REPRESENTATIONS OF QUIVERS AND MODULI OF LINEAR DYNAMICAL SYSTEMS

Michiel Hazewinkel
Econometric Inst. Erasmus University
Rotterdam, The Netherlands

1. PREFACE

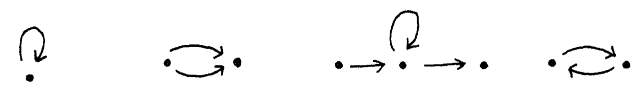
This note is the written version of the part which is not covered by [15] and [16] (cf. also [12], [13], [14], and [3] of the talks I gave at the Ames conference in June/July 1976. The main purpose of this part of the talks was to acquaint engineers and applied mathematicians with the fact that some of the problems they have been studying in (algebraic) system theory and identification theory are identical (or at least very similar to) a certain set of problems studied by algebraists belonging to representation theory or linear algebra (depending on one's taste and judgement) viz. the theory of representations of "quivers." Inversely it may be of interest to the algebraists that the two quivers for which results have been obtained in algebraic system theory are both of wild type.

2. QUIVERS AND THEIR REPRESENTATIONS

2.1 Definition

A quiver is a finite connected directed graph. i.e. a quiver Q consists of a finite set P_Q of points and a finite set A_Q of arrows between points of P_Q. Loops are allowed and also multiple arrows between the same points.

2.2 Some examples of quivers are



(a) (b) (c) (d)

(e) (f) (g) (h)

2.3 Definitions

A representation V over a field K of a quiver Q assigns to each $P \in P_Q$ a vector space V(P) and to each arrow $a \in A_Q$ a vector space homomorphism $V(a):V(s(a)) \to V(r(a))$ where a is an arrow from $s(a) \in P_Q$ to $r(a) \in P_Q$. The zero representation assigns to each $P \in P_Q$ the zero vector space. Given two representations V_1 and V_2 their direct sum $V_1 \oplus V_2$ assigns to each $P \epsilon P_Q$ the vector space $V_1(P) \oplus V_2(P)$ and to each arrow $a \epsilon A_Q$ the direct sum homomorphism $V_1(a) \oplus V_2(a)$. A representation V is called *indecomposable* if it cannot be written as a direct sum $V = V_1 \oplus V_2$ with V_1 and V_2 both unequal to the zero representation. Given a representation V a *subrepresentation* W consists of subspaces $W(P) \subset V(P)$ for all $P \epsilon P_Q$ such that $V(a)(W)(s(a)) \subset W(r(a))$ for all $a \epsilon A_Q$. A representation V is called *irreducible* if it has no other subrepresentations than itself and the zero representation. Finally two representations V, W are said to be isomorphic if there are isomorphisms $\psi(P) : V(P) \to W(P)$ for every $P \epsilon P_Q$ such that the following diagram commutes for all $a \epsilon A_Q$.

$$\begin{array}{ccc} V(s(a)) & \xrightarrow{V(a)} & V(r(a)) \\ \downarrow{\scriptstyle \psi(s(a))} & & \downarrow{\scriptstyle \psi(r(a))} \\ W(s(a)) & \xrightarrow{W(a)} & W(r(a)) \end{array}$$

2.4 The general problem is now: given a quiver, describe all isomorphism classes of (indecomposable) representations.

In the case of the quiver 2.2(a) above this is the familiar linear algebra problem of classifying square matrices up to similarity. The indecomposable representations are precisely those which have one Jordan block.

In the case of example 2.2(b) a representation consists of two matrices (A,B), and a second representation (C,D) is isomorphic to (A,B) if there are invertible matrices S, T such that C = SAT, D = SBT. Writing A + sB and C + sD for (A,B) and (C,D), where s is an indeterminate we see that the study of isomorphism classes of representations of the quiver 2.2(b) is the same as the study of pencils of matrices in the sense of Kronecker, who also solved this problem.

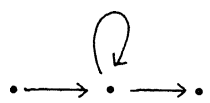
Similarly quiver 2.2(g) concerns the study of two dimensional pencils A + sB + tC. (These turn up when one studies control systems with delays.)

To conclude this section let us remark that quiver 2.2(c) is the study of pairs of matrices under simultaneous similarity a problem which has been around for some 150 years (and is still unsolved).

2.5 A special quiver from system theory

A linear dynamical system $\dot{x} = Fx + Gu, y = Hx$ or $x_{t+1} = Fx_t + Gu_t$, $y_t = Hx_t$ (discrete case) gives rise to a triple of matrices (F,G,H) with coefficients in $\underline{\underline{R}}$, $\underline{\underline{C}}$ in the continuous case or in any field (or ring for that matter) in the discrete case. Base change in all three of the spaces involved (input space, state space, output space) changes the triple (F,G,H) into $(T_2FT_2^{-1}, T_2GT_1^{-1}, T_3HT_2^{-1})$ where the T_i, $i = 1,2,3$, are invertible matrices of the appropriate sizes.

In other words the study of linear dynamical systems under base change in input space, state space and output space is the same as the study of the representations up to isomorphism of the quiver



which is the quiver 2.2(c). If one neglects outputs one obtains instead the quiver 2.2(f).

For a description of some of the results obtained recently for these quivers cf. section 4 below.

3. GABRIEL'S THEOREM AND ITS RELATIVES

One of the really beautiful results in the theory of representations of quivers (and also the result which started the business) is Gabriel's theorem which describes all quivers which have--up to isomorphism--only finitely many indecomposables representations. First a definition.

3.1 Definitions

A quiver Q is of <u>finite type</u> if there exist up to isomorphism only finitely many indecomposable representations; the quiver Q is <u>tame</u> if there are infinitely many isomorphism classes of indecomposable representations but these classes can be parametrized by a finite set of integers together with an irreducible polynomial (over the field k one happens to work over); the quiver Q is <u>wild</u> if given a finite dimensional k-algebra E there are infinitely many pair-wise non-isomorphic representations of Q with endomorphism algebra isomorphic to E.

These classes of quivers are clearly exclusive. They are also, as it turns out, exhaustive.

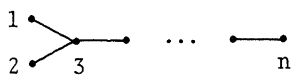
3.2 Gabriel's theorem

The quivers of finite type are those whose underlying undirected graph is of one of the following types

A_n:

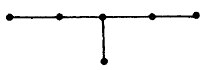
$n \geq 1$

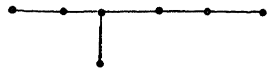
D_n:

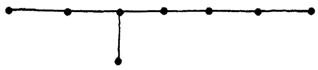


$n \geq 4$

E_6:



E_7: 

E_8: 

It is not an accident that the graphs above are Dynkin diagrams. For deatils cf. [7] and [1] and also [4] for where and how the other Dynkin diagrams fit.

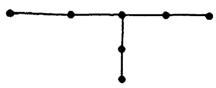
3.3 Nazarova [18] has similarly described all quivers which are tame. These have as underlying undirected graphs one of the following extended Dynkin diagrams.

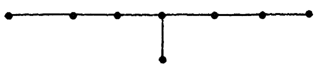
$\hat{A}_n$: 

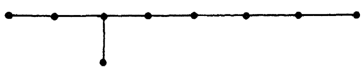
$n \geq 0$

$\hat{D}_n$: 

$n \geq 4$

$\hat{E}_6$: 

$\hat{E}_7$: 

$\hat{E}_8$: 

3.4 All other quivers are wild. So that in particular the quivers of algebraic system theory 2.2(c) and 2.2(f) are wild. Also wild are the quivers 2.2(g) and 2.2(e). The quivers 2.2(a), 2.2(b), 2.2(d) and 2.2(h) are all tame.

3.5 The quadratic form of a quiver

Let Q be a quiver. We attach to Q a quadratic form in as many variables X_P as there are elements in P_Q. The quadratic form is

$$K\,(\ldots,X_P,\ldots) = \sum_{P\varepsilon P_Q} X_P^2 - \sum_{a\varepsilon A_Q} X_{s(a)}X_{r(a)}$$

Thus e.g. if Q is of type A_4 we find a form

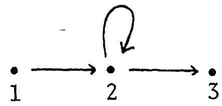
$$X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_1X_2 - X_2X_3 - X_3X_4 =$$

$$\tfrac{1}{2}X_1^2 + \tfrac{1}{2}(X_3-X_2)^2 + \tfrac{1}{2}(X_2-X_3)^2 + \tfrac{1}{2}(X_3-X_4)^{\frac{1}{2}} + \tfrac{1}{2}X_4^2$$

It now turns out that a quiver is respectively of finite type, tame or wild if this quadratic form K_Q is respectively positive definite, positive semidefinite, indefinite.

4. ON THE QUIVERS OF (ALGEBRAIC) LINEAR SYSTEM THEORY

We now return to the quiver 2.2(c) of linear system theory. Cf. also 2.5 above. The quiver in question is

(QL) 

4.1 A representation of this quiver with dim $V(1) = m$, $\dim V(2) = n$, $\dim V(3) = P$ is a linear dynamical system with m inputs, p outputs and state space dimension n. Let $L_{m,n,p}(k)$ be the space of all representations over the field k with these dimensions. The group $G(k) = GL_m(k) \times GL_n(k) \times GL_p(k)$ acts on $L_{m,m,p}(k)$ as $((T_1,T_2,T_3),(F,G,H)) \to (T_2FT_2^{-1},T_2GT_2\ ,T_3HT_2^{-1})$ and the isomorphism classes of representations correspond bijectively to the elements of the quotient set $L_{m,n,p}(k)/G(k)$.

Now most of the results which have been obtained recently are not about $L_{m,n,p}(k)/G(k)$ but the equally interesting related quotient $L_{m,n,p}(k)/GL_n(k)$ where $GL_n(k)$ is the subgroup $1 \times GL_n(k) \times 1$ of $G(k)$. This corresponds to a finer notion of isomorphism (more isomorphism classes); viz. two representations V, W of (QL) are isomorphic in the fine sense if there is an isomorphism $\psi : V \to W$ such that $\psi(1) = id$, $\psi(3) = id$. For later purposes we define the corresponding notions: a <u>fine subrepresentation</u> of a representation V of (QL) is a subrepresentation W such that $W(1) = V(1)$ and $V(3) = W(3)$

and we say that V is <u>finely irreducible</u> if the only fine subrepresentation of V is V itself.

4.2 Complete reachability

Recall that a triple $(F,G,H) \in L_{m,n,p}(k)$ is completely reachable if and only if the space spanned by the columns of the matrices $G, FG, \ldots F^nG$ is all of k^n = state space. Thus we see that a representation V = (F,G,H) of (QL) is completely reachable if and only if it is finely irreducible.

4.3 Some results on $L^{cr}_{m,n,p}(k)/GL_n(k)$

Let $L^{cr}_{m,n,p}(k)$ be the subspace of all completely reachable triples (F,G,H). First suppose that k is an algebraically closed field. Then one has:

4.3.1 $L^{cr}_{m,n,p}(k)/GL_n(k)$ is a connected nonsingular algebraic variety over k of dimension np + mn.

Let us write $M^{cr}_{m,n,p}(k)$ for this variety.

4.3.2 $M^{cr}_{1,n,p}(k) = \underline{\underline{A}}_k^{n+np}$ affine space over k of dimension n + np.

4.3.3 If $m \geq 2$ then $M^{cr}_{m,n,p}(k)$ is cohomologically nontrivial.

(For these and many related results cf. [12], [13], [14 [15], [16] and [3].)

In the special case $k = \underline{\underline{R}}$ one has that $M^{cr}_{m,n,p}(\underline{\underline{R}})$ is a smooth noncompact differentiable manifold diffeomorphic to $\underline{\underline{R}}^{n+np}$ if m = 1 and cohomologically nontrivial if $m \geq 2$.

In terms of representations of the quiver (QL) 4.3.1 says c.q. that every fine class of finely irreducible representations of dimensions (m,n,p) can be continuously deformed into any other fine class. In particular there are mn + np "moduli" for these classes of representations, which, of course, is not unexpected given that (QL) is of wild type.

BIBLIOGRAPHY

[1] J.N. Bernstein, I.M. Gelfand, V.A. Ponomarev, Coxeter functors and Gabriel's theorem, Usp. Mat. Nauk 28.2 (1973) Translation: Russ. Math. Surveys 28, 2 (1973), 17-30.

[2] S. Brenner, Quivers with commutativity conditions and some phenomenology of forms, Lect. Notes Math. 488, 1975, 29-53, Springer.

[3] C. Byrnes, N.E. Hurt, On the moduli of linear dynamical systems (To appear in Adv. Math.).

[4] V. Dlab, C.M. Ringel, On algebras of finite representation type, J. of Algebra 33, 2 (1975), 306-394.

[5] V. Dlab, C.M. Ringel, Representations of graphs and algebras, Carelton Math. Lecture Notes 8, 1974.

[6] P. Donovan, M.R. Freislich, The representation theory of finite graphs and associative algebras, Carleton Math. Lecture Notes 5, 1973.

[7] P. Gabriel, Unzerlegbare Darstellungen I. Manuscripta Math. 6, 71-103 (1972).

[8] P. Gabriel, Indecomposable representation II, Symp. Math. XI, INDAM, 81-104, Acad. Press, 1973.

[9] I.M. Gelfand, V.M. Ponomarev, Free modular structures and their representations, Usp. Mat. Nauk 29, 6 (1974), 3-58 (Translation: Russ. Math. Surveys 29, 6 (1974) 1-56).

[10] I.M. Gelfand, V.M. Ponomarev, Indecomposable representations of the Lorentz group, Usp. Mat. Nauk 23 (1968), 3-60.

[11] I.M. Gelfand, V.M. Ponomarev, Problems of linear algebra and classification of quadruples of subspaces in a finite dimensional vector space. Coll. Math. Soc. J. Bolyai 5 (1970), 163-237, North Holland Publ. Cy.

[12] M. Hazewinkel, R.E. Kalman, Moduli and canonical forms for linear dynamical systems (To appear; a preliminary version is available in preprint form as report 7504, Econometric Inst. Erasmus Univ. of Rotterdam, 1975).

[13] M. Hazewinkel, R.E. Kalman, On invariant, canonical forms and moduli for linear constant finite dimensional, dynamical systems. In: Roc. CNR-CJSM symposium on "Algebraic system theory," Udine 1975, Lect. Notes in Systems Theory 131, Springer 1976.

[14] M. Hazewinkel, On the nonexistence of canonical forms for real linear dynamical systems. (To appear in Bull. Calcutta Math. Soc.; available in preprint form as report 7520, Econometric Inst. Erasmus Univ., Rotterdam, 1975).

[15] M. Hazewinkel, Moduli and canonical forms for linear dynamical systems II: the topological case (Submitted to J. Math. Syst. Theory; available in preprint form as report 7609, Econometric Inst., Erasmus Univ., Rotterdam, 1975).

[16] M. Hazewinkel, Moduli and canonical forms for linear dynamical systems III: the algebraic geometric case, this volume.

[17] S. MacLane, An algebra of additive relations, Proc. Nat. Acad. Sci., U.S.A. 47 (1961), 1043-1051.

[18] L.A. Nazarova, Representations of quivers of infinite type, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 752-791.

[19] L.A. Nazarova, A.V. Roiter, On a problem of I.M. Gelfand, Funk. Analiz. i Priloz. 7 (1973), 54-69.

[20] L.A. Nazarova, Izv. Akad. Nauk SSR ser. Mat. 39 (1975).