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MODULI AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS. III: THE ALGEBRAIC-GEOMETRIC CASE

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1. INTRODUCTION.

In this paper we treat the algebraic-geometric version of the topological theory developed in [3]. That is we study linear dynamical systems over an algebraically closed field k

$$
(1.1) \qquad \begin{aligned} x_{t+1} &= Fx_t + Gu_t \ , \quad x_t \in k^n, \ u_t \in k^m \\ y_t &= Hx_t , \qquad\qquad y_t \in k^p \end{aligned}
$$

where F,G,H are matrices with coefficients in k of the appropriate sizes. A change of basis in state space changes the triple of matrices (F,G,H) into (TFT^{-1}, TG, HT^{-1}) and as in [3] we are interested in such questions as the following.

Does the set of orbits under this action have a (natural) structure of an algebraic variety? Do there exist continuous canonical forms? Similar questions for the case of two matrices were studied and answered in [1], cf. also [2].

Essentially the answers are as in [3]. This paper uses a moderate amount of algebraic geometry (nothing much beyond definitions). Appendices 1,2 and 3 of [1] provide sufficient background information for this paper. All schemes in this paper will be reduced and of finite type over k, and we shall identify them with their associated algebraic varieties of closed points. We use $\underline{\underline{A}}^r$ to denote affine space of dimension r over k, and we give the space of all triples of matrices (F,G,H) of dimensions n x n, n x m, p x n respectively, the algebraic variety structure of $\underline{\underline{A}}^{n(n+m+p)}$. Let $L_{m,n,p}$ denote this algebraic variety.

Then the assignment

$$
(1.2) \qquad (T,(F,G,H)) \mapsto (TFT^{-1}, TG, HT^{-1}) = (F,G,H)^T
$$

defines an action of the algebraic group GL_n of invertible n x n matrices with coefficients in k on $L_{m,n,p}$. Cf [1] Appendix 2. We can now define what a continuous algebraic canonical form on a subvariety $L' \subset L_{m,n,p}$ would be.

1.3. Definition.

A continuous algebraic canonical form on L' is an algebraic morphism $c: L' \to L'$ such that

(1.3.1) for every $(F,G,H) \in L'$ there is a $T \in GL_n$ such that
$$(F,G,H)^T = c(F,G,H)$$

(1.3.2) $c(F,G,H) = c(\bar{F},\bar{G},\bar{H})$ iff there is a $T \in GL_n$ such that
$$(F,G,H)^T = (\bar{F},\bar{G},\bar{H})$$

Again, as in [3], we have that continuous algebraic canonical forms on all of $L_{m,n,p}$ cannot exist for trivial reasons. ("Jump phenomena"). The conditions "completely reachable", "completely observable", "rank of G maximal and rank of H maximal and completely reachable and completely observable" all define open subvarieties of $L_{m,n,p}$ which we shall denote with $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{\rho}_{m,n,p}$ respectively. In addition we consider the condition "F is diagonalizable (i.e. semisimple) with distinct eigenvalues all different from zero" which defines a (non-open) subvariety $L^{\mu}_{m,n,p}$ of $L_{m,n,p}$. Combining different attributes we have the following list of (possibly interesting) subvarieties of $L_{m,n,p}$.

1.4. List of subvarieties.

$L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{cr,co}_{m,n,p} = L^{cr}_{m,n,p} \cap L^{co}_{m,n,p}$, $L^{\mu}_{m,n,p}$,

$L^{cr,co,\mu}_{m,n,p} = L^{\mu}_{m,n,p} \cap L^{cr,co}_{m,n,p}$, $L^{\rho}_{m,n,p}$, $L^{\rho,\mu}_{m,n,p} = L^{\rho}_{m,n,p} \cap L^{\mu}_{m,n,p}$

All these subvarieties of $L_{m,n,p}$ are GL_n-invariant. We now have the following theorem.

1.5. Theorem.

The following table gives necessary and sufficient conditions for the existence of continuous algebraic canonical forms on various subvarieties of $L_{m,n,p}$.

	variety L'	necessary and sufficient condition for the existence of an algebraic continuous canonical form
(i)	$L' = L^{cr}_{m,n,p}$	$m = 1$
(ii)	$L' = L^{co}_{m,n,p}$	$p = 1$
(iii)	$L' = L^{cr,co}_{m,n,p}$	$m = 1$ or $p = 1$
(iv)	$L' = L^{cr,co,\mu}_{m,n,p}$	$m = 1$ or $p = 1$
(v)	$L' = L^{\rho}_{m,n,p}$	$m = 1$ or $p = 1$ or $m = n$ or $p = n$
(vi)	$L' = L^{\rho,\mu}_{m,n,p}$	$m = 1$ or $p = 1$ or $m = n$ or $p = n$

This theorem is "identical" with theorem 1.7 of [3]. The proof is similar in spirit but different in details.

There is of course also a corollary similar to corollary 1.8 of [3].
We shall see that the "orbit space" $L^{cr}_{m,n,p}/GL_n$ has the structure of a quasi-projective algebraic variety and its open subvariety $L^{cr,co}_{m,n,p}/GL_n$ is in fact a quasi-affine algebraic variety. Let $M^{cr}_{m,n,p}$ denote this algebraic variety. Then we shall also see that $M^{cr}_{m,n,p}$ is a fine moduli variety for a suitable definition of (algebraic) families of linear dynamical systems.

As we said the field k we work over is supposed to be algebraically closed. This is mainly a matter of convenience: the varieties $L^{cr,co}_{m,n,p}$, $L^{cr}_{m,n,p}$, $L_{m,n,p}$, $M^{cr,co}_{m,n,p}$, $M^{cr}_{m,n,p}$, $M^{\rho}_{m,n,p}$ are all defined over any field k; in fact they are even defined over $\mathbb{Z}$. This also explains our notation $M^{cr}_{m,n,p}(\mathbb{R})$, etc. of [3]: the underlying sets of these real manifolds are simply the real points of the variety $M^{cr}_{m,n,p}$, etc. However, some care must be taken in interpreting the results of e.g. part (iii) of theorem 1.5 in this context.

Consider e.g. the following situation: let k be a finite field; let $L^{cr,co}_{m,n,p}(k)$ be the set of all k rational points of $L^{cr,co}_{m,n,p}$, i.e. the set of all completely reachable and completely controllable triples of

matrices with coefficients in k; let $GL_n(k)$ be the group of n x n matrices with coefficients in k acting on $L^{cr,co}_{m,n,p}(k)$ in the obvious way. Then part (iii) of theorem 1.5 does not say that there is no map $L^{cr,co}_{m,n,p}(k) \to L^{cr,co}_{m,n,p}(k)$ (locally) given by polynomials such that the analogues of (1.3.1) and (1.3.2) hold. E.g. such a map always exists when k is $\mathbb{F}_2$, the field of two elements. But part (iii) of theorem 1.5 does say that the map $L^{cr,co}_{m,n,p}(\bar{k}) \to L^{cr,co}_{m,n,p}(\bar{k})$ defined by the same polynomials, does not satisfy the analogues of (1.3.1) and (1.3.2). Here $\bar{k}$ is the algebraic closure of k.
A large part of the proofs and constructions of [3] can be carried through unchanged in the algebraic geometric case. In these cases we shall as a rule simply refer to the appropriate section of [3].

The contents of the paper are

2. THE QUOTIENT VARIETY $M^{cr}_{m,n,p}$

2.1. Nice Selections.

Let $(F,G,H) \in L_{m,n,p}$. The matrices $R(F,G)$ and $Q(F,H)$ are defined as in [3], 2.2. The conditions "$R(F,G)$ has rank n", i.e. "complete reachability" and "$Q(F,H)$ has rank n" i.e. "complete observability" define open subvarieties of $L_{m,n,p}$ which we denote by $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$ respectively.
In addition we put $L^{cr,co}_{m,n,p} = L^{cr}_{m,n,p} \cap L^{co}_{m,n,p}$ which is also an open subvariety of $L_{m,n,p}$. As in [3], 2.3 we let $J_{n,m}$ lenote the set of column indices of $R(F,G)$. Nice selections α (from $J_{n,m}$) and the successor indices $s(a,j)$, $j = 1, \ldots, m$ of the nice selection α are defined as in [3], 2.3. We again have (c.f. [1] 2.4.1 for a proof).

2.1.1. Lemma.

If $(F,G,H) \in L^{cr}_{m,n,p}$, then there is a nice selection α such that $\det(R(F,G)_\alpha \neq 0$.

2.2. The Local Quotients U_α/GL_n.

Let α be a nice selection. One defines the subvarieties of $L^{cr}_{m,n,p}$

(2.2.1) $$U_\alpha = \{(F,G,H) \in L_{m,n,p} | \det(R(F,G)_\alpha) \neq 0\}$$

(2.2.2) $$W_\alpha = \{(F,G,H) \in L_{m,n,p} | R(F,G)_\alpha = I_n\}$$

THe map ψ_α of [3], 2.4.5 now defines an isomorphism of algebraic varieties

(2.2.3) $$\psi_\alpha : \underline{\underline{A}}^{nm+np} \overset{\sim}{\to} W$$

We define a morphism $t_\alpha : U_\alpha \to GL_n x W_\alpha$

(2.2.4) $$t_\alpha : (F,G,H) \to (T^{-1}, (F,G,H)^T), \text{ where } T = R(F,G)^{-1}_\alpha$$

2.2.5. Lemma. t_α is a GL_n-invariant isomorphism of algebraic varieties (where GL_n acts on $GL_n \times W_\alpha$ by left multiplication on the left hand factor.

2.2.6. Corollary.

The (categorial) quotients U_α/GL_n exist (as algebraic varieties) and are isomorphic to the affine space $\underline{\underline{A}}^{mn+np}$.
This follows from 2.5.5 and the isomorphism ψ_α. For the notion of categorical quotient cf. [1] A.2.7. As a matter of fact U_α/GL_n is also a geometric quotient in the sense of [6]; we shall not need this fact.

2.3. The Quotient Variety $M^{cr}_{m,n,p}$.

We are now going to define a quotient prevariety $M^{cr}_{m,n,p}$ by glueing the local quotients U_α/GL_n together in a suitable way. For each nice selection α let $V_\alpha = \underline{\underline{A}}^{mn+np}$ and for each second nice selection β let $V_{\alpha\beta}$ be the open subvariety $V_{\alpha\beta} = \psi_\alpha^{-1}(W_\alpha \cap U_\beta)$. We define $\phi_{\alpha\beta}: V_{\alpha\beta} \to V_{\beta\alpha}$ by the formula (identical to [3], (2.5.4)).

$$(2.3.1)\quad \phi_{\alpha\beta}(x) = y \iff (F_\alpha(x),G_\alpha(x),H_\alpha(x))^T = (F_\beta(y),G_\beta(y),H_\beta(y))$$
$$\text{with } T = R(F_\alpha(x),G_\alpha(x))_\beta^{-1}$$

where we have written $\psi_\alpha(x) = (F_\alpha(x),G_\alpha(x),H_\alpha(x)) \in W_\alpha$ and similarly for $\psi_\beta(y)$. These $\phi_{\alpha\beta}$ are well defined and define isomorphisms of algebraic varieties $V_{\alpha\beta} \to V_{\beta\alpha}$, which moreover satisfy the cocycle condition $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$ whenever the left hand side is defined. This means that by glueing together the various V_α by means of the $\phi_{\alpha\beta}$ we obtain a certain prevariety which we shall denote $M^{cr}_{m,n,p}$. To prove that $M^{cr}_{m,n,p}$ is an (abstract) variety we have to prove that it is separated. This can either be done by using the algebraic geometric version of [3], 2.5.7 or by means an embedding argument. To carry this embedding argument through we first observe.

2.3.2. Lemma.

The natural projections $\pi_\alpha: U_\alpha \to V_\alpha$ combine to define an algebraic morphism $\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$, and π is a categorical quotient <u>in the category of prevarieties</u> for the action of GL_n on $L^{cr}_{m,n,p}$ defined by (1.2).

Proof. It is obvious that π: $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ kills the action of GL_n. Now let ϕ: $L^{cr}_{n,n,p} \to X$ be any morphism which kills the action of GL_n. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then we know that $U_\alpha \to V_\alpha$ and $U_{\alpha\beta} \to V_{\alpha\beta}$ are categorical quotients by 2.2.6. Let ϕ_α be the restriction of ϕ to U_α. By the categorical quotient property of $U_\alpha \to V_\alpha$ there are unique morphisms χ_α: $V_\alpha \to X$ such that $\phi_\alpha = \chi_\alpha\pi_\alpha$. Because $U_{\alpha\beta} \to V_{\alpha\beta}$ are categorical quotients we also know that $\chi_\beta\phi_{\alpha\beta}(x) = \chi_\alpha(x)$ for $x \in V_{\alpha\beta}$, where $\phi_{\alpha\beta}$ is as in (2.3.1). It follows that the χ_α combine to define a morphism χ: $M^{cr}_{m,n,p} \to X$ such that $\phi = \chi\pi$. The morphism χ is unique because on each V_α it must equal χ_α. Essentially the same proof was used for [1], 3.2.14.

2.3.3. <u>The Morphisms</u> h: $L_{m,n,p} \to \underline{\underline{A}}^r$ <u>and</u> g: $L_{m,n,p} \to G_{n,(n+1)m}$

Let $(F,G,H) \in L_{m,n,p}$. We let $h(F,G,H) \in \underline{\underline{A}}^r$, where $r = (n+1)^2mp$, be the block Hankel matrix

$$h(F,G,H) = \begin{pmatrix} HG & HFG & . & . & . & HF^nG \\ HFG & & & & & . \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ HF^nG & & . & . & . & HF^{2n}G \end{pmatrix} = Q(F,H)R(F,G)$$

This defines a morphism h: $L_{m,n,p} \to \underline{\underline{A}}^r$, which certainly kills the action of GL_n.

Restricting to $L^{cr}_{m,n,p}$ and applying lemma 2.3.2 we obtain an induced morphism

$$(2.3.4) \qquad \bar{h}: M^{cr}_{m,n,p} \longrightarrow \underline{\underline{A}}^r,\ r = (n+1)^2pm$$

Let $L^{cr}_{m,n}$ be the algebraic variety of all pairs of matrices (F,G) of sizes n x n and n x m. In [1] we constructed a morphism g: $L^{cr}_{m,n} \to G_{n,(n+1)m}$ which kills the action of GL_n on $L^{cr}_{m,n}$, where $G_{n,(n+1)m}$ is the Grassmann variety of n-planes in (n+1)m space:

g assigns to (F,G) the point of $G_{n(n+1)m}$ corresponding to the rank n matrix R(F,G) of size nx(n+1)m. We proved that the quotient variety $M_{m,n} = L^{cr}_{m,n}/GL_n$ exists and that g induces an embedding $\bar{g}: M_{m,n} \to G_{n,(n+1)m}$. Cf. [1] Theorem 3.2.13 and proposition 3.2.14. Now let $g': L^{cr}_{m,n,p} \to G_{n,(n+1)m}$ be the composed morphism $(F,G,H) \mapsto (F,G) \mapsto g(F,G)$. This morphisms kills the action of GL_n and hence induces a morphism

$$\hat{g}: M^{cr}_{m,n,p} \to G_{n,(n+1)m} \tag{2.3.5}$$

From the remarks make above we know that if (F,G,H), (F',G',H') (F',G',H') $\in L^{cr}_{m,n,p}$ are such that g'(F,G,H) = g'(F,G,H) then there is a $T \in GL_n$ such that $(F,G)^T = (\bar{F},\bar{G})$.

2.3.6. An embedding $M^{cr}_{m,n,p} \to G_{n,(n+1)m} \times \underline{\underline{A}}^r$

The morphisms $h,\hat{g}$ of (2.3.4) and (2.3.5) together define a morphism

$$i: M^{cr}_{m,n,p} \to G_{n,(n+1)m} \times \underline{\underline{A}}^r \tag{2.3.7}$$

We claim that i is injective. Indeed if (F,G,H), $(\bar{F},\bar{G},\bar{H}) \in L^{cr}_{m,n,p}$ are such that $h(F,G,H) = h(\bar{F},\bar{G},\bar{H})$ and $g'(F,G,H) = g'(\bar{F},\bar{G},\bar{H})$, then we know that there is a $T \in GL_n$ such that $(F,G)^T = (\bar{F},\bar{G})$ i.e. $TR(F,G) = R(\bar{F},\bar{G})$, and then because $h(F,G,H) = h(\bar{F},\bar{G},\bar{H})$ we have in particular $HR(F,G) = \bar{H}R(\bar{F},\bar{G})$ so that $\bar{H}TR(F,G) = HR(F,G)$ and hence $\bar{H} = HT^{-1}$ because R(F,G) has rank n. This concludes the proof that i is injective.

2.3.8. Corollary.

The prevariety $M^{cr}_{m,n,p}$ is separated, i.e. $M^{cr}_{m,n,p}$ is a variety.

2.3.9. Corollary. $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is a quotient for the cation of GL_n on $L^{cr}_{m,n,p}$ in the category of algebraic varieties.

In fact $M^{cr}_{m,n,p}$ is also a geometric quotient in the sense of [6], but we shall not need this.

2.3.10. Let $V^{co}_\alpha = \psi^{-1}_\alpha(W_\alpha \cap L^{cr,co}_{m,n,p})$ and $V^{co}_{\alpha\beta} = V^{co}_\alpha \cap V_{\alpha\beta}$. Then the $\phi_{\alpha\beta}: V_{\alpha\beta} \to V_{\beta\alpha}$ induce isomorphisms $\phi^{co}_{\alpha\beta}: V^{co}_{\alpha\beta} \to V^{co}_{\beta\alpha}$. Glueing together the V_α^{co} by means of the $\phi_{\alpha\beta}^{co}$ we obtain an open subvariety $M_{m,n,p}^{cr,co}$ of $M_{m,n,p}^{cr}$ which is the image of $L_{m,n,p}^{cr,co}$ under $\pi: L_{m,n,p}^{cr} \to M_{m,n,p}^{cr}$. It follows that the induced morphism $\pi^{co}: L_{m,n,p}^{cr,co} \to M_{m,n,p}^{cr,co}$ is also a categorical quotient.

2.3.11 Similarly, using $V_\alpha^\rho = \psi_\alpha^{-1}(W_\alpha \cap L_{m,n,p}^\rho)$, $V_\alpha^\mu = \psi_\alpha^{-1}(W_\alpha \cap L_{m,n,p}^\mu)$, $V_\alpha^{\mu\rho} = \psi_\alpha^{-1}(W_\alpha \cap L_{m,n,p}^{\mu,\rho})$ and the corresponding $V_{\alpha\beta}$ we obtain categorical quotients $L_{m,n,p}^\rho \to M_{m,n,p}^\rho$, $L_{m,n,p}^\mu \to M_{m,n,p}^\mu$, $L_{m,n,p}^{\rho,\mu} \to M_{m,n,p}^{\rho,\mu}$ where the $M_{m,n,p}^{\cdot\cdot}$ are subvarieties of $M_{m,n,p}^{cr}$.

2.4. Some Realization Theory.

The morphism $\bar{h}$ of (2.3.4) above induces a morphism $\hat{h} : M_{m,n,p}^{cr,co} \to \underline{\underline{A}}^r$. It is the purpose of this and the following subsection to show that $\hat{h}$ is injective and to derive equations for the subvariety $\hat{h}(M_{m,n,p}^{cr,co}) \subset \underline{\underline{A}}^r$. To do this we use some (partial) realization theory an embodied by proposition 2.4.3 below. First a definition

2.4.1. Definition.

Let $A_o, A_1, \ldots$ be a sequence of p x m matrices. Then $h_{q,r}(A)$ denotes the block Hankel matrix

$$h_{q,r}(A) = \begin{pmatrix} A_o & A_1 & \cdots & A_r \\ A_1 & & & \vdots \\ \vdots & & & \vdots \\ A_q & \cdots & \cdots & A_{r+q} \end{pmatrix}$$

2.4.2. Definition.

If A is a matrix and α_c is a subset of the column indices of A, and α_r is a subset of the row indices of A, then we define

A_{α_c} = matrix obtained from A by removing all columns whose index is not in α_c

A_{α_r} = matrix obtained from A by removing all rows whose index is not in α_r

A_{α_r,α_c} = matrix obtained from A by removing all rows and columns whose indices are not in α_r, α_c respectively.

2.4.3. Proposition.

Let $A_0, A_1, \ldots, A_{2n-1}$ be a sequence of 2n matrices with coefficients in k, all of size p x m, and suppose that

$\mathrm{rank}(h_{n-1,n-1}(A)) = \mathrm{rank}(h_{n,n-1}(A)) = \mathrm{rank}(h_{n-1,n}(A)) = n.$

Then there exists an $(F,G,H) \in L^{cr,co}_{m,n,p}$ such that $HF^iG = A_i$ for $i = 0,1, \ldots, 2n-1$.

Moreover if $(\bar{F},\bar{G},\bar{H}) \in L^{cr,co}_{m,n,p}$ is a second triple such that $\bar{H}\bar{F}^i\bar{G} = A_i$ for $i = 0, 1, \ldots, 2n-1$ then there is a $T \in GL_n$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^T$.

Proof. Existence of a triple $(F,G,H) \in L_{m,n,p}$ such that

$$HF^iG = A_i, \quad i = 0, \ldots, 2n-1 \tag{2.4.4}$$

holds is assured by the realizability criterion 11.32 of Chapter 10 of [4]. We define

$$\bar{R}(F,G) = (G \; FG \; \ldots \; F^{n-1}G), \tag{2.4.5}$$

$$\bar{Q}(F,H)' = (H' \; F'H' \; \ldots \; (F')^{n-1}H')$$

Then it follows from (2.4.3) that $\bar{Q}(F,H)\bar{R}(F,H) = h_{n-1,n-1}(A)$.

Now we have $\mathrm{rank}(R(F,G)) \leq n$, $\mathrm{rank}(Q(F,H)) \leq n$ and $\mathrm{rank}(h_{n-1,n-1}(A)) = n$.

It follows that $\mathrm{rank}(R(F,G)) = \mathrm{rank}(Q(F,H)) = n$, so that $(F,G,H) \in L^{cr,co}_{m,n,p}$. Now let $(\bar{F},\bar{G},\bar{H})$ be a second triple in $L_{m,n,p}$ such that

$$\bar{H}\bar{F}^i\bar{G} = A_i, \quad i = 0,1, \ldots, 2n-1 \tag{2.4.6}$$

Then as above we find $\bar{Q}(\bar{F},\bar{H})\bar{R}(\bar{F},\bar{G}) = h_{n-1,n-1}(A)$. Now because $\bar{R}(F,G)$ has rank n there is a subset α_c of size n of the column indices of $R(F,G)$ such that $R(F,G)_{\alpha_c}$ is invertible; further because

$\bar{Q}(F,H)$ has rank n there is a subset α_r fo size n of the row indices of $\bar{Q}(F,H)$ such that $\bar{Q}(F,H)_{\alpha_r}$ is invertible. We have

$$(2.4.7)\quad (h_{n-1,n-1}(A))_{\alpha_r,\alpha_c} = \bar{Q}(F,H)_{\alpha_r}\bar{R}(F,G)_{\alpha_c} =$$

$$\bar{Q}(\bar{F},\bar{H})_{\alpha_r}\bar{R}(\bar{F},\bar{G})_{\alpha_c}$$

so that it follows that all five n x n matrices occurring in (2.4.7) are invertible.

Now let

$$(2.4.8)\quad (F_1,G_1,H_1) = (F,G,H)^T,\text{ where } T = \bar{Q}(F,H)_{\alpha_r}$$

$$(2.4.9)\quad (\bar{F}_1,\bar{G}_1,\bar{H}_1) = (\bar{F},\bar{G},\bar{H})^{\bar{T}},\text{ where } \bar{T} = \bar{Q}(\bar{F},\bar{H})_{\alpha_r}$$

Then we have of course

$$(2.4.10)\qquad H_1F_1^iG_1 = \bar{H}_1\bar{F}_1^i\bar{G}_1 = A_i \text{ for } i = 0,\ \ldots,\ 2n-1$$

which means

$$(2.4.11)\qquad \bar{Q}(F_1,H_1)R(F_1,G_1) = \bar{Q}(\bar{F}_1,\bar{H}_1)R(\bar{F}_1,\bar{G}_1) = h_{n-1,n}(A)$$

and moreover because

$$\bar{Q}(F_1,H_1) = \bar{Q}(F,H)T^{-1},\ \bar{Q}(\bar{F}_1,\bar{H}_1) = \bar{Q}(\bar{F},\bar{H})\bar{T}^{-1}$$

we have

$$(2.4.12)\qquad \bar{Q}(F_1,H_1)_{\alpha_r} = I_n = \bar{Q}(\bar{F}_1,\bar{H}_1)_{\alpha_r}$$

Now combine (2.4.12) and (2.4.11) to obtain that $R(F_1,G_1) = R(\bar{F}_1,\bar{G}_1)$ which by corollary 2.4.2 of [1] means that $F_1 = \bar{F}_1$ and $G_1 = \bar{G}_1$. And because $R(\bar{F}_1,\bar{G}_1) = R(F_1,G_1)$ has rank n, it follows from (2.4.11) that also $H_1 = \bar{H}_1$. We therefore have $(F,G,H)^T = (F_1,G_1,H_1) = (\bar{F}_1,\bar{G}_1,\bar{H}_1) = (\bar{F},\bar{G},\bar{H})^{\bar{T}}$, which proves the second statement of the proposition.

2.4.13. Corollary.

The morphism $\hat{h}\colon M^{cr,co}_{m,n,p} \to \underline{\underline{A}}^r$ of (2.3,4) above is injective.

2.5. Equations for $M^{cr,co}_{m,n,p}$.

By means of the injective morphism $\hat{h}$ we can now consider $M^{cr,co}_{m,n,p}$ as a subvariety of $\underline{\underline{A}}^r$, $r = (n+1)^2pm$, where we write $x \in \underline{\underline{A}}^r$ as an $(n+1)p \times (n+1)m$ matrix. We now consider the following sets of polynomials in the coordinates of $\underline{\underline{A}}^r$.

(2.5.1) $P_a(x)$: these polynomials are such that $P_a(x) = 0$ for all a if and only if the matrix x is of block Hankel type (cf. 2.4.1) with the blocks of size $p \times m$.

(2.5.2) $Q_b(x)$: here $Q_b(x)$ runs through all determinants of $(n+1) \times (n+1)$ submatrices of x.

(2.5.3) $R_c(x)$: here $R_c(x)$ runs through all determinants of $n \times n$ submatrices of the submatrix x' of x which is obtained by removing the last p rows and the last m columns.

2.5.4. Lemma.

Let $(F,G,H) \in L^{cr,co}_{m,n,p}$, $x = h(F,G,H) \in \underline{\underline{A}}^r$. Then we have $P_a(x) = 0$ for all a, $Q_b(x) = 0$ for all b and there is a c such that $R_c(x) \neq 0$.

Proof. Obvious because $h(F,G,H) = Q(F,H)R(F,G)$.

2.5.5. Proposition.

$\hat{h}(M^{cr,co}_{m,n,p}) \subset \underline{\underline{A}}^r$ is the subvariety consisting of those $x \in \underline{\underline{A}}^r$ such that $P_a(x) = 0$ for all a, $Q_b(x) = 0$ for all b and such that these is a c such that $R_c(x) \neq 0$.

Proof. Because of lemma 2.5.4 we only have to show that if $x \in \underline{\underline{A}}^s$ satisfies $P_a(x) = 0$ all a, $Q_b(x) = 0$ all b and $R_c(x) \neq 0$ for some c, then x is in $\hat{h}(M^{cr,co}_{m,n,p})$. Write x as a block Hankel matrix

$$x = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ A_2 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ A_n & & \cdots & A_{2n} \end{pmatrix}$$

This can be done because $P_a(x) = 0$ for all a. Cf. 2.5.1. Then the matrices $A_1, \ldots, A_{2n-1}$ satisfy the conditions of proposition 2.4.3 so that there is a triple $(F,G,H) \in L^{cr,co}_{m,n,p}$ such that $HF^iG = A_i$ for $i = 0,1,2, \ldots, 2n-1$. To show that $h(F,G,H) = x$ it therefore only remains to show that $HF^{2n}G = A_{2n}$. This follows from lemma 2.5.6 below.

2.5.6. Lemma. Let E,E' be two partioned matrices

$$E = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \qquad E' = \left(\begin{array}{c|c} A & B \\ \hline C & D' \end{array}\right)$$

and suppose that rank(E) = rank(E') = rank(A). Then D = D'.

Proof. Let d be an element of D and d' the corresponding element of D'. Let A' be an n x n submatrix of A such that $\det(A') \neq 0$ where n = rank(A). Suppose $A' = E_{\alpha_r,\alpha_c}$, then also $A' = E'_{\alpha_r,\alpha_c}$.

Let $\beta_r = \alpha_r \cup \{i\}$ where i is the index of the row in E containing d and of the row in E' containing d') and $\beta_c = \alpha_c \cup \{j\}$ where j is the index of the column in E containing d (and of the column in E' containing d'). Then we have $\det(E_{\beta_r,\beta_c}) = 0 = \det(E'_{\beta_r,\beta_c})$.

All elements of E_{β_r,β_c} and E'_{β_r,β_c} except possibly the one in the right hadn lower corner are equal and $\det(A') \neq 0$. It follow that d = d'. (By expanding the determinants along the last row e.g .).

2.5.7. Corollary (of proposition 2.5.5).

$M^{cr,co}_{m,n,p}$ is a quasiaffine variety.

2.5.8. Using similar arguments as above combined with those used in [1] to find equations for the variety $M_{m,n}$ (cf. [1] section 3.2), it is not fifficult to find equations for the variety $M^{cr}_{m,n,p}$ (as a subvariety of $G_{n,(n+1)m} \times \underline{\underline{A}}^r$ or as a subvariety of $\underline{\underline{P}}^{r'} \times \underline{\underline{A}}^r$, where $r' = \binom{(n+1)m}{n} -1$).

$M^{cr}_{m,n,p}$ is a quasiprojective variety but not a quasi affine variety if $m > 1$. This last statement is seen as follows. The embedding $L^{cr}_{m,n} \to L^{cr}_{m,n,p}$ given by $(F,G) \to (F,G,0)$, where 0 is a zero matrix of appropriate size, induces an embedding $M_{m,n} \to M^{cr}_{m,n,p}$. Now according to [1] section 3.3 there is an embedding $\underline{\underline{P}}^1 \to M_{m,n}$. Combining these we find an embedding $\underline{\underline{P}}^1 \to M^{cr}_{m,n,p}$ which shows that $M^{cr}_{m,n,p}$ is not quasi affine. (Cf. also the proof of theorem 3.4.6 in [1]).

2.6. <u>The Algebraic Principal Fibre Bundle</u> π: $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$.

As in [3] we can now show that $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is an algebraic principal GL_n fibre bundle over the variety $M^{cr}_{m,n,p}$, and we could use an analysis of the nontriviality or triviality of this bundle to prove nonexistence and existence of algebraic continuous canonical forms. This can be done almost exactly as in [3] section 3 except that one has to construct a different example because the example of [3], section 3.2 is essentially nonalgebraic. Cf. also section 4.1 below for further comments. In this paper, however, we shall first discuss the fine moduli variety properties of $M^{cr}_{m,n,p}$ and then use these to investigate the existence of continuous algebraic canonical forms; this is the same procedure as in [2], cf. especially theorem 6.1 ~~of~~ [2]. The two approaches are essentially equivalent because the underlying vectorbundle of the universal family over $M^{cr}_{m,n,p}$ is the algebraic n-vectorbundle associated to the principal GL_n bundle $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$.

2.7. The codimension of $(M^{cr}_{m,n,p} \setminus M^{cr,co}_{m,n,p})$ in $M^{cr}_{m,n,p}$.

Let $K_{m,n,p}$ be the subvariety of $M^{cr}_{m,n,p}$ defined by the equations $\det(Q(F,H))_\beta) = 0$ for all subsets of size n of the row indices of $Q(F,H)$. I.e.

(2.7.1) $$K_{m,n,p} = M^{cr}_{m,n,p} \setminus M^{cr,co}_{m,n,p}$$

We want to find out something about the codimension of the closed subvariety $K_{m,n,p}$ of $M^{cr}_{m,n,p}$. The result is.

2.7.2. Proposition.

The codimension of $K_{m,n,p}$ in $M^{cr}_{m,n,p}$ is 1 if $p = 1$ and it is $\geq p$ if $p \geq 2$.

To prove proposition 2.7.2 we use the following combinatorial lemma

2.7.3. Lemma.

Let $X = \{a_1,\ldots,a_n\}$ be a finite set of n elements. Let X_o be a subset of X and $\sigma\colon X_o \to X$ an injective map with the following property

(2.7.4) If $Y \subset X_o$ then $\sigma(Y) \not\subset Y$ unless $Y = X_o = X$.

Then there exists a cyclic permutation $\tilde{\sigma}\colon X \to X$ of order n such that $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in X_o$.

Proof. If $X_o = X$ then condition (2.7.4) says that σ is already a cyclic permutation of order n. We can therefore assume that $X_o \neq X$. We are going to show that there is $b \in X \setminus X_o$ and an injective map $\sigma_1\colon X_1 \to X$ with $X_1 = X_o \cup \{b\}$ and $\sigma_1(a) = \sigma(a)$ for $a \in X_o$ such that (2.7.4) holds with X_o replaced by X_1. By induction (with respect to the number of elements in $X \setminus X_o$) this proves the lemma. Because $X_o \neq X$ there is an $a_1 \in X$ which is not in the image of σ. If $a_1 \in X_o$ let $a_2 = \sigma(a_1)$, if $a_1 \notin X_o$ stop; if $a_2 \in X_o$ let $a_3 = \sigma(a_2)$, if $a_2 \notin X_o$ stop; continuing in this way we find a sequence of elements $a_1, a_2, \ldots, a_r$, $r \geq 1$ such that

$$(2.7.5) \qquad a_1 \notin \mathrm{Im}(\sigma),\ a_i = \sigma(a_{i-1}) \text{ for } i = 1, \ldots, r-1,$$

$$a_r \notin X_o$$

Note that the $a_1, \ldots, a_r$ are all different from one another because σ is injective and $a_1 \notin \mathrm{Im}\sigma$. There now are two possibilities

(i) There is no $b \notin X \setminus \mathrm{Im}\sigma$ different from a_1.

In this case $\mathrm{Im}\sigma$ has $n - 1$ elements and hence so has X_o. Therefore $X \setminus X_o = \{a_r\}$. Let $Y = X \setminus \{a_1,\ldots,a_r\}$ and suppose $Y \neq \emptyset$. Then we have $Y \subset X_o$ because $X \setminus X_o = \{a_r\}$. We also have $\sigma(Y) \subset Y$ because $\sigma(\{a_1,\ldots,a_r\} \cap X_o) \subset \{a_1,\ldots,a_r\}$.
Therefore, because σ is injective, we would have $\sigma(Y) = Y$ contradicting (2.7.4). Therefore $Y = \emptyset$ and $X = \{a_1,\ldots,a_r\}$ in this case (i.e. $r = n$). We now take $b = a_1$ and define $\sigma_1(b) = a_1$. Then $X_1 = X_o \cup \{a_r\} = X$ and $\sigma_1 \colon X \to X$ is clearly the desired cyclic permutation.

(ii) There is a $b_1 \in X \setminus \mathrm{Im}\sigma$ which is different from a_1.
In this case we take $b = a_r$ and define $\sigma_1(b) = b_1$. The map σ_1 is injective because $b_1 \notin \mathrm{Im}\sigma$. Now suppose that $Y \subset X_1$ is such that $\sigma_1(Y) = Y$. Note that in this case $X \setminus \mathrm{Im}\sigma$ has at least two elements, hence so has $X - X_o$, so that $X_1 \neq X$.
There are two possibilities

(ii_1) $b = a_r \notin Y$. In this case $Y = \sigma_1(Y) = \sigma(Y)$ and $Y \subset X_o$ which contradicts (2.7.4).

(ii_2) $b = a_r \in Y$. Then because $\sigma_1(Y) = Y$ we must have $a_{r-1} \in Y$, $a_{r-2} \in Y, \ldots, a_1 \in Y \subset X_1$, which is a contradiction because there is no $c \in X$ such that $\sigma(c) = a_1$ because $a_1 \notin \mathrm{Im}\sigma_1 = \mathrm{Im}\sigma \cup \{b_1\}$. This concludes the proof of the lemma.

2.7.6. Now consider $x \in \underline{\underline{A}}^{mn}$; consider $(F_\alpha(x), G_\alpha(x)$, where α is a nice selection, $\alpha \subset J_{m,n}$. We recall how $F_\alpha(x)$ and $G_\alpha(x)$ are defined (cf.[1] section 2.3). Let $J = \alpha \cup \{s(\alpha,1), \ldots, s(\alpha,m)\}$ as an ordered subset of $J_{m,n}$. Let x_1 be the column vector consisting of the first n coordinates of x, x_2 the column vector consisting of the second n coordinates etc. We now define $n + m$ column vectors

y_i, $i = 1,2, \ldots, m+n$ of length n as follows

$$(2.7.7) \qquad y_i = \begin{cases} e_\ell & \text{if the i-th element of J is the } \ell\text{-th element of } \alpha \\ x_j & \text{if the i-th element of J is } s(\alpha,j) \end{cases}$$

where e_ℓ is the ℓ-th standard basis vector.

The matrices $G_\alpha(x)$ and $F_\alpha(x)$ are now defined by

$$(2.7.8) \qquad G_\alpha(x)_j = y_j,\ j = 1, \ldots, m;\ F_\alpha(x)_j = y_{m+j},\ j = 1, \ldots, n$$

It readily follows from this that $F_\alpha(x)$ is a matrix of the following type: the j-th column of $F_\alpha(x)$ is either a standard basis vector e_ℓ with $\ell > j$, or $F_\alpha(x)_j = x_i$ for some i; moreover if $F_\alpha(x)_{j_1} = e_{\ell_1}$, $F_\alpha(x)_{j_2} = e_{\ell_2}$ with $j_1 < j_2$ then $\ell_1 < \ell_2$. Applying lemma 2.7.3 we thus see that by specifying the x_i, $i = 1, \ldots, m$ to be suitable standard basis vectors one obtains

2.7.9. Lemma.

For every nice selection α, there is an $x \in \underline{\underline{A}}_{mn}$ such that $F_\alpha(x)$ is a cyclic permutation of order n of the standard basis vectors.

2.7.10. Let α be a nice selection. Now consider $K_{m,n,p} \cap V_\alpha = V_\alpha \setminus V_\alpha^{co}$ where α is a nice selection. This closed subvariety of U_α is defined by the equations $\det(Q(F_\alpha(x),H_\alpha(x))_\beta) = 0$ for all subsets β of size n of the row indices of $Q(F_\alpha(x),H_\alpha(x))$. We number the rows of $Q(F_\alpha(x),H_\alpha(x))$ as follows

$$((0,1), \ldots, (0,p); (1,p), \ldots, (1,p); \ldots$$
$$\ldots; (n,1), \ldots, (n,p))$$

Take $\beta_1 = \{(0,1),(1,1), \ldots, (n-1,1)\}$. Write $x \in V_\alpha = \underline{\underline{A}}^{mn} \times \underline{\underline{A}}^{pn}$ as $x = (y,z)$ and write z as the matrix (z_{ij}), $i = 1, \ldots, p$, $j = 1, \ldots, n$. We write $F_\alpha(x) = F_\alpha(y)$, $H_\alpha(x) = z$. Now consider the equation

$$(2.7.11) \qquad \det(Q(F_\alpha(x), H_\alpha(x))_{\beta_1}) = 0$$

Now specify the y such that $F_\alpha(x)$ is a cyclic permutation matrix of order n and suppose that the first row vector of $F_\alpha(x)$ under this specification is the ℓ-th standard basis vector. Now take $z_{ij} = 0$ for $j \neq \ell$. Then (2.7.11) becomes

$$(2.7.12) \qquad \pm z^n_{1\ell} = 0$$

If p = 1, equation (2.7.11) defines $K_{m,n,p} \cap V_\alpha$ (Because if rank Q(F,H) = n then there is a nice "selection" β from the row indices of Q(F,H) such that $\det(Q(F,H)_\beta) \neq 0$ by the transposed version of lemma 2.1.1). Equation (2.7.12) which is obtained from (2.7.11) by a suitable specification of some of the variables shows that (2.7.11) is non trivial, so that the codimension of $K_{m,n,p} \cap V_\alpha$ in V_α is one for each nice selection α proving that the codimension of $K_{m,n,1}$ in $M^{cr}_{m,n,1}$ is one. Now suppose that p > 1. And consider the selections

$$\beta_i = \{(0,i), (1,i), \ldots, (n-1,i)\} \quad i = 1, \ldots, p$$

Specifying the y and z as before (NB the specification to be used depends on α!), the equations

$$(2.7.13) \qquad \det(Q(F_\alpha(x), H_\alpha(x))_{\beta_i}) = 0 \quad i = 1, \ldots, p$$

specify to

$$(2.7.14) \qquad \pm z^n_{i\ell} = 0 \quad i = 1, \ldots, p$$

The equations (2.7.14) are independent, hence so are the equations (2.7.13) proving that the codimension of $K_{m,n,p} \cap V_\alpha$ in V_α is $\geq$ p. This holds for all nice selections α so that the codimension of $K_{m,n,p}$ in $M^{cr}_{m,n,p}$ is always $\geq$ p. We have now proved assertion 2.7.2.

2.7.15. *Remark*.

To prove 2.7.2 all one really needs is the existence of a triple $(F,G,H) \in W_\alpha$ for each α such that F' is a cyclic matrix. This can be seen as follows: U_α is a nonempty open subvariety of $L_{m,n,p}$. Let $L' = \{(F,G,H) \in L_{m,n,p} | F' \text{ is cyclic}\}$ this also defines a nonempty open subvariety of $L_{m,n,p}$. Because $L_{m,n,p}$ is irreducible $L' \cap U_\alpha \neq \phi$. Let $(F,G,H) \in L' \cap U_\alpha$ and let $(F,G,H) = (F,G,H)^T$ where $T = R(F,G)_\alpha^{-1}$. Then $(F,G,H) \in W_\alpha$ and F' is cyclic.

3. THE FINE MODULI VARIETY $M^{cr}_{m,n,p}$.

We now proceed to study families of linear dynamical systems. Some motivation as to why one would like to study families is given in section 1.8 of [3]. Morover, in this paper we shall use families to investigate whether there exist continuous canonical forms or not. This is not necessary; one can also use the principal algebraic GL_n bundle $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$. Cf. also 2.6 above. This part of the theory in the algebraic geometric case is practically completely analogous to the corresponding part of the topological case which was treated in section 4 of [3].

3.1. *Families of Linear Dynamical Systems*.

3.1.1. *Definition*.

A *family of linear dynamical systems over a variety* S of *dimensions* (n,m,p) consists of

(i) an algebraic n-dimensional vectorbundle $p:E \to S$
(ii) an algebraic vectorbundle endomorphism $F:E \to E$
(iii) an algebraic vectorbundle homomorphism $G:S \times \underline{\underline{A}}^m \to E$
(iv) an algebraic vectorbundle homomorphism $H:E \to S \times \underline{\underline{A}}^p$.

Let $s \in S$, then F,G,H induce homomorphisms $F_s\colon E_s \to E_s$, $G_s\colon s \times \underline{\underline{A}}^m \to E_s$, $H_s\colon E_s \to s \times \underline{\underline{A}}^p$; $E_s = p^{-1}(s)$ is the fibre over s.

(Cf. Appendix 3 of [1]). Choosing a basis $e_1(s), \ldots, e_n(s)$ of E_s and taking the obvious bases in $s \times \underline{\underline{A}}^m$ and $s \times \underline{\underline{A}}^p$ we calculate the matrices of F_s, G_s, H_s relative these bases. Let the result be

$(F(s,e), G(s,e), H(s,e))$. This triple depends on $e_1(s), \ldots, e_n(s)$ only to the extent that a different choice of $e_1(s), \ldots, e_n(s)$ gives a triple in the same orbit (under GL_n) as $(F(s,e), G(s,e), H(s,e))$.

The family Σ is said to be completely reachable if $(F(s,e), G(s,e)$ $H(s,e)) \in L^{cr}_{m,n,p}$ for all s. (This is well defined because $L^{cr}_{m,n,p}$ is GL_n invariant).

3.1.2 The Canonical Morphism Associated to Completely Reachable Family.

Now let Σ be a completely reachable family. Then F_s, G_s, H_s define a unique orbit in $L^{cr}_{m,n,p}$ and thus a unique point in $M^{cr}_{m,n,p}$ which we shall denote $f_\Sigma(s)$. Thus we have a map $f_\Sigma : S \to M^{cr}_{m,n,p}$. Using the local triviality of the bundle E one shows by means of the algebraic analogues of the constructions in 4.1.2 - 4.1.8 of [3] that f_Σ is a morphism in the category of varieties.

3.1.3. In the topological case we associated a continuous map $f_\Sigma: X \to M_{m,n,p}(\mathbb{R})$ to every family Σ, and used this map to define complete reachability of families. This cannot be done in the algebraic geometric case because the variety $M_{m,n,p}$ does not exist.

3.2. The Universal Family Σ^u over $M_{m,n,p}$

Let α be a nice selection. Let $E_\alpha = V_\alpha \times \underline{\underline{A}}^n$, $p_\alpha: E_\alpha \to V_\alpha$ the obvious projection. We define families Σ_α of linear dynamical systems with underlying bundles E_α by the formulas

$$(3.2.1) \quad F_\alpha(x,v) = (x, F_\alpha(x)v), \; G_\alpha(x,u) = (x, G_\alpha(x)u), \\ H_\alpha(x,v) = (x, H_\alpha(x)v)$$

where for $x \in V_\alpha$, $\psi_\alpha(x) = (F_\alpha(x), G_\alpha(x), H_\alpha(x))$, cf. [3] 2.4.5. Now let $E_{\alpha\beta} = V_{\alpha\beta} \times \underline{\underline{A}}^n$ and define the isomorphisms $\phi_{\alpha\beta}: E_{\alpha\beta} \to E_{\beta\alpha}$ by formula (4.3.6) of [3]. Then glueing together the E_α by means of the $\phi_{\alpha\beta}$ we obtain an algebraic vectorbundle E^u. The $F_\alpha, G_\alpha, H_\alpha$ are compatible with the $\phi_{\alpha\beta}$ in the sense of (4.3.9) - (4.3.11) of [3] and thus define homomorphisms $F^u: E^u \to E^u$, $g^u: M^{cr}_{m,n,p} \times \underline{\underline{A}}^m \to E^u$,

H^u: $E^u \to M^{cr}_{m,n,p} \times \underline{\underline{A}}^p$. This defines the family Σ^u. The family Σ^u is completely reachable (because this is true for the families Σ_α), and the associated map f_{Σ^u}: $M^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is the identity map (because the triple $(F_\alpha(x), G_\alpha(x), H_\alpha(x))$ maps to $x \in V_\alpha \subset M_{m,n,p}$ under π: $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$).

3.3. The Fine Moduli Variety $M^{cr}_{m,n,p}$.

3.3.1. Two families Σ, $\bar{\Sigma}$ are isomorphic if there is an algebraic vectorbundle isomorphism ϕ: $E \to E$ such that $\bar{F}\phi = \phi F$, $\phi G = \bar{G}$, $H = \bar{H}\phi$. For each $S \in \underline{\underline{Sch}}_k$, the category of algebraic varieties over k, let $\Phi_{m,n,p}(S)$ be the set of isomorphism classes of completely reachable families of linear dynamical systems over S. By means of the pullback construction we turn $\Phi_{m,n,p}(S)$ into a functor $\Phi_{m,n,p}$: $\underline{\underline{Sch}}_k^{opp} \to \underline{\underline{Set}}$.

3.3.2. Theorem.

The variety $M^{cr}_{m,n,p}$ is a fine moduli variety for $\Phi_{m,n,p}$ or, in other words, the functor $\Phi_{m,n,p}$ is representable by $M^{cr}_{m,n,p}$. More precisely, the assignment $\Sigma \mapsto f_\Sigma$ induces a functorial isomorphism $\Phi_{m,n,p}(S) \to \underline{\underline{Sch}}_k(S, M^{cr}_{m,n,p})$; the inverse isomorphism assigns the isomorphism class of $f^{!}\Sigma^u$ to f: $S \to M^{cr}_{m,n,p}$.

Proof. Identical with the proof of the corresponding theorem 4.5.2 of [3].

4. EXISTENCE AND NONEXISTENCE OF ALGEBRAIC CONTINUOUS CANONICAL FORMS.

In [1] we used the fact that $M_{m,n}$ admits an embedding $\underline{\underline{P}}^1 \to M_{m,n}$ if $m \geq 2$ to show that there is no algebraic continuous form for completely reachable pairs of matrices. This cannot be used to prove e.g. part (iii) of theorem 1.5 because as we have seen $M^{cr,co}_{m,n,p}$ is a quasi affine algebraic variety. Further the example we used in [3] to prove nonexistence of continuous canonical forms for real linear dynamical systems if $m \geq 2$ and $p \geq 2$ is essentially nonalgebraic. There is, however, a three (instead of one) dimensional version of it which is algebraic and that is the example we shall use in this paper. We proceed via moduli varieties as in [2].

4.1. Triviality of E^u and Existence of Continuous Algebraic Canonical Forms.

4.1.1. Theorem.

Let $L \subset L^{cr}_{m,n,p}$ be a GL_n-invariant subvariety of $L^{cr}_{m,n,p}$ and let $M = \pi(L)$. Then there exists a continuous algebraic canonical form on L if and only if the algebraic vector bundle $E^u|M$ is trivial.

Proof. Let $\Phi^L_{m,n,p}$ be the subfunctor of $\Phi_{m,n,p}$ defined by considering only isomorphism classes of families Σ over S such that f_Σ maps S into $M = \pi(L)$. It follows directly from theorem 3.3.2 that $\Sigma \mapsto f_\Sigma$ then defines a functorial isomorphism $\Phi^L_{m,n,p}(S) \overset{\sim}{\to} Sch_k(S,M)$ and that the inverse isomorphism is given by $f \mapsto f^{!}(\Sigma^u|M)$ where $\Sigma^u|M = (E^u|M, F^u|M, G^u|M, H^u|M)$ is the restriction of Σ^u to M. Now suppose that there exists a continuous algebraic canonical form $c\colon L \to L$. Because c kills the action of GL_n there is a unique morphism $\bar{c}\colon M \to L$ such that $c = \bar{c}\pi$. For each $x \in M$ we write $\bar{c}(x) = (F_c(x), G_c(x), H_c(x))$. Note that $\pi\bar{c} = id$, by condition (1.3.1) of the definition of canonical form.

We now define a family Σ^c over M as follows; $\Sigma^c = (E^c, F^c, G^c, H^c)$ with $E^c = M \times A^n$, $F^c(x,v) = (x, F_c(x)v)$, $G^c(x,u) = (x, G_c(x)u)$, $H^c(x,v) = (x, H_c(x)v)$. Because $\pi c = id$ and $c(x) = (F_c(x), G_c(x), H_c(x))$ we have that $f_{\Sigma^c}\colon M \to M$ is the identity morphism, cf. 3.1.2.

But, according to theorem 3.3.2, or rather the relative version discussed in the beginning of this proof, we have that $(f_{\Sigma^c})^{!}(\Sigma^u|M)$ is isomorphic to Σ^c, which in particular means that $(f_{\Sigma^c})^{!}(E^u|M) \simeq E^c = M \times \underline{\underline{A}}^n$; but $f_{\Sigma^c} = id$, hence $E^u|M$ is trivial.

Inversely suppose that $E^u|M$ is trivial. Then we can find n algebraic sections $e_1, \ldots, e_n\colon M \to E^u|M$ such that $e_1(x), \ldots, e_n(x)$ is a basis for E^u_x for all $x \in M$. Let $F(x,e), G(x,e), H(x,e)$ be the matrices of $F_x\colon E^u_x \to E^u_x$, $G_x\colon \{x\} \times A^m \to E^u_x$, $H_x\colon E^u_x \to x \times A^p$ relative the obvious bases in $x \times \underline{\underline{A}}^m$ and $x \times \underline{\underline{A}}^p$ and the basis $\{e_1(x), \ldots, e_n(x)\}$ of E^u_x.

We now define a morphism $c\colon L \to L$ as follows

$$c(F,G,H) = (F(x,e),\ G(x,e),\ H(x,e)) \text{ where } x = \pi(F,G,H)$$

One easily checks that this is a continuous algebraic canonical form.

4.1.2. The Local Canonical Forms $c_{\ast\alpha}$.

Let α be a nice selection. The bundle $E^u|U_\alpha$ is trivial (by the definition of E^u cf. 3.2) hence by theorem 4.1.1 there exist continuous algebraic canonical forms on U_α. Such canonical forms are wellknown. An example is the canonical form $c_{\ast\alpha}$ defined by

(4.1.3) $$c_{\ast\alpha}(F,G,H) = (F,G,H)^T,\ T = R(F,G)_\alpha^{-1}$$

4.1.3. <u>Corollary.</u>

If $m = 1$ there is a continuous algebraic canonical form on $L^{cr}_{m,n,p}$.

Proof. If $m = 1$ there is only one nice selection α, and hence $L^{cr}_{m,n,p} = U_\alpha$ by lemma 2.1.1.

4.2. <u>Duality.</u>

The assignment δ: $(F,G,H) \to (F',H',G')$ defines an isomorphism of algebraic varieties $L_{m,n,p} \to L_{p,n,m}$. If $L \subset L_{m,n,p}$ is GL_n-invariant then so is $\delta(L) \subset L_{p,n,m}$ (but δ is not GL_n-invariant). As in [3], 3.1.6 one now easily shows that there is a continuous algebraic canonical form on L if and only if there is a continuous canonical form on $\delta(L)$.

4.2.1. <u>Corollary.</u>

There is an algebraic continuous canonical form on $L^{cr}_{m,n,p}$ if $p = 1$.

4.3. <u>Example of a Nontrivial Algebraic Line Bundle.</u>

Let $U_1 = \underline{\underline{A}}^1 \times (\underline{\underline{A}}^2 \setminus (0,0))$, $U_2 = \underline{\underline{A}}^1 \times (\underline{\underline{A}}^2 \setminus (0,0))$. We give U_1 coordinates (t,y_1,y_2) and U_2 coordinates (s,x_1,x_2). Let $U_{12} = \{(t,y_1,y_2) \in U_1 \mid t \neq 0\}$, $U_{21} = \{(s,x_1,x_2) \in U_2 | s \neq 0\}$. We define an isomorphism ϕ: $U_{12} \to U_{21}$ by $(t,y_1,y_2) \to (t^{-1},y_1t,y_2t)$. Let X be the prevariety obtained by glueing U_1 and U_2 together by means of ϕ. In fact X is a variety viz. the quasi affine subvariety of $\mathbb{A}^4 = \{(z_1,z_2,z_3,z_4)\}$ given by $z_1z_4 = z_2z_3$ and ($z_1 \neq 0$ or $z_2 \neq 0$ or $z_3 \neq 0$ or $z_4 \neq 0$). The embeddings of U_1 and U_2 in this subvariety are given by $(t,y_1,y_2) \to (y_1t,y_1,y_2t,y_2)$, $(s,x_1,x_2) \to (x_1,x_1s,x_2,x_2s)$. It is easy to check that this respects the identification ϕ given above.

We now define an algebraic line bundle V over X by glueing $U_1 \times \mathbb{A}^1$ and $U_2 \times \mathbb{A}^1$ together by means of the isomorphism

$$\tilde{\phi}: U_{12}\times\mathbb{A}^1 \to U_{21}\times\mathbb{A}^1,\ (t,y_1,y_2,u) \mapsto (s,x_1,x_2,v) \quad \text{iff} \tag{4.3.1}$$

$$ts = 1,\ x_1 = ty,\ x_2 = ty_2,\ v = t^{-1}u$$

Now suppose that this line bundle is trivial. Then there must be everywhere non zero sections $U_1 \to U_1 \times \mathbb{A}^1$, $(t,y_1,y_2) \mapsto ((t,y_1,y_2),\ g_1(t,y_1,y_2))$; $U_2 \to U_2 \times \mathbb{A}^1$, $(s,x_1,x_2) \mapsto ((s,x_1,x_2),\ g_2(s,x_1,x_2))$ compatible with the identification $\tilde{\phi}$. Now g_1 and g_2 are morphisms $\mathbb{A}_1 \times (\mathbb{A}_2 \setminus (0,0)) \to \mathbb{A}_1$. Because $A_1 \times (\ )$ is of codimension 2 in $\mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3$ this means that g_1 and g_2 extend to morphisms on all of $\mathbb{A}^3$, i.e. g_1 and g_2 are polynomials. Putting everything together we therefore have that C is a trivial line bundle iff there are polynomials $g_1(t,y_1,y_2)$, $g_2(s,x_1,x_2)$ such that $g_1(t,y_1,y_2) \neq 0$ if $y_1 \neq 0$ or $y_2 \neq 0$ and $g_2(s,x_1,x_2) \neq 0$ if $x_1 \neq 0$ or $x_2 \neq 0$ and such that moreover

$$tg_1(t,y_1,y_2) = g_2(t^{-1},ty_1,ty_2) \tag{4.3.2}$$

for all points (t,y_1,y_2) such that $t \neq 0$ and $y_1 \neq 0$ or $y_2 \neq 0$. One easily checks that the only polynomials $g_1(t,y_1,y_2)$ such that $g_1(t,y_1,y_2) \neq 0$ for all (t,y_1,y_2) for which $y_1 \neq 0$ or $y_2 \neq 0$ are constants. Similarly $g_2(s,x_1,x_2)$ is a constant. But then (4.3.2) is a contradiction. So we have proved

4.3.2. Lemma. The line bundle V defined by 4.3.1 is nontrivial.

4.4. Examples.

Let $p \geq 2$ and $m \geq 2$. We write down a number of G,F and H matrices as follows

(4.4.1) If $n = 1$, $m \geq 2$ $\qquad G_{1,m}(t,s) = (t \;\; s \;\; 0 \;\; \ldots \;\; 0)$

(4.4.2) If $n > 2 < m < n$

$$G_{n,m}(t,s) = \left(\begin{array}{cc|ccc} t & s & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \hline a & 1 & & & \\ \vdots & \vdots & & B & \\ a & 1 & & & \end{array}\right)$$

where a is a nonzero element of k different from 1, and where B is an $(n-2) \times (m-2)$ matrix with coefficients in k such that the columns of B and the column vector $(1, \ldots, 1)'$ together span an $m - 1$ dimensional subspace of k^{n-2}. Such a B exists because $2 < m < n$.

(4.4.3) If $n > 2 = m$

$$G_{n,2}(t,s) = \begin{pmatrix} t & s \\ 1 & 1 \\ a & 1 \\ \vdots & \vdots \\ a & 1 \end{pmatrix}$$

(4.4.4) If $m > n \geq 2$

$$G_{n,m}(t,s) = \begin{pmatrix} t & s & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & \vdots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & 0 & \underbrace{0 \; \cdots}_{} & & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad \underbrace{\qquad\qquad}_{n-1} \qquad \underbrace{\qquad\qquad}_{m-n-1}$$

(4.4.5) If $m = n \geq 2$

$$G_{n,n}(t,s) = \left(\begin{array}{ccccc} t & s & 0 & \ldots & 0 \\ 1 & 1 & \vdots & & \vdots \\ a & 1 & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ a & 1 & \underbrace{0 \quad \ldots \quad 0}_{n-2} \end{array}\right) \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} n-2 \end{array}$$

$$(4.4.6) \qquad F_n = \begin{pmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & - & - & 0 & a_n \end{pmatrix}$$

where $a_1, \ldots, a_n$ are n different elements of k which are all different from zero

$$(4.4.7) \qquad H_{p,n}(y_1,y_2) = G_{n,p}(y_1,y_2)',$$

4.5 An Embedding $X \to M^{cr}_{m,n,p}$

Let U_1, U_2 be as in 4.3 above. We define for all n,m,p with $m \geq 2$ and $p \geq 2$

$$(4.5.1) \qquad \begin{aligned} \sigma_{n,m,p}&: U_1 \to L^{cr}_{m,n,p}, \ (t,y_1,y_2) \to (F_n, G_{n,m}(t,1), H_{p,n}(y_1,y_2)) \\ \bar{\sigma}_{n,m,p}&: U_2 \to L^{cr}_{m,n,p}, \ (s,x_1,x_2) \to (F_n, G_{n,m}(1,s), H_{p,n}(x_1,x_2)) \end{aligned}$$

We now note that if $ts = 1$, $x_1 = y_1 t$, $x_2 = y_2 t$

$$(4.5.2) \qquad (F_n, G_{n,m}(t,1), H_{p,n}(y_1,y_2))^{T(t)} = (F_n, G_{n,m}(1,s), H_{p,n}(x_1,x_2))$$

where

$$T(t) = \begin{pmatrix} t^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & - & - & 0 & 1 \end{pmatrix}$$

This means that the morphisms $U_1 \to M^{cr}_{m,n,p}$, $U_2 \to M^{cr}_{m,n,p}$, obtained from the morphisms $\sigma_{n,m,p}$ and $\bar{\sigma}_{n,m,p}$ of (4.5.1) by composing with $\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$, combine to define a morphism

$$(4.5.3) \qquad \tau_{m,n,p}: X \to M^{cr}_{m,n,p}$$

where X is the variety defined in 4.3 above.

4.5.4. Let α be the nice selection $\{(0,2), (1,2), \ldots, (n-1,2)\}$ then we see from 4.4 that $\sigma_{m,n,p}(U_1) \subset U_\alpha$ and hence $\tau_{m,n,p}(U_1) \subset V_\alpha$. Let β be the nice selection $\{(0,1), (1,1), \ldots, (n-1,1)\}$ then we see from 4.4 that $\bar{\sigma}_{m,n,p}(U_2) \subset U_\beta$ and hence $\tau_{m,n,p}(U_2) \subset V_\beta$. It follows that the pullback of E^u by means of $\tau_{m,n,p}$ is an algebraic vectorbundle over X whose restrictions to U_1 and U_2 are trivial, and the glueing data of this bundle are given by (Cf. [1] Appendix 3.6)

$$\tilde{\psi} : U_{12} \times \underline{\underline{A}}^n \to U_{21} \times \underline{\underline{A}}^n \tag{4.5.5}$$

$$((t,y_1,y_2),u) \to ((t^{-1},ty_1,ty_2),\ T(t,y_1,y_2)u)$$

where $T(t,y_1,y_2)$ is equal to the matrix

$$R(F_n,G_{n,m}(t,1))_\beta^{-1} R(F_n,G_{n,m}(t,1))_\alpha \tag{4.5.6}$$

where α and β are the nice selections $\{(0,2),(1,2), \ldots, (n-1,2)\}$ and $\{(0,1),(1,1), \ldots, (n-1,1)\}$. Let $E \to X$ be this bundle. The exterior product bundle $\Lambda^n E \to X$ is then the line bundle obtained by glueing together $U_1 \times A^1$ and $U_2 \times A^1$ by means of the isomorphism

$$\psi: U_{12} \times \underline{\underline{A}}^1 \to U_{21} \times \underline{\underline{A}}^1 \tag{4.5.7}$$

$$((t,y_1,y_2),u) \to ((t^{-1},ty_1,ty_2),\ \det(T(t,y_1,y_2))u)$$

and from (4.5.6) we see that

$$\det(T(t,y_1,y_2)) = \begin{cases} t^{-1} & \text{if } n \leq 2 \\ t^{-1}a^{n-2} & \text{if } n \geq 2 \end{cases} \tag{4.5.8}$$

It follows that the line bundle defined by $\hat{\psi}$ is nontrivial. Cf. 4.3 above.

4.5.9. <u>Proposition</u>.

The algebraic vectorbundle $\tau^!_{n,m,p}E^u$ is nontrivial if $p \geq 2$, $m \geq 2$.

Proof. This follows from the above because if $E \to X$ is a trivial algebraic n dimensional vector bundle then $\overset{n}{\wedge} E \to X$ is a trivial line bundle.

4.5.10. Corollary.

Let M be a subvariety of $M^{cr}_{m,n,p}$ such that $\tau_{n,m,p}(X) \subset M$. Then $E^u|M$ is a nontrivial algebraic vectorbundle.

4.6. Nonexistence of Continuous Algebraic Canonical Forms.

We can now prove theorem 1.5

4.6.1. Proof of Theorem 1.5

First let $m \geq 2$ and $p \geq 2$. Let $M^w = \pi(L^w_{m,n,p})$ where $L^w_{m,n,p}$ runs through the subvarieties listed in 1.4. Then we see from 4.4

$$(4.6.2) \qquad \tau_{m,n,p}(X) \subset M^{\rho,\mu}$$

if $m \neq n$ and $p \neq n$, and that in any case (still assuming $p \geq 2$ and $m \geq 2$)

$$(4.6.3) \qquad \tau_{m,n,p}(X) \subset M^{cr,co,\mu}$$

By corollary 4.5.10 and theorem 4.4.1 this takes care of the only if parts of statements (iii), (iv), (v), (vi) of theorem 1.5. (Because $L^{\rho,\mu}_{m,n,p} \subset L^{\rho}_{m,n,p}$ and $L^{cr,co}_{m,n,p} \subset L^{cr,co,\rho}_{m,n,p}$). On the other hand if $m = 1$ in cases (iii) and (iv) and $m = 1$ or n in cases (v) and (vi) then the respective subvarieties are contained in one U_α for a certain nice selection α. By 4.1.2 there are therefore continuous algebraic canonical forms in these cases. The corresponding fact for $p = 1$ in cases (iii), (iv) and $p = 1$ or n in cases (v), (vi) follows by duality. Cf. 4.2. This proves (iii) - (vi) of theorem 1.5.

The if part of (i) is corollary 4.1.3; the if part of (ii) follows by duality. Cf. 4.2. To prove the only if part of (i) observe that if $m \geq 2$ $\pi((F_n,G_{n,m}(t,s),0)$, where $t \neq 0$ or $s \neq 0$, depends only on the point $(t:s) \in \underline{\underline{P}}^1$ and not on the actual t and s. Thus

$$\tau : (t;s) \to \pi((F_n,G_{n,m}(t,s),0)$$

defines a morphism $\underline{\underline{P}}^1 \to M^{cr}_{m,n,p}$ for all (m,n,p) such that $m \geq 2$. As in 4.5 one now proves that $\tau^{!}E^u$ is nontrivial. By 4.5.10 and 4.4.1 this proves the only if part of (i). The only if part of (ii) follows by duality. Cf. 4.2. This concludes the proof of theorem 4.5.

4.7. <u>On Relations between Various Local Canonical Forms</u>.

Let $U \subset L^{cr}_{m,n,p}$ be a GL_n invariant subvariety of $L^{cr}_{m,n,p}$, and suppose that there is a continuous algebraic canonical form $c: U \to U$. Let $\kappa: U \to \underline{\underline{A}}^1$ be a morphism, e.g. a "coordinate function". Then $\kappa c: U \to A^1$ is GL_n invariant, showing that "the coordinate functions of a canonical form are invariants".

4.7.1. Now let $a: U \to GL_n$ be a morphism which kills the action of GL_n on U. Then if $c: U \to U$ is a continuous algebraic canonical form so is $c^a: U \to U$ which is defined by $(F,G,H) \mapsto c(F,G,H)^{a(F,G,H)}$. Inversely if c' is a second continuous algebraic canonical form on U then $c' = c^a$ for some morphism $a: U \to GL_n$ which kills the action of GL_n on U. All this is proved as in section 3.6 of [1].

4.7.2. Now let $U = U_\alpha$. We have the canonical forms $c_{*\alpha}$. Every other canonical form is given by a morphism $a: U_\alpha \to GL_n$ which kills the action of GL_n, i.e. by a morphism $\hat{a}: V_\alpha \to GL_n$. Because $V_\alpha \simeq A^{mn+np}$ we must have that $\det(\hat{a}(x))$ is a nonzero constant independant of $x \in V_\alpha$. Cf. also section 3.6.7 of [1].

4.7.3. The situation becomes slightly more complicated if we take $U = U^{co}_\alpha$. We still have the canonical forms $c_{*\alpha}$ and all other canonical forms are obtained by means of a morphism $\hat{a}: V^{co}_\alpha \to GL_n$. Now if $p = 1$ then $\det(\hat{a}(x))$ need not be a constant independant of $x \in V^{co}_\alpha$, because the codimension of $V_\alpha \setminus V^{co}_\alpha$ in V_α is one if $p = 1$. An example of this is found by taking $m = 1 = p$ and comparing the canonical form $c_{*\alpha}$ and its dual on $M^{cr,co}_{1,n,1}$. However if $p \geq 2$, then the codimension of $V_\alpha \setminus V^{co}_\alpha$ in V_α is ≥ 2 (cf. section 2.7) above), which means that in this case we again have that $\hat{a}: V_{\alpha}^{co} \to GL_n$ is given by n^2 polynomials such that $\det(\hat{a}(x))$ is a constant independant of $x \in V_{\alpha}^{co}$.

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