (F_\alpha(x),G_\alpha(x),H_\alpha(x)) \in W_\alpha$ and similarly for $\psi_\beta(y)$. These $\phi_{\alpha\beta}$ are well defined and define isomorphisms of algebraic varieties $V_{\alpha\beta} \to V_{\beta\alpha}$, which moreover satisfy the cocycle condition $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$ whenever the left hand side is defined. This means that by gluing together the various $V_\alpha$ by means of the $\phi_{\alpha\beta}$ we obtain a certain prevariety which we shall denote $M^{cr}_{m,n,p}$. To prove that $M^{cr}_{m,n,p}$ is an (abstract) variety we have to prove that it is separated. This can either be done by using the algebraic geometric version of [3], 2.5.7 or by means an embedding argument. To carry this embedding argument through we first observe.

2.3.2. Lemma

The natural projections $\pi_\alpha$: $U_\alpha \to V_\alpha$ combine to define an algebraic morphism $\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$, and $\pi$ is a categorical quotient in the category of prevarieties for the action of $GL_n$ on $L^{cr}_{m,n,p}$ defined by (1.2).

Proof. It is obvious that $\pi$: $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ kills the action of $GL_n$. Now let $\phi: L^{cr}_{n,n,p} \to X$ be any morphism which kills the action of $GL_n$. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then we know that $U_\alpha \to V_\alpha$ and $U_{\alpha\beta} \to V_{\alpha\beta}$ are categorical quotients by 2.2.6. Let $\phi_\alpha$ be the restriction of $\phi$ to $U_\alpha$. By the categorical quotient property of $U_\alpha \to V_\alpha$ there are unique morphisms $\chi_\alpha$: $V_\alpha \to X$ such that $\phi_\alpha = \chi_\alpha\pi_\alpha$. Because $U_{\alpha\beta} \to V_{\alpha\beta}$ are categorical quotients we also know that $\chi_\beta\phi_{\alpha\beta}(x) = \chi_\alpha(x)$ for $x \in V_{\alpha\beta}$, where $\phi_{\alpha\beta}$ is as in (2.3.1). It follows that the $\chi_\alpha$ combine to define a morphism

$\chi\colon M^{cr}_{m,n,p} \to X$ such that $\phi = \chi\pi$. The morphism $\chi$ is unique because on each $V_\alpha$ it must equal $\chi_\alpha$. Essentially the same proof was used for [1], 3.2.14.

### 2.3.3. The Morphisms

$h\colon L_{m,n,p} \to \underline{A}^r$ and $g\colon L_{m,n,p} \to G_{n,(n+1)m}$. Let $(F,G,H) \in L_{m,n,p}$. We let $h(F,G,H) \in \underline{A}^r$, where $r = (n+1)^2 mp$, be the block Hankel matrix

$$h(F,G,H) = \begin{pmatrix} HG & HFG & \cdot & \cdot & \cdot & HF^nG \\ HFG & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \\ \cdot & & & & & \cdot \\ HF^nG & & \cdot & \cdot & \cdot & HF^{2n}G \end{pmatrix} = Q(F,H)R(F,G)$$

This defines a morphism $h\colon L_{m,n,p} \to \underline{A}^r$, which certainly kills the action of $GL_n$.

Restricting to $L^{cr}_{m,n,p}$ and applying lemma 2.3.2 we obtain an induced morphism

$$\bar{h}\colon M^{cr}_{m,n,p} \to \underline{A}^r,\ r = (n+1)^2 pm \qquad (2.3.4)$$

Let $L^{cr}_{m,n}$ be the algebraic variety of all pairs of matrices $(F,G)$ of sizes $n \times n$ and $n \times m$. In [1] we constructed a morphism $g\colon L^{cr}_{m,n} \to G_{n,(n+1)m}$ which kills the action of

$GL_n$ on $L^{cr}_{m,n}$, where $G_{n,(n+1)m}$ is the Grassmann variety of n-planes in (n+1)m space; g assigns to (F,G) the point of $G_{n(n+1)m}$ corresponding to the rank n matrix R(F,G) of size nx(n+1)m. We proved that the quotient variety $M_{m,n} = L^{cr}_{m,n}/GL_n$ exists and that g induces an embedding $\bar{g}$: $M_{m,n} \to G_{n,(n+1)m}$. Cf. [1] Theorem 3.2.13 and proposition 3.2.14. Now let $g'$: $L^{cr}_{m,n,p} \to G_{n,(n+1)m}$ be the composed morphism $(F,G,H) \to (F,G) \to g(F,G)$. This morphism kills the action of $GL_n$ and hence induces a morphism

$$\hat{g}: M^{cr}_{m,n,p} \to G_{n,(n+1)m} \tag{2.3.5}$$

From the remarks made above we know that if (F,G,H), (F',G',H') (F',G',H') $\varepsilon$ $L^{cr}_{m,n,p}$ are such that g'(F,G,H) = g'(F,G,H) then there is a T $\varepsilon$ $GL_n$ such that $(F,G)^T = (\bar{F},\bar{G})$.

2.3.4 An embedding $M^{cr}_{m,n,p} \to G_{n,(n+1)m} \times \underline{\underline{A}}^r$

The morphisms h,g of (2.3.4) and (2.3.5) together define a morphism

$$i: M^{cr}_{m,n,p} \to G_{n,(n+1)m} \times \underline{\underline{A}}^r \tag{2.3.6}$$

We claim that i is injective. Indeed if (F,G,H), $(\bar{F},\bar{G},\bar{H})$ $\varepsilon$ $L^{cr}_{m,n,p}$ are such that $h(F,G,H) = h(\bar{F},\bar{G},\bar{H})$ and g'(F,G,H) $= g'(\bar{F},\bar{G},\bar{H})$, then we know that there is a T $\varepsilon$ $GL_n$ such that $(F,G)^T = (\bar{F},\bar{G})$ i.e. $TR(F,G) = R(\bar{F},\bar{G})$, and then because $h(F,GH) = h(\bar{F},\bar{G},\bar{H})$ we have in particular $HR(F,G) = \bar{H}R(\bar{F},\bar{G})$

so that $\overline{H}TR(F,G) = HR(F,G)$ and hence $\overline{H} = HT^{-1}$ because $R(F,G)$ has rank n. This concludes the proof that i is injective.

2.3.5. Corollary

The prevariety $M^{cr}_{m,n,p}$ is separated, i.e. $M^{cr}_{m,n,p}$ is a variety.

2.3.6. Corollary

$L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is a quotient for the action of $GL_n$ on $L^{cr}_{m,n,p}$ in the category of algebraic varieties. In fact $M^{cr}_{m,n,p}$ is also a geometric quotient in the sense of [6], but we shall not need this.

2.3.7..

Let $V^{co}_{\alpha} = \psi^{-1}_{\sigma}(W_{\alpha} \cap L^{cr,co}_{m,n,p})$ and $V^{co}_{\alpha\beta} = V^{co}_{\alpha} \cap V_{\alpha\beta}$. Then the $\phi_{\alpha\beta}$: $V_{\alpha\beta} \to V_{\beta\alpha}$ induce isomorphisms $\phi^{co}_{\alpha\beta}$: $V^{co}_{\alpha\beta} \to V^{co}_{\beta\alpha}$. Gluing together the $V^{co}_{\alpha}$ by means of the $\phi^{co}_{\alpha\beta}$ we obtain an open subvariety $M^{cr,co}_{m,n,p}$ of $M^{cr}_{m,n,p}$ which is the image of $L^{cr,co}_{m,n,p}$ under $\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$. It follows that the induced morphism $\pi^{co}$: $L^{cr,co}_{m,n,p} \to M^{cr,co}_{m,n,p}$ is also a categorical quotient.

2.3.8.

Similarly, using $V^{\rho}_{\alpha} = \psi^{-1}_{\alpha}(W_{\alpha} \cap L^{\rho}_{m,n,p})$,

$V_\alpha^{\mu\rho} = \psi_\alpha^{-1}(W_\alpha \cap L^\mu_{m,n,p})$, $V_\alpha^{\mu\rho} = \psi_\alpha^{-1}(W_\alpha \cap L^{\mu,\rho}_{m,n,p})$ and the corresponding $V^-_{\alpha\beta}$ we obtain categorical quotients $L^\rho_{m,n,p} \to M^\rho_{m,n,p}$, $L^\mu_{m,n,p} \to M^\mu_{m,n,p}$, $L^{\rho,\mu}_{m,n,p}$ $M^{\rho,\mu}_{m,n,p}$ where the $M^-_{m,n,p}$ are subvarieties of $M^{cr}_{m,n,p}$.

## 2.4. Some Realization Theory

The morphism $\bar{h}$ of (2.3.4) above induces a morphism $\hat{h}: M^{cr,co}_{m,n,p} \to \underline{\underline{A}}^r$. It is the purpose of this and the following subsection to show that $\hat{h}$ is injective and to derive equations for the subvariety $\hat{h}(M^{cr,co}_{m,n,p}) \subset \underline{\underline{A}}^r$. To do this we use some (partial) realization theory an embodied by proposition 2.4.3 below. First a definition

### 2.4.1. Definition

Let $A_o, A_1, \ldots$ be a sequence of $p \times m$ matrices. Then $h_{q,r}(A)$ denotes the block Hankel matrix

$$h_{q,r}(A) = \begin{pmatrix} A_o & A_1 & \cdot & \cdot & \cdot & A_r \\ A_1 & & & & & \\ \vdots & & & & & \vdots \\ A_q & \cdot & & \cdot & \cdot & A_{r+q} \end{pmatrix}$$

### 2.4.2. Definition

If A is a matrix and $\alpha_c$ is a subset of the column indices of A, and $\alpha_r$ is a subset of the row indices of A, then we define

$A_{\alpha_c}$ = matrix obtained from A by removing all columns whose index is not in $\alpha_c$

$A_{\alpha_r}$ = matrix obtained from A by removing all rows whose index is not in $\alpha_r$

$A_{\alpha_r,\alpha_c}$ = matrix obtained from A by removing all rows and columns whose indices are not in $\alpha_r$, $\alpha_c$ respectively.

2.4.3. Proposition

Let $A_0, A_1, \ldots, A_{2n-1}$ be a sequence of 2n matrices with coefficients in k, all of size $p \times m$, and suppose that $\operatorname{rank}(h_{n-1,n-1}(A)) = \operatorname{rank}(h_{n,n-1}(A)) = \operatorname{rank}(h_{n-1,n}(A)) = n$. Then there exists an $(F,G,H) \in L^{cr,co}_{m,n,p}$ such that $HF^iG = A_i$ for $i = 0,1,\ldots,2n-1$.

Moreover, if $(\bar{F},\bar{G},\bar{H}) \in L^{cr.co}_{m,n,p}$ is a second triple such that $\overline{HF^iG} = A_i$ for $i = 0,1,\ldots,2n-1$ then there is a $T \in GL_n$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^T$.

Proof. Existence of a triple $(F,G,H) \in L_{m,n,p}$ such that

$$HF^iG = A_i, \quad i = 0,\ldots,2n-1 \tag{2.4.4}$$

holds is assured by the realizability criterion 11.32 of

Chapter 10 of [4]. We define

$$\overline{R}(F,G) = (G,FG\ldots F^{n-1}G),$$
$$\overline{Q}(F,H)' = (H'F'H'\ldots(F')^{n-1}H') \qquad (2.4.5)$$

Then it follows from (2.4.3) that $\overline{Q}(F,H)\overline{R}(F,H) = h_{n-1,n-1}(A)$. Now we have rank $(R(F,G)) \leq n$, rank$(Q(F,H)) \leq n$ and rank$(h_{n-1,n-1}(A)) = n$. It follows that rank$(R(F,G))$ = rank$(Q(F,H)) = n$, so that $(F,G,H) \in L^{cr,co}_{m,n,p}$. Not let $(\overline{F},\overline{G},\overline{H})$ be a second triple in $L_{m,n,p}$ such that

$$\overline{H}\overline{F}^{i}\overline{G} = A_i, \; i = 0,1,\ldots,2n-1 \qquad (2.4.6)$$

Then as above we find $\overline{Q}(\overline{F},\overline{H})\overline{R}(\overline{F},\overline{G}) = h_{n-1,n-1}(A)$. Now because $\overline{R}(F,G)$ has rank n there is a subset $\alpha_c$ of size n of the column indices of $R(F,G)$ such that $R(F,G)_{\alpha_c}$ is invertible; further because $\overline{Q}(F,H)$ has rank n there is a subset $\alpha_r$ of size n of the row indices of $\overline{Q}(F,H)$ such that $\overline{Q}(F,H)_{r}$ is invertible. We have

$$(h_{n-1,n-1}(A))_{\alpha_r,\alpha_c} = \overline{Q}(F,H)_{\alpha_r}\overline{R}(F,G)_{\alpha_c} = \qquad (2.4.7)$$
$$\overline{Q}(\overline{F},\overline{H})_{\alpha_r}\overline{R}(\overline{F},\overline{G})_{\alpha_c}$$

so that it follows that all five $n \times n$ matrices occurring in (2.4.7) are invertible.

Now let

$$(F_1,G_1,H_1) = (F,G,H)^T, \text{ where } T = \overline{Q}(F,H)_{\alpha_r} \quad (2.4.8)$$

$$(\overline{F}_1,\overline{G}_1,\overline{H}_1) = (\overline{F},\overline{G},\overline{H})^T, \text{ where } \overline{T} = \overline{Q}(\overline{F},\overline{H})_{\alpha_r} \quad (2.4.9)$$

Then we have of course

$$H_1F_1^iG_1 = \overline{H}_1\overline{F}_1^i\overline{G}_1 = A_i \text{ for } i = 0,\dots,2n-1 \quad (2.4.10)$$

which means

$$\overline{Q}(F_1,H_1)R(F_1,G_1) = \overline{Q}(\overline{F}_1,\overline{H}_1)R(\overline{F}_1,\overline{G}_1) = h_{n-1,n}(A) \quad (2.4.11)$$

and moreover because

$$\overline{Q}(F_1,H_1) = \overline{Q}(F,H)T^{-1}, \ \overline{Q}(\overline{F}_1,\overline{H}_1) = \overline{Q}(\overline{F},\overline{H})\overline{T}^{-1}$$

we have

$$Q(F_1,H_1)_{\alpha_r} = I_n = Q(F_1,H_1)_{\alpha_r} \quad (2.4.12)$$

Now combine (2.4.12) and (2.4.11) to obtain that $R(F_1,G_1) = R(F_1,G_1)$ which be corollary 2.4.2 of [1] means that $F_1 = \overline{F}_1$ and $G_1 = \overline{G}_1$. and because $R(\overline{F}\ ,\overline{G}\ ) = R(F\ ,G\ )$ has rank n, it follows from (2.4.11) that also $H_1 = \overline{H}_1$. We therefore have $(F,G,H)^T = (F_1,G_1,H_1) = (\overline{F}_1,\overline{G}_1,\overline{H}_1) = (\overline{F},\overline{G},\overline{H})^T$, which proves the second statement of the proposition.

2.4.4. <u>Corollary</u>

The morphism $\hat{h}: M^{cr,co}_{m,n,p} \to \underline{\underline{A}}^r$ of (2.3.4) above is injective.

## 2.5. Equations for $M^{cr,co}_{m,n,p}$

By means of the injective morphism $\hat{h}$ we can now consider $M^{cr,co}_{m,n,p}$ as a subvariety of $\underline{\underline{A}}^r$, $r = (n+1)^2pm$, where we write $x \in \underline{\underline{A}}^r$ as an $(n+1)n \times (n+1)m$ matrix. We now consider the following sets of polynomials in the coordinates of $\underline{\underline{A}}^r$.

> $P_a(x)$: these polynomials are such that $P_a(x) = 0$ for all a if and only if the matrix x is of block Hankel type (cf. 2.4.1) with the blocks of size $p \times m$. (2.5.1)
>
> $Q_b(x)$: here $Q_b(x)$ runs through all determinants of $(n+1) \times (n+1)$ submatrices of x. (2.5.2)
>
> $R_c(x)$: here $R_c(x)$ runs through all determinants of $n \times n$ submatrices of the submatrix x' of x which is obtained by removing the last p rows and the last m columns.

### 2.5.4. Lemma

Let $(F,G,H) \in L^{cr,co}_{m,n,p}$, $x = h(F,G,H) \in \underline{\underline{A}}^r$. Then we have $P_a(x) = 0$ for all a, $Q_b(x) = 0$ for all b and there is a c such that $R_c(x) \neq 0$.

Proof. Obvious because $h(F,G,H) = Q(F,H)R(F,G)$.

2.5.5. Proposition

$\hat{h}(M^{cr,co}_{m,n,p})$ $\underline{\underline{A}}^r$ is the subvariety consisting of those $x \in \underline{\underline{A}}^r$ such that $P_a(x) = 0$ for all a, $Q_b(x) = 0$ for all b and such that these is a c such that $R_c(x) \neq 0$.

Proof. Because of lemma 2.5.4 we only have to show that if $x \in \underline{\underline{A}}^s$ satisfies $P_a(x) = 0$ all a, $Q_b(x) = 0$ all b and $R_c(x) \neq 0$ for some c, then x is in $\hat{h}(M^{cr,co}_{m,n,p})$. Write x as a block Hankel matrix

$$x = \begin{bmatrix} A_1 & A_2 & \cdot & \cdot & \cdot & A_n \\ A_2 & & & & & \\ \vdots & & & & & \vdots \\ A_n & & \cdot & \cdot & \cdot & A_{2n} \end{bmatrix}$$

This can be done because $P_a(x) = 0$ for all a. Cf. 2.5.1. Then the matrices $A_1,\ldots,A_{2n-1}$ satisfy the conditions of proposition 2.4.3 so that there is a triple $(F,G,H) \in L^{cr,co}_{m,n,p}$ such that $HF^iG = A_i$ for $i = 0,1,2,\ldots,2n-1$. To show that $h(F,G,H) = x$ it therefore only remains to show that $HF^{2n}G = A_{2n}$. This follows from lemma 2.5.6 below.

2.5.6 Lemma

Let E,E' be two partitioned matrices

$$E = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \qquad E' = \left(\begin{array}{c|c} A & B \\ \hline C & D' \end{array}\right)$$

and suppose that rank(E) = rank(E') = rank(A). Then D = D'.

Proof. Let d be an element of D and d' the corresponding element of D'. Let A' be an $n \times n$ submatrix of A such that $\det(A') \neq 0$ where $n = \mathrm{rank}(A)$. Suppose $A' = E_{\alpha_r,\alpha_c}$, then also $A' = E'_{\alpha_r,\alpha_c}$. Let $\beta_r = \alpha_r \cup \{i\}$ where i is the index of the row in E containing d (and of the row in E' containing d') and $\beta_c = \alpha_c \cup \{j\}$ where j is the index of the column in E containing d (and of the column in E' containing d'). Then we have $\det(E_{\beta_r,\beta_c}) = 0 = \det(E'_{\beta_r,\beta_c})$. All elements of $E_{\beta_r,\beta_c}$ and $E'_{\beta_r,\beta_c}$ except possibly the one in the right hand lower corner are equal and $\det(A') \neq 0$. It follows that d = d'. (By expanding the determinants along the last row e.g.).

2.5.7. Corollary (of proposition 2.5.5.)

$M^{cr,co}_{m,n,p}$ is a quasiaffine variety.

2.5.8.

Using similar arguments as above combined with those used in [1] to find equations for the variety $M_{m,n}$ (cf.

[1] section 3.2), it is not difficult to find equations for the variety $M^{cr}_{m,n,p}$ (as a subvariety of $G_{n,(n+1)m} \times \underline{A}^r$ or as a subvariety of $\underline{P}^{r'} \times \underline{A}^r$, where $r' = \binom{(n+1)m}{n} - 1$).

$M^{cr}_{m,n,p}$ is a quasiprojective variety but not a quasi affine variety if $m > 1$. This last statement is seen as follows. The embedding $L^{cr}_{m,n,p} \to L^{cr}_{m,n,p}$ given by $(FG) \to (F,G,0)$ where 0 is zero matrix of appropriate size, induces an embedding $M_{m,n} \to M^{cr}_{m,n,p}$. Now according to [1] section 3.3 there is an embedding $\underline{P}^1 \to M_{m,n}$. Combining these we find an embedding $\underline{P}^1 \to M^{cr}_{m,n,p}$ which shows that $M^{cr}_{m,n,p}$ is not quasi affine. (Cf. also the proof of theorem 3.4.6 in [1]).

## 2.6 The Algebraic Principal Fiber Bundle

$\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$. As in [3] we can now show that $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is an algebraic principal $GL_n$ fibre bundle over the variety $M^{cr}_{m,n,p}$, and we could use an analysis of the nontriviality or triviality of this bundle to prove nonexistence and existence of algebraic continuous canonical forms. This can be done almost exactly as in [3] section 3 except that one has to construct a different example because the example of [3], section 3.2 is essentially nonalgebraic. Cf. also section 4.1 below for further comments. In this paper, however, we shall first discuss the fine moduli variety properties of $M^{cr}_{m,n,p}$ and then use these to investigate the existence of continuous algebraic canonical forms; this is the same procedure as in [2], cf. especially theorem 6.1 of [2]. The two approaches are essentially equivalent because the underlying vectorbundle of the universal family over $M^{cr}_{m,n,p}$ is the algebraic n-vectorbundle associated to the principal $GL_n$ bundle $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$.

## 2.7 <u>The Codimension of $(M^{cr}_{m,n,p} \setminus M^{cr,co}_{m,n,p})$ in $M^{cr}_{m,n,p}$.</u>

Let $K_{m,n,p}$ be the subvariety of $M^{cr}_{m,n,p}$ defined by the equations $\det(Q(F,H))_{\beta}) = 0$ for all subsets of size n of the row indices of $Q(F,H)$. i.e.

$$K_{m,n,p} = M^{cr}_{m,n,p} \setminus M^{cr,co}_{m,n,p} \qquad (2.7.1)$$

We want to find out something about the codimension of the closed subvariety $K_{m,n,p}$ of $M^{cr}_{m,n,p}$. The result is:

2.7.2. <u>Proposition</u>

The codimension of $K_{m,n,p}$ in $M^{cr}_{m,n,p}$ is 1 if $p = 1$ and it is $\geq p$ if $p \geq 2$.

To prove proposition 2.7.2 we use the following combinatorial lemma.

2.7.3 Lemma

Let $X = \{a_1, \ldots, a_n\}$ be a finite set of n elements. Let $X_o$ be a subset of X and $\sigma: X_o \to X$ an injective map with the following property

$$\text{If } Y \subset X_o \text{ then } \sigma(Y) \not\subset Y \text{ unless } Y = X_o = X. \quad (2.7.2)$$

Then there exists a cyclic permutation $\tilde{\sigma}: X \to X$ of order n such that $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in X_o$.

Proof. If $X_o = X$ then condition (2.7.4) says that $\sigma$ is already a cyclic permutation of order n. We can therefore assume that $X_o \neq X$. We are going to show that there is $b \in X \setminus X_o$ and an injective map $\sigma_1: X_1 \to X$ with $X_1 = X_o \cup \{b\}$ and $\sigma_1(a) = \sigma(a)$ for $a \in X_o$ such that (2.7.4) holds with $X_o$ replaced by $X_1$. By induction (with respect to the number of elements in $X \setminus X_o$) this proves the lemma. Because $X_o \neq X$ there is an $a_1 \in X$ which is not in the image of $\sigma$. If $a_1 \in X_o$ let $a_2 = \sigma(a_1)$, if $a_1 \notin X_o$ stop; if $a_2 \in X_o$ let $a_3 = \sigma(a_2)$, if $a_2 \notin X_o$ stop; continuing in this way we find a sequence of elements $a_1, a_2, \ldots, a_r$, $r \geq 1$ such that

$$a_1 \notin \mathrm{Im}(\sigma),\ a_i = \sigma(a_{i-1}) \text{ for } i = 1, \ldots, r-1,$$

$$a_r \notin X_o$$

Note that the $a_1, \ldots, a_r$ are all different from one another

$a_{r-1} \in Y$, $a_{r-2} \in Y, \ldots, a_1 \in Y \subset X_1$, which is a contradiction because there is no $c \in X$ such that $\sigma(c) = a_1$ because $a_1 \notin \mathrm{Im}\sigma_1 = \mathrm{Im}\sigma \cup \{b_1\}$. This concludes the proof of the lemma.

2.7.6.

Now consider $x \in \underline{A}^{mn}$; consider $(F_\alpha(x), G_\alpha(x)$, where $\alpha$ is a nice selection, $\alpha \subset J_{m,n}$. We recall how $F_\alpha(x)$ and $G_\alpha(x)$ are defined (cf. 1 section 2.3). Let $J = \alpha \cup \{s(\alpha,1),\ldots,s(\alpha,m)\}$ as an ordered subset of $J_{m,n}$. Let $x_1$ be the column vector consisting of the first n coordinates of x, $x_2$ the column vector consisting of the second n coordinates, etc. We now define n + m column vectors $y_i$, $i = 1,2,\ldots,m+n$ of length n as follows

$$y = \begin{cases} e_\ell & \text{if the i-th element of J is the } \ell\text{-th element of } \alpha \\ x_j & \text{if the i-th element of J is } s(\alpha,j) \end{cases} \qquad (2.7.3)$$

where $e_\ell$ is the $\ell$-th standard basis vector.

The matrices $G_\alpha(x)$ and $F_\alpha(x)$ are now defined by

$$G_\alpha(x)_j = y_j,\ j = 1,\ldots,m; \quad F_\alpha(x)_j = y_{m+j},\ j = 1,\ldots,n \qquad (2.7.4)$$

It readily follows from this that $F_\alpha(x)$ is a matrix of because $\sigma$ is injective and $a_1 \notin \mathrm{Im}\sigma$. There now are two possibilities

(i) There is no $b \notin X \setminus \mathrm{Im}\sigma$ different from $a_1$.

In this case $\mathrm{Im}\sigma$ has n-1 elements and hence so has $X_o$. Therefore $X \setminus X_o = \{a_r\}$. Let $Y = X \setminus \{a_1,...,a_r\}$ and suppose $Y \neq \emptyset$. Then we have $Y \subset X_o$ because $X \setminus X_o = \{a_r\}$. We also have $\sigma(Y) \subset Y$ because $\sigma(\{a_1,...a_r\} \cap X_o) \subset \{a_1,...,a_r\}$.

Therefore, because is injective, we would have $\sigma(Y) = Y$ contradicting (2.7.2). Therefore $Y = \emptyset$ and $X = \{a_1,...,a_r\}$ in this case (i.e. $r = n$). We now take $b = a_1$ and define $\sigma_1(b) = a_1$. Then $X_1 = X_o \cup \{a_r\} = X$ and $\sigma_1 : X \to X$ is clearly the desired cyclic permutation.

(ii) There is a $b_1 \in X$ Im which is different from $a_1$. In this case we take $b = a_r$ and define $\sigma_1(b) = b_1$. The map $\sigma_1$ is injective because $b_1 \notin \mathrm{Im}\sigma$. Now suppose $Y \subset X_1$ is such that $\sigma_1(Y) = Y$. Note that in this case $X \setminus \mathrm{Im}\sigma$ has at least two elements, hence so has $X - X_o$, so that $X_1 \neq X$. There are two possibilities.

($ii_1$) $b = a_r \notin Y$. In this case $Y = \sigma_1(Y) = \sigma(Y)$ and $Y \subset X_o$ which contradicts (2.7.4).

($ii_2$) $b = a_r \in Y$. Then because $\sigma_1(Y) = Y$ we must have the following type: the j-th column of $F_\alpha(x)$ is either a standard basis vector $e_\ell$ with $\ell > j$, or $F_\alpha(x)_j = x_i$ for some i; moreover if $F_\alpha(x)_{j_1} = e_{\ell_1}$, $F_\alpha(x)_{j_2} = e_{\ell_2}$ with $j_1 < j_2$ then $\ell_1 < \ell_2$. Applying lemma 2.7.3 we thus see that by specifying the $x_i$, $i = 1,\dots,m$ to be suitable standard basis vectors one obtains

2.7.9 Lemma

For every nice selection $\alpha$, there is an $x \in \underline{\underline{A}}_{mn}$ such that $F_\alpha(x)$ is a cyclic permutation of order n of the standard basis vectors.

2.7.10.

Let $\alpha$ be a nice selection. Now consider $K_{m,n,p} \cap V_\alpha = V_\alpha \setminus V_\alpha^{co}$ where $\alpha$ is a nice selection. This closed subvariety of $U_\alpha$ is defined by the equations $\det(Q(F_\alpha(x), H_\alpha(x))_\beta) = 0$ for all subsets $\beta$ of size n of the row indices of $Q(F_\alpha(x),H_\alpha(x))$. We number the rows of $Q(F_\alpha(x),H_\alpha(x))$ as follows

$$(0,1),\dots,(0,p);\ (1,p),\dots,(1,p);\dots$$

$$\dots,(n,1),\dots,(n,p))$$

Take $\beta_1 = \{(0,1),(1,1),\dots,(n-1,1)\}$. Write $x \in V_\alpha = \underline{\underline{A}}^{mn} \times \underline{\underline{A}}^{pn}$ as $x = (y,z)$ and write z as the matrix $(z_{ij})$,

$i = 1,\ldots,p$, $j = 1,\ldots,n$. We write $F_\alpha(x) = F_\alpha(y)$, $H_\alpha(x) = z$. Now consider the equation

$$\det(Q(F_\alpha(x),H_\alpha(x))_{\beta_1}) = 0 \qquad (2.7.5)$$

Now specify the y such that $F_\alpha(x)$ is a cyclic permutation matrix of order n and suppose that the first row vector of $F_\alpha(x)$ under this specification is the $\ell$-th standard basis vector. Now take $z_{ij} = 0$ for $j \neq \ell$. Then (2.7.5) becomes

$$\pm\, z_{1\ell}^{n} = 0 \qquad (2.7.6)$$

If $p = 1$, equation (2.7.11) defines $K_{m,n,p} \cap V_\alpha$ (because if rank $Q(F,G) = n$ then there is a nice "selection" $\beta$ from the row indices of $Q(F,H)$ such that $\det(Q(F,H)_\beta) \neq 0$ by the transposed version of lemma 2.1.1). Equation (2.7.12) which is obtained from (2.7.11) by a suitable specification of some of the variables shows that (2.7.5) is nontrivial, so that the codimension of $K_{m,n,p} \cap V_\alpha$ in $V_\alpha$ is one for each nice selection $\alpha$ proving that the codimension of $K_{m,n,1}$ in $M^{cr}_{m,n,p}$ is one. Now suppose that $p > 1$. And consider the selections

$$\beta_i = \{(0,i),(1,i),\ldots,(n-1,i)\} \quad i = 1,\ldots,p$$

Specifying the y and z as before (NB the specification to be used depends on $\alpha$!), the equations

$$\det(Q(F_\alpha(x),H_\alpha(x))_{\beta_i}) = 0 \quad i = 1,\ldots,p \qquad (2.7.7)$$

specify to

$$\pm z^n_{i\ell} = 0 \quad i = 1,\ldots,p \qquad (2.7.8)$$

The equations (2.7.14) are independent, hence so are the equations (2.7.13) proving that the codimension of $K_{m,n,p} \cap V_\alpha$ in $V_\alpha$ is $\geq p$. This holds for all nice selections $\alpha$ so that the codimension of $K_{m,n,p}$ in $M^{cr}_{m,n,p}$ is always $\geq p$. We have now proved assertion 2.7.2.

2.7.12. Remark

To prove 2.7.2 all one really needs is the existence of a triple $(F,G,H) \in W_\alpha$ for each $\alpha$ such that $F'$ is a cyclic matrix. This can be seen as follows: $U$ is a non-empty open subvariety of $L_{m,n,p}$. Let $L' = \{(F,G,H) \in L_{m,n,p} | F' \text{ is cyclic}\}$ this also defined a nonempty open subvariety of $L_{m,n,p}$. Because $L_{m,n,p}$ is irreducible $L' \cap U_\alpha \neq \phi$. Let $(F,G,H) \in L' \cap U$ and let $(F,G,H) = (F,G,H)^T$ where $T = R(F,G)^{-1}_\alpha$. Then $(F,G,H) \in W_\alpha$ and $F'$ is cyclic.

## 3. THE FINE MODULI VARIETY $M^{cr}_{m,n,p}$

We now proceed to study families of linear dynamical systems. Some motivation as to why one would like to study families is given in section 1.8 of [3]. Moreover, in this paper we shall use families to investigate whether there exist continuous canonical forms or not. This is not necessary; one can also use the principal algebraic $GL_n$ bundle $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$. Cf. also 2.6 above. This part of the theory in the algebraic geometric case is practically completely analogous to the corresponding part of the topological case which was treated in section 4 of [3].

## 3.1. Families of Linear Dynamical Systems

### 3.1.1. Definition

A family of linear dynamical systems over a variety S of dimensions (n,m,p) consists of

(i) an algebraic n-dimensional vectorbundle $p:E \to S$

(ii) an algebraic vectorbundle endomorphism $F:E \to E$

(iii) an algebraic vectorbundle homomorphism $G:S\times A^m \to E$

(iv) an algebraic vectorbundle homomorphism $H:E \to S\times A^p$.

Let $s \in S$, then F,G,H induce homomorphisms $F_s:E_s \to E_s$, $G_s:s\times\underline{A}^m \to E_s$, $H_s:E_s \to s\times\underline{A}^p$; $E_s = p^{-1}(s)$ is the fibre over s. (Cf. Appendix 3 of [1]). Choosing a basis $e_1(s),\dots,e_n(s)$ of $E_s$ and taking the obvious bases in $s\times\underline{A}^m$ and $s\times\underline{A}^p$ we calculate the matrices of $F_s,G_s,H_s$ relative these bases. Let the result be

$(F(s,e),G(s,e),H(s,e)$. This triple depends on $e_1(s),\ldots,$ $e_n(s)$ only to the extent that a different choice of $e_1(s),\ldots,e_n(s)$ gives a triple in the same orbit (under $GL_n$) as $(F(s,e),G(s,e),H(s,e))$.

The family $\Sigma$ is said to be <u>completely reachable</u> if $(F(s,e),G(s,e)H(s,e)) \in L^{cr}_{m,n,p}$ for all s. (This is well defined because $L^{cr}_{m,n,p}$ is $GL_n$ invariant).

### 3.1.2. The Canonical Morphism Associated to Completely Reachable Family

Now let $\Sigma$ be a completely reachable family. Then $F_s,G_s,H_s$ define a unique orbit in $L^{cr}_{m,n,p}$ and thus a unique point in $M^{cr}_{m,n,p}$ which we shall denote $f_\Sigma(s)$. Thus we have a map $f_\Sigma : S \to M^{cr}_{m,n,p}$. Using the local triviality of the bundle E one shows by means of the algebraic analogues of the constructions in 4.1.2 - 4.1.8 of [3] that $f_\Sigma$ is a morphism in the category of varieties.

### 3.1.3

In the topological case we associated a continuous map $f_\Sigma : X \to M_{m,n,p}(\mathbb{R})$ to every family $\Sigma$, and used this map to define complete reachability of families. This cannot be done in the algebraic geometric case because the variety $M_{m,n,p}$ does not exist.

3.2. The Universal Family $\Sigma^u$ over $M_{m,n,p}$.

Let $\alpha$ be a nice selection. Let $E_\alpha = V_\alpha \times \underline{\underline{A}}^n$, $p_\alpha$: $E_\alpha \to V_\alpha$ the obvious projection. We define families $\Sigma_\alpha$ of linear dynaminal systems with underlying bundles $E_\alpha$ by the formulas

$$F_\alpha(x,v) = (x,F_\alpha(x)v),\ G_\alpha(x,u) = (x,G_\alpha(x)u),$$
$$H_\alpha(x,v) = (x,H_\alpha(x)v) \qquad (3.2.1)$$

where for $x \in V_\alpha$, $\psi_\alpha(x) = (F_\alpha(x),G_\alpha(x),H_\alpha(x))$, cf. [3]. 2.4.5

Now let $E_{\alpha\beta} = V_{\alpha\beta} \times \underline{\underline{A}}^n$ and define the isomorphisms $\phi_{\alpha\beta}$: $E_{\alpha\beta} \to E_{\beta\alpha}$ by formula (4.3.6) of [3]. Then glueing together the $E_\alpha$ by means of the $\phi_{\alpha\beta}$ we obtain an algebraic vectorbundle $E^u$. The $F_\alpha, G_\alpha, H_\alpha$ are compatible with the $\phi_{\alpha\beta}$ in the sense of (4.3.9) - (4.3.11) of [3] and thus define homomorphisms $F^u$: $E^u \to E^u$, $g^u$: $M^{cr}_{m,n,p} \times \underline{\underline{A}}^m \to E^u$, $H^u$: $E^u \to M^{cr}_{m,n,p} \times \underline{\underline{A}}^p$. This defines the family $\Sigma^u$. The family $\Sigma^u$ is completely reachable (because this is true for the families $\Sigma_\alpha$), and the associated map $f_{\Sigma^u}$: $M^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is the identity map (because the triple $(F_\alpha(x), G_\alpha(x),H_\alpha(x))$ maps to $x \in V_\alpha \subset M_{m,n,p}$ under $\pi$: $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$).

3.3. The Fine Moduli Variety $M^{cr}_{m,n,p}$

3.3.1

Two families $\Sigma$, $\overline{\Sigma}$ are isomorphic if there is an algebraic vectorbundle isomorphism $\phi:E \to E$ such that $\overline{F}\phi = \phi F$, $\phi G = \overline{G}$, $H = \overline{H}\phi$. For each $S \varepsilon$ $\underline{\underline{Sch}}_k$, the category of algebraic varieties over k, let $\Phi_{m,n,p}(S)$ be the set of isomorphism classes of completely reachable families of linear dynamical systems over S. By means of the pullback construction we turn $\Phi_{m,n,p}(S)$ into a functor $\Phi_{m,n,p}:\underline{\underline{Sch}}_k^{opp} \to \underline{\underline{Set}}$.

3.3.2. Theorem

The variety $M^{cr}_{m,n,p}$ is a fine moduli variety for $\Phi_{m,n,p}$ or, in other words, the functor $\Phi_{m,n,p}$ is representable by $M^{cr}_{m,n,p}$. More precisely, the assignment $\Sigma \to f_\Sigma$ induces a functorial isomorphism $\Phi_{m,n,p}(S) \to \underline{\underline{Sch}}_k(S,M^{cr}_{m,n,p})$; the inverse isomorphism assigns the isomorphism class of $f^!\Sigma^u$ to $f: S \to M^{cr}_{m,n,p}$.

Proof. Identical with the proof of the corresponding theorem 4.5.2 of [3].

## 4. EXISTENCE AND NONEXISTENCE OF ALGEBRAIC CONTINUOUS CANONICAL FORMS

In [1] we used the fact that $M_{m,n}$ admits an embedding $\mathbb{P}^1 \to M_{m,n}$ if $m \geq 2$ to show that there is no algebraic continuous form for completely reachable pairs of matrices. This cannot be used to prove e.g. part (iii) of theorem 1.5 because as we have seen $M^{cr,co}_{m,n,p}$ is a quasi-affine algebraic variety. Further the example we used in [3] to prove nonexistence of continuous canonical forms for real linear dynamical systems if $m \geq 2$ and $p \geq 2$ is essentially nonalgebraic. There is, however, a three (instead of one) dimensional version of it which is algebraic and that is the example we shall use in this paper. We proceed via moduli varieties as in [2].

### 4.1. Triviality of $E^u$ and Existence of Continuous Algebraic Canonical Forms

#### 4.1.1. Theorem

Let $L \subset L^{cr}_{m,n,p}$ be a $GL_n$-invariant subvariety of $L^{cr}_{m,n,p}$ and let $M = \pi(L)$. Then there exists a continuous algebraic canonical form on L if and only if the algebraic vector bundle $E^u|M$ is trivial.

Proof. Let $\Phi^L_{m,n,p}$ be the subfunctor of $\Phi_{m,n,p}$ defined by considering only isomorphism classes of families $\Sigma$ over S such that $f_\Sigma$ maps S into $M = \pi(L)$. It follows directly from theorem 3.3.2 that $\Sigma \to f_\Sigma$ then defines a functorial isomorphism $\Phi^L_{m,n,p}(S) \overset{\sim}{\to} Sch_k(S,M)$ and that the inverse isomorphism is given by $f \to f^!(\Sigma^u|M)$ where $\Sigma^u|M = (E^u|M, F^u|M, G^u|M, H^u|M)$ is the restriction of $\Sigma^u$ to

M. Now suppose that there exists a continuous algebraic canonical form $c: L \to L$. Because c kills the action of $GL_n$ there is a unique morphism $\bar{c}: M \to L$ such that $c = \bar{c}\pi$. For each $x \in M$ we write $\bar{c}(x) = (F_c(x), G_c(x), H_c(x))$. Note that $\pi\bar{c} = id$, by condition (1.3.1) of the definition of canonical form.

We now define a family $\Sigma^c$ over M as follows: $\Sigma^c = (C^c, F^c, G^c, H^c)$ with $E^c = M \times A^n$, $F^c(x,v) = (x, F_c(x)v)$, $G^c(x,u) = (x, G_c(x)u)$, $H^c(x,v) = (x, H_c(x)v)$. Because $\pi c =$ id and $c(x) = (F_c(x), G_c(x), H_c(x))$ we have that $f_{\Sigma^c}: M \to M$ is the identity morphism, cf. 3.1.2. But, according to theorem 3.3.2, or rather the relative version discussed in the beginning of this proof, we have that $(F_{\Sigma c})^!(E^u|M)$ is isomorphic to $\Sigma^c$. which in particular means that $(f_{\Sigma^c})^!(E^u|M) = E^c = M \times \underline{A}^n$; but $f_{\Sigma^c} =$ id, hence $E^u|M$ is trivial.

Inversely suppose that $E^u|M$ is trivial. Then we can find n algebraic sections $e_1, \ldots, e_n: M \to E^u|M$ such that $e_1(x), \ldots, e_n(x)$ is a basis for $E^u_x$ for all $x \in M$. Let $F(x,e), G(x,e), H(x,e)$ be the matrices of $F_x: E^u_x \to E^u_x$, $G_x: \{x\} \times A^m \to E^u_x$, $H_x: E^u_x \to x \times A^p$ relative the obvious bases in $x \times \underline{A}^m$ and $x \times \underline{A}^p$ and the basis $\{e_1(x), \ldots, e_n(x)\}$ of $E^u_x$. We now define a morphism $c: L \to L$ as follows

$$c(F,G,H) = (F(x,e), G(x,e), H(x,e)) \text{ where } x = \pi(F,G,H)$$

One easily checks that this is a continuous algebraic canonical form.

4.1.2. The Local Canonical Froms $c_{\#\alpha}$.

Let $\alpha$ be a nice selection. The bundle $E^u|U_\alpha$ is trivial (by the definition of $E^u$ cf. 3.2) hence by theorem 4.1.1 there exist continuous algebraic canonical forms on $U_\alpha$. Such canonical forms are well known. An example is the canonical form $c_{\#\alpha}$ defined by

$$c_{\#\alpha}(F,G,H) = (F,G,H)^T, T = R(F,G)_\alpha^{-1} \qquad (4.1.3)$$

4.1.3 Corollary

If $m = 1$ there is a continuous algebraic canonical form on $L^{cr}_{m,n,p}$.

Proof. If $m = 1$ there is only one nice selection $\alpha$, and hence $L^{cr}_{m,n,p} = U_\alpha$ by lemma 2.1.1.

4.2. Duality

The assignment $\delta$: $(F,G,H) \to (F',H',G')$ defines an isomorphism of algebraic varieties $L_{m,n,p} \to L_{p,n,m}$. If $L \subset L_{m,n,p}$ is $GL_n$-invariant then so is $\delta(L) \subset L_{p,\dot{n},m}$ (but $\delta$ is not $GL_n$-invariant). As in [3], 3.1.6 one now easily shows that there is a continuous canonical form on $\delta(L)$.

### 4.2.1. Corollary

There is an algebraic continuous canonical form on $\dot{L}^{cr}_{m,n,p}$ if $p = 1$.

### 4.3. Example of a Nontrivial Algebraic Line Bundle

Let $U_1 = \underline{\underline{A}}^1 \times (\underline{\underline{A}}^2 \setminus (0,0))$, $U_2 = \underline{\underline{A}}^1 \times (\underline{\underline{A}}^2 \setminus (0,0))$. We give $U_1$ coordinates $(t,y_1,y_2)$ and $U_2$ coordinates $(s,x_1,x_2)$. Let $U_{12} =' \{(t,y_1,y_2) \in U_1 \; t \neq 0\}$, $U_{21} =' \{(s,x_1,x_2) \in U_2 | s \neq 0\}$. We define an isomorphism $\phi$: $U_{12}$ $U_{21}$ by $(t,y_1,y_2) \to (t^{-1},y_1,t,y_2t)$. Let X be the prevariety obtained by glueing $U_1$ and $U_2$ together by means of $\phi$. In fact X is a variety viz. the quasi-affine subvariety of $\underline{\underline{A}}^4 = \{(z_1,z_2,z_3,z_4)\}$ given by $z_1z_4 = z_2z_3$ and $(z_1 \neq 0$ or $z_2 \neq 0$ or $z_3 \neq 0$ or $z_4 \neq 0)$. The embeddings of $U_1$ and $U_2$ is this subvariety are given by $(t,y_1,y_2) \to (y_1t,y_2t,y_2)$, $(s,x_1,x_2)$ $(x_1,x_1s,x_2,x_2s)$. It is easy to check that this respects the identification $\phi$ given above.

We now define an algebraic line bundle V over X by glueing $U_1 \times \underline{\underline{A}}^1$ and $U_2 \times \underline{\underline{A}}^1$ together by means of the isomorphism

$$\tilde{\phi}: U_{12}\times\underline{\underline{A}}^1 \to U_{21}\times\underline{\underline{A}}^1, \; (t,y_1,y_2,u) \to (s,x_1,x_2,v) \text{ iff}$$

$$ts = 1, \; x_1 = ty, \; x_2 = ty_2, \; v = t^{-1}u \qquad (4.3.1)$$

Now suppose that this line bundle is trivial. Then there must be everywhere nonzero sections $U_1 \to U_1 \times \underline{\underline{A}}^1$, $(t,y_1,y_2) \to ((t,y_1,y_2), g_1(t,y_1,y_2))$; $U_2 \to U_2 \times \underline{\underline{A}}^1$, $(s,x_1,x_2) \to ((s,x_1,x_2), g_2(s,x_1,x_2))$ compatible with the identification $\phi$. Now $g_1$ and $g_2$ are morphisms $\underline{\underline{A}}_1 \times (\underline{\underline{A}}_2 \times (0,0)) \to \underline{\underline{A}}_1$. Because $A_1 \times (\ )$ is of codimension 2 $\underline{\underline{A}}^1 \times \underline{\underline{A}}^2 = \underline{\underline{A}}^3$ this means that $g_1$ and $g_2$ extend to morphisms on all of $\underline{\underline{A}}^3$, i.e. $g_1$ and $g_2$ are polynomials. Putting everything together we therefore have that C is a trivial line bundle iff there are polynomials $g_1(t,y_1,y_2)$, $g_2(s,x_1,x_2)$ such that $g_1(t,y_1,y_2) \neq 0$ if $y_1 \neq 0$ or $y_2 \neq 0$ and $g_2(s,x_1,x_2) \neq 0$ if $x_1 \neq 0$ or $x_2 \neq 0$ and such that moreover

$$tg_1(t,y_1,y_2) = g_2(t^{-1},ty_1,ty_2) \tag{4.3.2}$$

for all points $(t,y_1,y_2)$ such that $t \neq 0$ and $y_1 \neq 0$ or $y_2 \neq 0$. One easily checks that the only polynomials $g_1(t,y_1,y_2)$ such that $g_1(t,y_1,y_2) \neq 0$ for all $(t,y_1,y_2)$ for which $y_1 \neq 0$ or $y_2 \neq 0$ are constants. Similarly $g_2(s,x_1,x_2)$ is a constant. But then (4.3.2) is a contradiction. So we have proved

4.3.2 Lemma

The line bundle V defined by 4.3.1 is nontrivial.

## 4.4. Examples

Let $p \geq 2$ and $m \geq 2$. We write down a number of G,F and H matrices as follows

$$\text{If } n = 1,\ m \geq 2 \qquad G_{1,m}(t,s) = (t\ s\ 0\ \ldots\ 0) \tag{4.4.1}$$

$$\text{If } n < 2 < m < n \qquad G_{n,m}(t,s) = \left(\begin{array}{cc|c} t & s & 0\ .\ .\ .\ 0 \\ 1 & 1 & 0\ .\ .\ .\ 0 \\ \hline a & 1 & \\ \vdots & \vdots & B \\ a & 1 & \end{array}\right) \tag{4.4.2}$$

where a is a nonzero element of k different from 1, and where B is an $(n-2) \times (m-2)$ matrix with coefficients in k such that the columns of B and the column vector $(1,\ldots,1)'$ together span an $m-1$ dimensional subspace of $k^{n-2}$. Such a B exists because $2 < m < n$.

$$\text{If } n > 2 = m \qquad G_{n,2}(t,s) = \begin{pmatrix} t & s \\ 1 & 1 \\ a & 1 \\ \vdots & \vdots \\ a & 1 \end{pmatrix} \tag{4.4.3}$$

If $m > n \geq 2$

$$G_{n,m}(t,s) = \begin{pmatrix} t & s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 \cdots & 0 & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & & \cdot & & \cdot \\ \vdots & \vdots & \vdots & \cdots & 0 & \cdot & & \cdot \\ 0 & 0 & \underbrace{0 \cdots 0}_{n-1} & & 1 & \underbrace{0 \cdots 0}_{m-n-1} & & \end{pmatrix} \tag{4.4.4}$$

If $m = n \geq 2$

$$G_{n,n}(t,s) = \begin{pmatrix} t & s & 0 & \cdots & 0 \\ 1 & 1 & \cdot & & \cdot \\ a & 1 & \cdot & & \cdot \\ \vdots & \vdots & \cdot & & \cdot \\ a & 1 & \underbrace{0 \cdots \cdots 0}_{n-2} & & \end{pmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n-2 \tag{4.4.5}$$

$$F_n = \begin{pmatrix} a_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & a_2 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & a_n \end{pmatrix} \tag{4.4.6}$$

where $a_1, \ldots, a_n$ are n different elements of k which are all different from zero

$$H_{p,n}(y_1,y_2) = G_{n,p}(y_1,y_2)', \qquad (4.4.7)$$

## 4.5 An Embedding $X \to M^{cr}_{m,n,p}$

Let $U_1$, $U_2$ be as in 4.3 above. We define for all $n,m,p$ with $m \geq 2$ and $p \geq 2$

$$\sigma_{n,m,p}: U_1 \to L^{cr}_{m,n,p}, (t,y_1,y_2) \to (F_n, G_{n,m}(t,1), H_{p,n}(y_1,y_2)) \qquad (4.5.1)$$

$$\bar{\sigma}_{n,m,p}: U_2 \to L^{cr}_{m,n,p}, (s,x_1,x_2) \to (F_n, G_{n,m}(1,s), H_{p,n}(x_1,x_2))$$

We now note that if $ts = 1$, $x_1 = y_1 t$, $x_2 = y_2 t$

$$(F_n, G_{n,m}(t,1), H_{p,n}(y_1,y_2))^{T(t)} = (F_n, G_{n,m}(1,s), H_{p,n}(x_1,x_2)) \qquad (4.5.2)$$

where

$$T(t) = \begin{pmatrix} t^{-1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix}$$

This means that the morphisms $U_1 \to M^{cr}_{m,n,p}$, $U_2 \to M^{cr}_{m,n,p}$, obtained from the morphisms $\sigma_{n,m,p}$ and $\overline{\sigma}_{n,m,p}$ of (4.5.1) be composing with $\pi$: $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$, combine to define a morhpism

$$\tau_{m,n,p}: X \to M^{cr}_{m,n,p} \tag{4.5.3}$$

where X is the variety defined in 4.3 above.

4.5.4.

Let $\alpha$ be the nice selection' {(0,2),(1,2),...,(n-1,2)} then we see from 4.4 that $\sigma_{m,n,p}(U_1) \subset U_\alpha$ and hence $\tau_{m,n,p}(U_1) \subset V_\alpha$. Let $\beta$ be the nice selection' {(0,1),(1,1), ...,(n-1,1)} then we see from 4.4 that $\overline{\sigma}_{m,n,p}(U_2) \subset U_\beta$ and hence $\tau_{m,n,p}(U_2) \subset V$ . It follows that the pullback of $E^u$ by means of $\tau_{m,n,p}$ is an algebraic vectorbundle over X whose restrictions to $U_1$ and $U_2$ are trivial, and the gluing data of this bundle are given by (Cf. [1] Appendix 3.6).

$$\begin{gathered} \mathfrak{F}: U_{12} \times \underline{\underline{A}}^n \to U_{21} \times \underline{\underline{A}}^n \\ ((t,y_1,y_2),u) \to ((t^{-1},ty_1,ty_2),T(t,y_1,y_2)u) \end{gathered} \tag{4.5.5}$$

where $T(t,y_1,y_2)$ is equal to the matrix

$$R(F_n,G_{n,m}(t,1))_\beta^{-1}R(F_n,G_{n,m}(t,1))_\alpha \tag{4.5.6}$$

where $\alpha$ and $\beta$ are the nice selections' {(0,2),(1,2),..., (n-1,2)} and' {(0,1),(1,1),...,(n-1,1)}. Let $E \to X$ be this bundle. The exterior product bundle $\overset{n}{\Lambda} E \to X$ is then the line bundle obtained by gluing together $U_1 \times \underline{\underline{A}}^1$ and $U_2 \times \underline{\underline{A}}^1$ by means of the isomorphism

$$\psi: U_{12} \times \underline{\underline{A}}^1 \to U_{21} \times \underline{\underline{A}}^1 \tag{4.5.7}$$

$$((t,y_1,y_2),u) \quad ((t^{-1},ty_1,ty_2),\ \det(T(t,y_1,y_2))u)$$

and from (4.5.6) we see that

$$\det(T(t,y_1,y_2)) = \begin{cases} t^{-1} & \text{if } n \leq 2 \\ t^{-1}a^{n-2} & \text{if } n \leq 2 \end{cases} \tag{4.5.8}$$

It follows that the line bundle defined by $\hat{\psi}$ is nontrivial. Cf. 4.3 above.

4.5.9. Proposition

The algebraic vectorbundle $\tau^!_{n,m,p}F^u$ is nontrivial if $p \geq 2$, $m \geq 2$.

Proof. This follows from the above because if $E \to X$ is a trivial algebraic n-dimensional vector bundle then $\overset{n}{\Lambda} E \to X$ is a trivial line bundle.

4.5.10. Corollary

Let M be a subvariety of $M^{cr}_{m,n,p}$ such that $\tau_{n,m,p}(x) \subset M$. Then $E^u|M$ is a nontrivial algebraic vectorbundle.

## 4.6. Nonexistence of Continuous Algebraic Canonical Forms

We can now prove theorem 1.5.

### 4.6.1. Proof of Theorem 1.5

First let $m \geq 2$ and $p \geq 2$. Let $M^W = \pi(L^W_{m,n,p})$ where $L^W_{m,n,p}$ runs through the subvarieties listed in 1.4. Then we see from 4.4

$$\tau_{m,n,p}(X) \subset M^{\rho,\mu} \qquad (4.6.1)$$

if $m \neq n$ and $p \neq n$, and that in any case (still assuming $p \geq 2$ and $m \geq 2$)

$$\tau_{m,n,p}(X) \subset M^{cr,co,\mu} \qquad (4.6.2)$$

By corollary 4.5.10 and theorem 4.4.1 this takes care of the only if parts of statements (iii), (iv), (v), (vi) of theorem 1.5. (Because $L^{\rho,\mu}_{m,n,p} \subset L^{\rho}_{m,n,p}$ and $L^{cr,co}_{m,n,p} \subset L^{cr,co,\rho}_{m,n,p}$). On the other hand if $m = 1$ in cases (iii) and (iv) and $m = 1$ or $n$ in cases (v) and (vi) then the respective subvarities are contained in one $U_\alpha$ for a certain nice selection $\alpha$. By 4.1.2 there are therefore continuous algebraic canonical forms in these cases. The corresponding fact for $p = 1$ in cases (iii), (iv) and $p = 1$ or $n$ in cases (v), (vi) follows by duality. Cf. 4.2. This proves (iii) - (vi) of theorem 1.5.

The if part of (i) is corollary 4.1.3; the if part of (ii) follows by duality. Cf. 4.2. To prove the only if part of (i) observe that if $m \geq 2$ $\pi((F_n, G_{n,m}(t,s),0)$, where $t \neq 0$ or $s \neq 0$, depends only on the point $(t:s) \in \underline{\underline{P}}^1$ and not on the actual t and s. Thus

$$\tau: (t:s) \to \pi((F_n, G_{n,m}(t,s),0)$$

defines a morphism $\underline{\underline{P}}^1 \to M^{cr}_{m,n,p}$ for all (m,n,p) such that $m \geq 2$. As in 4.5 one now proves that $\tau^! E^u$ is nontrivial. By 4.5.10 and 4.4.1 this proves the only if part of (i). The only if part of (ii) follows by duality. Cf. 4.2. This concludes the proof of theorem 4.5.

## 4.7. On Relations between Various Local Canonical Forms

Let $U \subset L^{cr}_{m,n,p}$ be a $GL_n$ invariant subvariety of $L^{cr}_{m,n,p}$, and suppose that there is a continuous algebraic canonical form $c: U \to U$. Let $\kappa: U \to \underline{\underline{A}}^1$ be a morphism, e.g. a "coordinate function." Then $\kappa c: U \to A^1$ is $GL_n$ invariant, showing that "the coordinate functions of a canonical form are invariants."

4.7.1

Now let $a: U \to GL_n$ be a morphism which kills the action of $GL_n$ on U. Then if $c: U \to U$ is a continuous algebraic canonical form so is $c^a: U \to U$ which is defined by $(F,G,H) \to c(F,G,H)^{a(F,G,H)}$. Inversely if $c'$ is a second continuous algebraic canonical form on U then $c' = c^a$ for some morphism $a: U \to GL_n$ which kills the action of $GL_n$ on U. All this is proved as in section 3.6 of [1].

4.7.3

The situation becomes slightly more complicated if we take $U = U_\alpha^{co}$. We still have the canonical forms $c_{\#\alpha}$ and all other canonical forms are obtained by means of a morphism $\hat{a}: V_\alpha^{co} \to GL_n$. Now if $p = 1$ then $\det(\hat{a}(x))$ need not be a constant independent of $x \in V_\alpha^{co}$, because the codimension of $V_\alpha \setminus V_\alpha^{co}$ in $V_\alpha$ is one if $p = 1$. An example of this is found by taking $m = 1 = p$ and comparing the canonical form $c_{\#\alpha}$ and its dual on $M_{1,n,1}^{cr,co}$. However if $p \geq 2$, then the codimension of $V_\alpha \setminus V_\alpha^{co}$ in $V_\alpha$ is $\geq 2$ (cf. section 2.7 above), which means that in this case we again have that $\hat{a}: V_\alpha^{co} \to GL_n$ is given by $n^2$ polynomials such that $\det(\hat{a}(x))$ is a constant independent of $x \in V_\alpha^{co}$.

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