MODULI AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS, III: THE ALGEBRAIC-GEOMETRIC CASE

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1. INTRODUCTION

In this paper we treat the algebraic-geometric version of the topological theory developed in [3].   That is we study linear dynamical systems over an algebraically closed field k

$$x_{t+1} = Fx_t + Gu_t \ , \ x_t \in k^n, \ u_t \in k^m$$

$$y_t = Hx_t \ , \qquad y_t \in k^p \qquad\qquad (1.1)$$

where F,G,H are matrices with coefficients in k of the appropriate sizes. A change of basis in state space changes the triple of matrices (F,G,H) into (TFT^{-1}, TG, HT^{-1}) and as in [3] we are interested in such questions as the following.

Does the set of orbits under this action have a (natural) structure of an algebraic variety? Do there exist continuous canonical forms? Similar questions for

the case of two matrices were studied and answered in [1], cf. also [2].

Essentially the answers are as in [3]. This paper used a moderate amount of algebraic geometry (nothing much beyond definitions). Appendices 1, 2 and 3 of [1] provide sufficient background information for this paper. (Related results, usually couched in more sophisticated algebraic-geometric language can be found in [7].) All schemes in this paper will be reduced and of finite type over k, and we shall identify them with their associated algebraic varieties of closed points. We use $\underline{A}^r$ to denote affine space of dimension r over k, and we give the space of all triples of matrices (F,G,H) of dimension n × n, n × m, p × n respectively, the algebraic variety structure of $\underline{A}^{n(n+m+p)}$. Let $L_{m,n,p}$ denote this algebraic variety.

Then the assignment

$$(T,(F,G,H)) \rightarrow (TFT^{-1},TG,HT^{-1}) = (F,G,H)^T \qquad (1.2)$$

defines an action of the algebraic group GL_n of invertible n × n matrices with coefficients in k on $L_{m,n,p}$. Cf. [1] Appendix 2. We can now define what a continuous algebraic canonical form on a subvariety $L' \subset L_{m,n,p}$ would be.

1.3. Definition

A continuous algebraic canonical form on L' is an algebraic morphism c: L' $\cap$ L' such that

for every (F,G,H) ε L' there is a T ε GL_n such
$$\text{that } (F,G,H)^T = c(F,G,H) \qquad (1.3.1)$$

$c(F,G,H) = c(\overline{F},\overline{G},\overline{H})$ iff there is a T ε GL_n such
$$\text{that } (F,G,H)^T = (\overline{F},\overline{G},\overline{H}) \qquad (1.3.2)$$

Again, as in [3], we have that continuous algebraic canonical forms on all of $L_{m,n,p}$ cannot exist for trivial reasons. ("Jump phenomena"). The conditions "completely reachable," "completely observable," "rank of G maximal and rank of H maximal and completely reachable and completely observable" all define open subvarieties of $L_{m,n,p}$ which we shall denote with $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{\rho}_{m,n,p}$ respectively. In addition, we consider the condition "F is diagonalizable (i.e. semisimple) with distinct eigenvalues all different from zero" which defines a (non-open) subvariety $L^{\mu}_{m,n,p}$ of $L_{m,n,p}$. Combining different attributes we have the following list of (possibly interesting) subvarieties of $L_{m,n,p}$.

1.4. List of subvarieties

$L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{cr,co}_{m,n,p} = L^{cr}_{m,n,p} \cap L^{co}_{m,n,p}$, $L^{\mu}_{m,n,p}$

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$$L_{m,n,p}^{cr,co,\mu} = L_{m,n,p}^{\mu} \cap L_{m,n,p}^{cr,co}, \quad L_{m,n,p}^{\rho}, \quad L_{m,n,p}^{\rho,\mu} = L_{m,n,p}^{\rho} \cap L_{m,n,p}^{\mu}$$

All these subvarieties of $L_{m,n,p}$ are GL_n-invariant. We now have the following theorem.

1.5. Theorem

The following table gives necessary and sufficient conditions for the existence of continuous algebraic canonical forms on various subvarieties of $L_{m,n,p}$.

	variety L'	necessary and sufficient condition for the existence of an algebraic continuous canonical form
(i)	$L' = L_{m,n,p}^{cr}$	m=1
(ii)	$L' = L_{m,n,p}^{co}$	p=1
(iii)	$L' = L_{m,n,p}^{cr,co}$	m=1 or p=1
(iv)	$L' = L_{m,n,p}^{cr,co,\mu}$	m=1 or p=1
(v)	$L' = L_{m,n,p}^{\rho}$	m=1 or p=1 or m=n or p=n
(vi)	$L' = L_{m,n,p}^{\rho,\mu}$	m=1 or p=1 or m=n or p=n

This theorem is "identical" with theorem 1.7 of [3]. The proof is similar in spirit but different in details.

There is of course also a corollary similar to corollary 1.8 of [3]. We shall see that the "orbit space"

$L^{cr}_{m,n,p}/GL_n$ has the structure of a quasi-projective alge-
braic variety and its open subvariety $L^{cr,co}_{m,n,p}/GL_n$ is in
fact a quasi-affine algebraic variety. Let $M^{cr}_{m,n,p}$ denote
this algebraic variety. Then we shall also see that
$M^{cr}_{m,n,p}$ is a fine moduli variety for a suitable definition
of (algebraic) families of linear dynamical systems.

As we said the field k we work over is supposed to
be algebraically closed. This is mainly a matter of con-
venience: the varieties $L^{cr,co}_{m,n,p}$, $L^{cr}_{m,n,p}$, $L_{m,n,p}$, $M^{cr,co}_{m,n,p}$,
$M^{cr}_{m,n,p}$, $M^{o}_{m,n,p}$ are all defined over any field k; in fact
they are even defined over $\mathbb{Z}$. This also explains our
notation $M^{cr}_{m,n,p}(\mathbb{R})$, etc. of [3]: the underlying sets of
these real manifolds are simply the real points of the
variety $M^{cr}_{m,n,p}$, etc. However, some care must be taken
in interpreting the results of e.g. part (iii) of theorem
1.5 in this context.

Consider e.g. the following situation: let k be a
finite field; let $L^{cr,co}_{m,n,p}(k)$ be the set of all k rational
points of $L^{cr,co}_{m,n,p}$, i.e. the set of all completely reachable
and completely controllable triples of matrices with
coefficients in k; let $GL_n(k)$ be the group of n × n
matrices with coefficients in k acting on $L^{cr,co}_{m,n,p}(k)$ in
the abvious way. Then part (iii) of theorem 1.5 does not
say that there is no map $L^{cr,co}_{m,n,p}(k) \to L^{cr,co}_{m,n,p}(k)$ (locally)
given by polynomials such that the analogues of (1.3.1)

and (1.3.2) hold. E.g. such a map always exists when k

is $\mathbb{F}_2$, the field of two elements. But part (iii) of

theorem 1.5 does say that the map $L_{m,n,p}^{cr,co}(\bar{k}) \to L_{m,n,p}^{cr,co}(\bar{k})$

defined by the same polynomials, does not satisfy the

analogues of (1.3.1) and (1.3.2). Here $\bar{k}$ is the alge-

braic closure of k.

A large part of the proofs and constructions of [3]

can be carried through unchanged in the algebraic geo-

metric case. In these cases we shall as a rule simply

refer to the appropriate section of [3].

2. THE QUOTIENT VARIETY $M^{cr}_{m,n,p}$

2.1. Nice Selections

Let $(F,G,H) \in L_{m,n,p}$. The matrices $R(F,G)$ and $Q(F,H)$
are defined as in [3], 2.2. The conditions "$R(F,G)$ has
rank n" i.e. "complete observability" define open subvari-
ties of $L_{m,n,p}$ which we denote by $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$ respec-
tively.

In addition we put $L^{cr,co}_{m,n,p} = L^{cr}_{m,n,p} \quad L^{co}_{m,n,p}$ which
is also an open subvariety of $L_{m,n,p}$. As in [3], 2.3 we

let $J_{n,m}$ denote the set of column indices of $R(F,G)$. Nice selections α (from $J_{n,m}$) and the successor indices $s(a,j)$, $j = 1,\ldots,m$ of the nice selection α are defined as in [3], 2.3. We again have (c.f. [1] 2.4.1 for a proof).

2.1.1. <u>Lemma</u>

If $(F,G,H) \in L^{cr}_{m,n,p}$, then there is a nice selection such that $\det(R(F,G)_{\alpha} \neq 0$.

2.2. <u>The Local Quotients</u> U_{α}/GL_{n}.

Let α be a nice selection. One defines the subvarieties of $L^{cr}_{m,n,p}$

$$U_{\alpha} =' \{(F,G,H) \in L_{m,n,p} | \det(R(F,G)_{\alpha}) \neq 0\} \quad (2.2.1)$$

$$W_{\alpha} =' \{(F,G,H) \in L_{m,n,p} | R(F,G)_{\alpha} = I_{n}\} \quad (2.2.2)$$

The map ψ_{α} of [3], 2.4.5 now defines an isomorphism of algebraic varieties

$$\psi_{\alpha}: \underline{A}^{nm+np} \overset{\sim}{\to} W \quad (2.2.3)$$

We define a morphism $t_{\alpha} : U_{\alpha} \to GL_{n} \times W_{\alpha}$

$$t_{\alpha} : (F,G,H) \to (T^{-1}, (F,G,H)^{T}), \text{ where } T = R(F,G)_{\alpha}^{-1}$$
$$(2.2.4)$$

2.2.5. <u>Lemma</u>

t_{α} is a GL_{n}-invariant isomorphism of algebraic

varieties (where GL_n acts on $GL_n \times W_\alpha$ by left multiplication on the left hand factor.)

2.2.6. Corollary

The (categorial) quotients U_α/GL_n exist (as algebraic varieties) and are isomorphic to the affine space $\underline{A}^{nm+np}$.

This follows from 2.5.5 and the isomorphism ψ_α. For the notion of categorical quotient cf. [1] A.2.7. As a matter of fact U_α/GL_n is also a geometric quotient in the sense of [6]; we shall not need this fact.

2.3. The Quotient Variety $M^{cr}_{m,n,p}$

We are now going to define a quotient prevariety $M^{cr}_{m,n,p}$ by gluing the local quotients U_α/GL_n together in suitable way. For each nice selection α let $V_\alpha = \underline{A}^{mn+np}$ and for each second nice selection β let $V_{\alpha\beta}$ be the open subvariety $V_{\alpha\beta} = \psi_\alpha^{-1}(W_\alpha \cap U_\beta)$. We define $\phi_{\alpha\beta} : V_{\alpha\beta} \rightarrow V_{\beta\alpha}$ by the formula (identical of [3], (2.5.4)).

$$\phi_{\alpha\beta}(x) = y \leftrightarrow (F_\alpha(x),G_\alpha(x),H_\alpha(x))^T = (F_\beta(y),G_\beta(y),H_\beta(y))$$

$$\text{with } T = R(F_\alpha(x),G_\alpha(x))^{-1}_\beta \qquad (2.3.1)$$

where we have written $\psi_\alpha(x) = (F_\alpha(x),G_\alpha(x),H_\alpha(x)) \in W_\alpha$ and similarly for $\psi_\beta(y)$. These $\phi_{\alpha\beta}$ are well defined and define isomorphisms of algebraic varieties $V_{\alpha\beta} \rightarrow V_{\beta\alpha}$,

which moreover satisfy the cocycle condition $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$ whenever the left hand side is defined. This means that by gluing together the various V_α by means of the $\phi_{\alpha\beta}$ we obtain a certain prevariety which we shall denote $M^{cr}_{m,n,p}$. To prove that $M^{cr}_{m,n,p}$ is an (abstract) variety we have to prove that it is separated. This can either be done by using the algebraic geometric version of [3], 2.5.7 or by means an embedding argument. To carry this embedding argument through we first observe.

2.3.2. Lemma

The natural projections $\pi_\alpha: U_\alpha \rightarrow V_\alpha$ combine to define an algebraic morphism $\pi: L^{cr}_{m,n,p} \rightarrow M^{cr}_{m,n,p}$, and π is a categorical quotient in the category of prevarieties for the action of GL_n on $L^{cr}_{m,n,p}$ defined by (1.2).

Proof. It is obvious that $\pi: L^{cr}_{m,n,p} \rightarrow M^{cr}_{m,n,p}$ kills the action of GL_n. Now let $\phi: L^{cr}_{n,n,p} \rightarrow X$ be any morphism which kills the action of GL_n. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then we know that $U_\alpha \rightarrow V_\alpha$ and $U_{\alpha\beta} \rightarrow V_{\alpha\beta}$ are categorical quotients by 2.2.6. Let ϕ_α be the restriction of ϕ to U_α. By the categorical quotient property of $U_\alpha \rightarrow V_\alpha$ there are unique morphisms $\chi_\alpha: V_\alpha \rightarrow X$ such that $\phi_\alpha = \chi_\alpha\pi_\alpha$. Because $U_{\alpha\beta} \rightarrow V_{\alpha\beta}$ are categorical quotients we also know that $\chi_\beta\phi_{\alpha\beta}(x) = \chi_\alpha(x)$ for $x \in V_{\alpha\beta}$, where $\phi_{\alpha\beta}$ is as in (2.3.1). It follows that the χ_α combine to define a morphism

χ: $M^{cr}_{m,n,p}$ $\to$ X such that $\phi = \chi\pi$. The morphism χ is unique because on each V_α it must equal χ_α. Essentially the same proof was used for [1], 3.2.14.

2.3.3. The Morphisms

h: $L_{m,n,p}$ $\to$ $\underline{A}^r$ $\underline{and}$ g: $L_{m,n,p}$ $\to$ $G_{n,(n+1)m}$. Let (F,G,H) ε $L_{m,n,p}$. We let h(F,G,H) ε $\underline{A}^r$, where r = $(n+1)^2 mp$, be the block Hankel matrix

$$h(F,G,H) = \begin{pmatrix} HG & HFG & . & . & . & HF^nG \\ HFG & & & & & . \\ . & & & & & . \\ . & & & & & \\ . & & & & & . \\ HF^nG & & . & . & . & HF^{2n}G \end{pmatrix} = Q(F,H)R(F,G)$$

This defines a morphism h: $L_{m,n,p}$ $\to$ $\underline{A}^r$, which certainly kills the action of GL_n.

Restricting to $L^{cr}_{m,n,p}$ and applying lemma 2.3.2 we obtain an induced morphism

$$\bar{h}: M^{cr}_{m,n,p} \to \underline{A}^r, \quad r = (n+1)^2 pm \qquad (2.3.4)$$

Let $L^{cr}_{m,n}$ be the algebraic variety of all pairs of matrices (F,G) of sizes n × n and n × m. In [1] we constructed a morphism g: $L^{cr}_{m,n}$ $\to$ $G_{n,(n+1)m}$ which kills the action of

GL_n on $L_{m,n,}^{cr}$, where $G_{n,(n+1)m}$ is the Grassmann variety
of n-planes in (n+1)m space; g assigns to (F,G) the point
of $G_{n(n+1)m}$ corresponding to the rank n matrix R(F,G) of
size nx(n+1)m. We proved that the quotient variety $M_{m,n}$
$= L_{m,n}^{cr}/GL_n$ exists and that g induces an embedding
$\bar{g}$: $M_{m,n} \to G_{n,(n+1)m}$. Cf. [1] Theorem 3.2.13 and proposi-
tion 3.2.14. Now let g': $L_{m,n,p}^{cr} \to G_{n,(n+1)m}$ be the com-
posed morphism (F,G,H) $\to$ (F,G) $\to$ g(F,G). This morphism
kills the action of GL_n and hence induces a morphism

$$\hat{g}: M_{m,n,p}^{cr} \to G_{n,(n+1)m} \qquad (2.3.5)$$

From the remarks made above we know that if (F,G,H),
(F',G',H') (F',G',H') ε $L_{m,n,p}^{cr}$ are such that g'(F,G,H)
= g'(F,G,H) then there is a T ε GL_n such that $(F,G)^T =$
$(\bar{F},\bar{G})$.

2.3.4 An embedding $M_{m,n,p}^{cr} \to G_{n,(n+1)m}$ x $\underline{A}^r$
    The morphisms h,g of (2.3.4) and (2.3.5) together
define a morphism

$$i: M_{m,n,p}^{cr} \to G_{n,(n+1)m} \text{ x } \underline{A}^r \qquad (2.3.6)$$

We claim that i is injective. Indeed if (F,G,H), $(\bar{F},\bar{G},\bar{H})$
ε $L_{m,n,p}^{cr}$ are such that h(F,G,H) = $h(\bar{F},\bar{G},\bar{H})$ and g'(F,G,H)
= $g'(\bar{F},\bar{G},\bar{H})$, then we know that there is a T ε GL_n such
that $(F,G)^T = (\bar{F},\bar{G})$ i.e. TR(F,G) = $R(\bar{F},\bar{G})$, and then because
h(F,GH) = $h(\bar{F},\bar{G},\bar{H})$ we have in particular HR(F,G) = $\bar{H}R(\bar{F},\bar{G})$

so that $\overline{HTR}(F,G) = HR(F,G)$ and hence $\overline{H} = HT^{-1}$ because

$R(F,G)$ has rank n. This concludes the proof that i is

injective.

2.3.5. Corollary

The prevariety $M^{cr}_{m,n,p}$ is separated, i.e. $M^{cr}_{m,n,p}$ is

a variety.

2.3.6. Corollary

$L^{cr}_{m,n,p} \rightarrow M^{cr}_{m,n,p}$ is a quotient for the action of GL_n

on $L^{cr}_{m,n,p}$ in the category of algebraic varieties. In

fact $M^{cr}_{m,n,p}$ is also a geometric quotient in the sense of

[6], but we shall not need this.

2.3.7.

Let $V^{co}_\alpha = \psi^{-1}_\sigma(W_\alpha \cap L^{cr,co}_{m,n,p})$ and $V^{co}_{\alpha\beta} = V^{co}_\alpha \cap V_{\alpha\beta}$. Then

the $\phi_{\alpha\beta}: V_{\alpha\beta} \rightarrow V_{\beta\alpha}$ induce isomorphisms $\phi^{co}_{\alpha\beta}: V^{co}_{\alpha\beta} \rightarrow V^{co}_{\beta\alpha}$.

Gluing together the V^{co}_α by means of the $\phi^{co}_{\alpha\beta}$ we obtain an

open subvariety $M^{cr,co}_{m,n,p}$ of $M^{cr}_{m,n,p}$ which is the image of

$L^{cr,co}_{m,n,p}$ under $\pi: L^{cr}_{m,n,p} \rightarrow M^{cr}_{m,n,p}$. It follows that the

induced morphism $\pi^{co}: L^{cr,co}_{m,n,p} \rightarrow M^{cr,co}_{m,n,p}$ is also a cate-

gorical quotient.

2.3.8.

Similarly, using $V^\rho_\alpha = \psi^{-1}_\alpha(W_\alpha \cap L^\rho_{m,n,p})$,

$V_\alpha^{\mu\rho} = \psi_\alpha^{-1}(W_\alpha \cap L_{m,n,p}^\mu)$, $V_\alpha^{\mu\rho} = \psi_\alpha^{-1}(W_\alpha \cap L_{m,n,p}^{\mu,\rho})$ and the

corresponding $V_{\alpha\beta}^{\cdot\cdot}$ we obtain categorical quotients $L_{m,n,p}^\rho$

$\to M_{m,n,p}^\rho$, $L_{m,n,p}^\mu \to M_{m,n,p}^\mu$, $L_{m,n,p}^{\rho,\mu}$ $M_{m,n,p}^{\rho,\mu}$ where the

$M_{m,n,p}^{\cdot\cdot}$ are subvarieties of $M_{m,n,p}^{cr}$.

2.4. Some Realization Theory

The morphism $\bar{h}$ of (2.3.4) above induces a morphism

$\hat{h}: M_{m,n,p}^{cr,co} \to \underline{A}^r$. It is the purpose of this and the follow-

ing subsection to show that $\hat{h}$ is injective and to derive

equations for the subvariety $\hat{h}(M_{m,n,p}^{cr,co}) \subset \underline{A}^r$. To do this

we use some (partial) realization theory an embodied by

proposition 2.4.3 below. First a definition

2.4.1. Definition

Let $A_o, A_1, \ldots$ be a sequence of $p \times m$ matrices. Then

$h_{q,r}(A)$ denotes the block Hankel matrix

$$h_{q,r}(A) = \begin{pmatrix} A_o & A_1 & \cdot & \cdot & \cdot & A_r \\ A_1 & & & & & \\ \vdots & & & & & \vdots \\ A_q & \cdot & \cdot & & \cdot & A_{r+q} \end{pmatrix}$$

2.4.2. Definition

If A is a matrix and α_c is a subset of the column

indices of A, and α_r is a subset of the row indices of
A, then we define

A_{α_c} = matrix obtained from A by removing all columns whose

index is not in α_c

A_{α_r} = matrix obtained from A by removing all rows whose

index is not in α_r

A_{α_r,α_c} = matrix obtained from A by removing all rows and

columns whose indices are not in α_r, α_c respec-

tively.

2.4.3. <u>Proposition</u>

Let $A_0, A_1, \ldots, A_{2n-1}$ be a sequence of 2n matrices
with coefficients in k, all of size p × m, and suppose
that $\operatorname{rank}(h_{n-1,n-1}(A)) = \operatorname{rank}(h_{n,n-1}(A)) = \operatorname{rank}(h_{n-1,n}(A))$
= n. Then there exists an $(F,G,H) \in L_{m,n,p}^{cr,co}$ such that
$HF^i G = A_i$ for i = 0,1,...,2n-1.

Moreover, if $(\bar{F},\bar{G},\bar{H}) \in L_{m,n,p}^{cr.co}$ is a second triple such
that $\bar{H}\bar{F}^i\bar{G} = A_i$ for i = 0,1,...,2n-1 then there is a
$T \in GL_n$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^T$.

<u>Proof.</u> Existence of a triple $(F,G,H) \in L_{m,n,p}$ such
that

$$HF^i G = A_i, \quad i = 0,\ldots,2n-1 \qquad (2.4.4)$$

holds is assured by the realizability criterion 11.32 of

Chapter 10 of [4]. We define

$$\bar{R}(F,G) = (G, FG \ldots F^{n-1}G),$$

$$\bar{Q}(F,H)' = (H'F'H' \ldots (F')^{n-1}H') \qquad (2.4.5)$$

Then it follows from (2.4.3) that $\bar{Q}(F,H)\bar{R}(F,H) =$
$h_{n-1,n-1}(A)$. Now we have rank $(R(F,G)) \leq n$,
rank $(Q(F,H)) \leq n$ and rank $(h_{n-1,n-1}(A)) = n$. It follows
that rank $(R(F,G)) = $ rank $(Q(F,H)) = n$, so that (F,G,H)
$\in L_{m,n,p}^{cr,co}$. Not let $(\bar{F},\bar{G},\bar{H})$ be a second triple in $L_{m,n,p}$
such that

$$\bar{H}\bar{F}^i\bar{G} = A_i, \quad i = 0,1,\ldots,2n-1 \qquad (2.4.6)$$

Then as above we find $\bar{Q}(\bar{F},\bar{H})\bar{R}(\bar{F},\bar{G}) = h_{n-1,n-1}(A)$. Now
because $\bar{R}(F,G)$ has rank n there is a subset α_c of size n
of the column indices of $R(F,G)$ such that $R(F,G)_{\alpha_c}$ is
invertible; further because $\bar{Q}(F,H)$ has rank n there is a
subset α_r of size n of the row indices of $\bar{Q}(F,H)$ such
that $\bar{Q}(F,H)_r$ is invertible. We have

$$(h_{n-1,n-1}(A))_{\alpha_r,\alpha_c} = \bar{Q}(F,H)_{\alpha_r} \bar{R}(F,G)_{\alpha_c} =$$
$$\qquad (2.4.7)$$
$$\bar{Q}(\bar{F},\bar{H})_{\alpha_r} \bar{R}(\bar{F},\bar{G})_{\alpha_c}$$

so that it follows that all five $n \times n$ matrices occurring
in (2.4.7) are invertible.

     Now let

$$(F_1,G_1,H_1) = (F,G,H)^T, \text{ where } T = \overline{Q}(F,H)_{\alpha_r} \qquad (2.4.8)$$

$$(\overline{F}_1,\overline{G}_1,\overline{H}_1) = (\overline{F},\overline{G},\overline{H})^T, \text{ where } \overline{T} = \overline{Q}(\overline{F},\overline{H})_{\alpha_r} \qquad (2.4.9)$$

Then we have of course

$$H_1 F_1^i G_1 = \overline{H}_1 \overline{F}_1^i \overline{G}_1 = A_i \text{ for } i = 0,\ldots,2n-1 \qquad (2.4.10)$$

which means

$$\overline{Q}(F_1,H_1)R(F_1,G_1) = \overline{Q}(\overline{F}_1,\overline{H}_1)R(\overline{F}_1,\overline{G}_1) = h_{n-1,n}(A)$$

$$(2.4.11)$$

and moreover because

$$\overline{Q}(F_1,H_1) = \overline{Q}(F,H)T^{-1}, \quad \overline{Q}(\overline{F}_1,\overline{H}_1) = \overline{Q}(\overline{F},\overline{H})\overline{T}^{-1}$$

we have

$$Q(F_1,H_1)_{\alpha_r} = I_n = Q(F_1,H_1)_{\alpha_r} \qquad (2.4.12)$$

Now combine (2.4.12) and (2.4.11) to obtain that $R(F_1,G_1)$ = $R(F_1,G_1)$ which be corollary 2.4.2 of [1] means that $F_1 = \overline{F}_1$ and $G_1 = \overline{G}_1$. and because $R(\overline{F},\overline{G}) = R(F,G)$ has rank n, it follows from (2.4.11) that also $H_1 = \overline{H}_1$. We therefore have $(F,G,H)^T = (F_1,G_1,H_1) = (\overline{F}_1,\overline{G}_1,\overline{H}_1) = (\overline{F},\overline{G},\overline{H})^T$, which proves the second statement of the proposition.

2.4.4. <u>Corollary</u>

The morphism $\hat{h}: M_{m,n,p}^{cr,co} \rightarrow \underline{A}^r$ of (2.3.4) above is injective.

2.5. <u>Equations for</u> $M_{m,n,p}^{cr,co}$

By means of the injective morphism $\hat{h}$ we can now consider $M_{m,n,p}^{cr,co}$ as a subvariety of $\underline{A}^r$, $r = (n+1)^2 pm$, where we write $x \in \underline{A}^r$ as an $(n+1)n \times (n+1)m$ matrix. We now consider the following sets of polynomials in the coordinates of $\underline{A}^r$.

$P_a(x)$: these polynomials are such that $P_a(x) = 0$ for all a if and only if the matrix x is of block Hankel type (cf. 2.4.1) with the blocks of size $p \times m$.   (2.5.1)

$Q_b(x)$: here $Q_b(x)$ runs through all determinants of $(n+1) \times (n+1)$ submatrices of x.   (2.5.2)

$R_c(x)$: here $R_c(x)$ runs through all determinants of $n \times n$ submatrices of the submatrix x' of x which is obtained by removing the last p rows and the last m columns.

2.5.4. <u>Lemma</u>

Let $(F,G,H) \in L_{m,n,p}^{cr,co}$, $x = h(F,G,H) \in \underline{A}^r$. Then we have $P_a(x) = 0$ for all a, $Q_b(x) = 0$ for all b and there is a c such that $R_c(x) \neq 0$.

Proof. Obvious because $h(F,G,H) = Q(F,H)R(F,G)$.

2.5.5. <u>Proposition</u>

$\hat{h}(M_{m,n,p}^{cr,co})$ $\underline{A}^r$ is the subvariety consisting of those $x \in \underline{A}^r$ such that $P_a(x) = 0$ for all a, $Q_b(x) = 0$ for all b and such that these is a c such that $R_c(x) \neq 0$.

<u>Proof</u>. Because of lemma 2.5.4 we only have to show that if $x \in \underline{A}^s$ satisfies $P_a(x) = 0$ all a, $Q_b(x) = 0$ all b and $R_c(x) \neq 0$ for some c, then x is in $\hat{h}(M_{m,n,p}^{cr,co})$. Write x as a block Hankel matrix

$$x = \begin{pmatrix} A_1 & A_2 & \cdot & \cdot & \cdot & A_n \\ A_2 & & & & & \\ \vdots & & & & & \vdots \\ A_n & & \cdot & \cdot & \cdot & A_{2n} \end{pmatrix}$$

This can be done because $P_a(x) = 0$ for all a. Cf. 2.5.1. Then the matrices $A_1, \ldots, A_{2n-1}$ satisfy the conditions of proposition 2.4.3 so that there is a triple (F,G,H) $\in L_{m,n,p}^{cr,co}$ such that $HF^iG = A_i$ for i = 0,1,2,...,2n-1. To show that h(F,G,H) = x it therefore only remains to show that $HF^{2n}G = A_{2n}$. This follows from lemma 2.5.6 below.

2.5.6 <u>Lemma</u>

Let E,E' be two partitioned matrices

$$E = \begin{pmatrix} A & | & B \\ \hline C & | & D \end{pmatrix} \qquad\qquad E' = \begin{pmatrix} A & | & B \\ \hline C & | & D' \end{pmatrix}$$

and suppose that rank(E) = rank(E') = rank(A). Then
D = D'.

Proof. Let d be an element of D and d' the corre-
sponding element of D'. Let A' be an n × n submatrix of
A such that det(A') $\neq$ 0 where n = rank(A). Suppose A' =
E_{α_r,α_c}, then also A' = E'_{α_r,α_c}. Let β_r = $\alpha_r \cup \{i\}$ where
i is the index of the row in E containing d (and of the
row in E' containing d') and β_c = $\alpha_c \cup \{j\}$ where j is the
index of the column in E containing d (and of the column
in E' containing d'). Then we have det(E_{β_r,β_c}) = 0 =
det(E'_{β_r,β_c}). All elements of E_{β_r,β_c} and E'_{β_r,β_c} except
possibly the one in the right hand lower corner are equal
and det(A') $\neq$ 0. It follows that d = d'. (By expanding
the determinants along the last row e.g.).

2.5.7. Corollary (of proposition 2.5.5.)

   $M_{m,n,p}^{cr,co}$ is a quasiaffine variety.

2.5.8.

   Using similar arguments as above combined with those
used in [1] to find equations for the variety $M_{m,n}$ (cf.

[1] section 3.2), it is not difficult to find equations for the variety $M_{m,n,p}^{cr}$ (as a subvariety of $G_{n,(n+1)m} \times \underline{A}^r$ or as a subvariety of $\underline{P}^{r'} \times \underline{A}^r$, where $r' = \binom{(n+1)m}{n} - 1$).

$M_{m,n,p}^{cr}$ is a quasiprojective variety but not a quasi affine variety if $m > 1$. This last statement is seen as follows. The embedding $L_{m,n,p}^{cr} \to L_{m,n,p}^{cr}$ given by (FG) $\to$ (F,G,O) where O is zero matrix of appropriate size, induces an embedding $M_{m,n} \to M_{m,n,p}^{cr}$. Now according to [1] section 3.3 there is an embedding $\underline{P}^1 \to M_{m,n}$. Combining these we find an embedding $\underline{P}^1 \to M_{m,n,p}^{cr}$ which shows that $M_{m,n,p}^{cr}$ is not quasi affine. (Cf. also the proof of theorem 3.4.6 in [1]).

2.6 The Algebraic Principal Fiber Bundle

$\pi: L_{m,n,p}^{cr} \to M_{m,n,p}^{cr}$. As in [3] we can now show that $L_{m,n,p}^{cr} \to M_{m,n,p}^{cr}$ is an algebraic principal GL_n fibre bundle over the variety $M_{m,n,p}^{cr}$, and we could use an analysis of the nontriviality or triviality of this bundle to prove nonexistence and existence of algebraic continuous canonical forms. This can be done almost exactly as in [3] section 3 except that one has to construct a different example because the example of [3], section 3.2 is essentially nonalgebraic. Cf. also section 4.1 below for further comments. In this paper, however, we shall first

discuss the fine moduli variety properties of $M^{cr}_{m,n,p}$ and
then use these to investigate the existence of continuous
algebraic canonical forms; this is the same procedure as
in [2], cf. especially theorem 6.1 of [2]. The two
approaches are essentially equivalent because the under-
lying vectorbundle of the universal family over $M^{cr}_{m,n,p}$ is
the algebraic n-vectorbundle associated to the principal
GL_n bundle $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$.

2.7 The Codimension of $(M^{cr}_{m,n,p} \setminus M^{cr,co}_{m,n,p})$ in $M^{cr}_{m,n,p}$.

Let $K_{m,n,p}$ be the subvariety of $M^{cr}_{m,n,p}$ defined by the
equations $\det(Q(F,H))_\beta) = 0$ for all subsets of size n of
the row indices of $Q(F,H)$. i.e.

$$K_{m,n,p} = M^{cr}_{m,n,p} \setminus M^{cr,co}_{m,n,p} \qquad (2.7.1)$$

We want to find out something about the codimension of the
closed subvariety $K_{m,n,p}$ of $M^{cr}_{m,n,p}$. The result is:

2.7.2. Proposition

The codimension of $K_{m,n,p}$ in $M^{cr}_{m,n,p}$ is 1 if $p = 1$ and
it is $\geq p$ if $p \geq 2$.

To prove proposition 2.7.2 we use the following com-
binatorial lemma.

2.7.3 <u>Lemma</u>

Let $X = \{a_1,\ldots,a_n\}$ be a finite set of n elements.
Let X_0 be a subset of X and $\sigma:X_0 \to X$ an injective map with
the following property

$$\text{If } Y \subset X_0 \text{ then } \sigma(Y) \not\subset Y \text{ unless } Y = X_0 = X. \quad (2.7.2)$$

Then there exists a cyclic permutation $\tilde{\sigma}:X \to X$ of order n
such that $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in X_0$.

<u>Proof</u>. If $X_0 = X$ then condition (2.7.4) says that
σ is already a cyclic permutation of order n. We can
therefore assume that $X_0 \neq X$. We are going to show that
there is $b \in X \setminus X_0$ and an injective map $\sigma_1: X_1 \to X$ with
$X_1 = X_0 \cup \{b\}$ and $\sigma_1(a) = \sigma(a)$ for $a \in X_0$ such that
(2.7.4) holds with X_0 replaced by X_1. By induction (with
respect to the number of elements in $X \setminus X_0$) this proves
the lemma. Because $X_0 \neq X$ there is an $a_1 \in X$ which is not
in the image of σ. If $a_1 \in X_0$ let $a_2 = \sigma(a_1)$, if $a_1 \notin X_0$
stop; if $a_2 \in X_0$ let $a_3 = \sigma(a_2)$, if $a_2 \notin X_0$ stop; continu-
ing in this way we find a sequence of elements $a_1, a_2, \ldots,$
a_r, $r \geq 1$ such that

$$a_1 \notin \text{Im}(\sigma), \quad a_i = \sigma(a_{i-1}) \text{ for } i = 1,\ldots,r-1,$$

$$a_r \notin X_0$$

Note that the $a_1,\ldots,a_r$ are all different from one another

because σ is injective and $a_1 \notin \text{Im}\sigma$. There now are two possibilities

(i)   There is no $b \notin X \setminus \text{Im}\sigma$ different from a_1.

In this case $\text{Im}\sigma$ has $n-1$ elements and hence so has X_0. Therefore $X \setminus X_0 = \{a_r\}$. Let $Y = X \setminus \{a_1,\ldots,a_r\}$ and suppose $Y \neq \emptyset$. Then we have $Y \subset X_0$ because $X \setminus X_0 = \{a_r\}$. We also have $\sigma(Y) \subset Y$ because $\sigma'(\{a_1,\ldots a_r\} \cap X_0)$ $\subset \{a_1,\ldots,a_r\}$.

Therefore, because is injective, we would have $\sigma(Y) = Y$ contradicting (2.7.2). Therefore $Y = \emptyset$ and $X = \{a_1,\ldots,a_r\}$ in this case (i.e. $r = n$). We now take $b = a_1$ and define $\sigma_1(b) = a_1$. Then $X_1 = X_0 \cup \{a_r\} = X$ and $\sigma_1 : X \to X$ is clearly the desired cyclic permutation.

(ii)   There is a $b_1 \in X$ Im which is different from a_1.
       In this case we take $b = a_r$ and define $\sigma_1(b) = b_1$.
       The map σ_1 is injective because $b_1 \notin \text{Im}\sigma$. Now sup-
       pose $Y \subset X_1$ is such that $\sigma_1(Y) = Y$. Note that in
       this case $X \setminus \text{Im}\sigma$ has at least two elements, hence
       so has $X - X_0$, so that $X_1 \neq X$. There are two possi-
       bilities.

(ii$_1$) $b = a_r \notin Y$. In this case $Y = \sigma_1(Y) = \sigma(Y)$ and
         $Y \subset X_0$ which contradicts (2.7.4).

(ii$_2$) $b = a_r \in Y$. Then because $\sigma_1(Y) = Y$ we must have

$a_{r-1} \in Y$, $a_{r-2} \in Y, \ldots, a_1 \in Y \subset X_1$, which is a contradiction because there is no $c \in X$ such that

$\sigma(c) = a_1$ because $a_1 \notin \text{Im}\sigma_1 = \text{Im}\sigma \cup \{b_1\}$. This concludes the proof of the lemma.

2.7.6.

Now consider $x \in \underline{A}^{mn}$; consider $(F_\alpha(x), G_\alpha(x)$, where α is a nice selection, $\alpha \subset J_{m,n}$. We recall how $F_\alpha(x)$ and $G_\alpha(x)$ are defined (cf. 1 section 2.3). Let $J = \alpha \cup \{s(\alpha,1), \ldots, s(\alpha,m)\}$ as an ordered subset of $J_{m,n}$. Let x_1 be the column vector consisting of the first n coordinates of x, x_2 the column vector consisting of the second n coordinates, etc. We now define n + m column vectors y_i, $i = 1, 2, \ldots, m+n$ of length n as follows

$$ y = \begin{cases} e_\ell & \text{if the i-th element of J is the } \ell\text{-th element} \\ & \hspace{4cm} \text{of } \alpha \\ x_j & \text{if the i-th element of J is } s(\alpha,j) \end{cases} \quad (2.7.3) $$

where e_ℓ is the ℓ-th standard basis vector.

The matrices $G_\alpha(x)$ and $F_\alpha(x)$ are now defined by

$$ G_\alpha(x)_j = y_j, \ j = 1, \ldots, m; \quad F_\alpha(x)_j = y_{m+j}, \ j = 1, \ldots, n \tag{2.7.4} $$

It readily follows from this that $F_\alpha(x)$ is a matrix of

the following type: the j-th column of $F_\alpha(x)$ is eithe

a standard basis vector e_ℓ with $\ell > j$, or $F_\alpha(x)_j = x_i$

for some i; moreover if $F_\alpha(x)_{j_1} = e_{\ell_1}$, $F_\alpha(x)_{j_2} = e_{\ell_2}$ w

$j_1 < j_2$ then $\ell_1 < \ell_2$. Applying lemma 2.7.3 we thus se

that by specifying the x_i, $i = 1,\ldots,m$ to be suitable

standard basis vectors one obtains

2.7.9 <u>Lemma</u>

For every nice selection α, there is an $x \in \underline{A}_{mn}$ su

that $F_\alpha(x)$ is a cyclic permutation of order n of the st

dard basis vectors.

2.7.10.

Let α be a nice selection. Now consider $K_{m,n,p}$ $\cap$

$= V_\alpha \setminus V_\alpha^{co}$ where α is a nice selection. This closed su

variety of U_α is defined by the equations $\det(Q(F_\alpha(x),$

$H_\alpha(x))_\beta) = 0$ for all subsets β of size n of the row

indices of $Q(F_\alpha(x),H_\alpha(x))$. We number the rows of

$Q(F_\alpha(x),H_\alpha(x))$ as follows

$$((0,1),\ldots,(0,p);\ (1,p),\ldots,(1,p);\ldots$$

$$\ldots;(n,1),\ldots,(n,p))$$

Take $\beta_1 =' \{(0,1),(1,1),\ldots,(n-1,1)\}$. Write $x \in V_\alpha =$

$\underline{A}^{mn} \times \underline{A}^{pn}$ as $x = (y,z)$ and write z as the matrix (z_{ij}),

$i = 1,\ldots,p$, $j = 1,\ldots,n$. We write $F_\alpha(x) = F_\alpha(y)$,
$H_\alpha(x) = z$. Now consider the equation

$$\det(Q(F_\alpha(x),H_\alpha(x))_{\beta_1}) = 0 \qquad (2.7.5)$$

Now specify the y such that $F_\alpha(x)$ is a cyclic permutation
matrix of order n and suppose that the first row vector
of $F_\alpha(x)$ under this specification is the ℓ-th standard
basis vector. Now take $z_{ij} = 0$ for $j \neq \ell$. Then (2.7.5)
becomes

$$\pm z_{1\ell}^n = 0 \qquad (2.7.6)$$

If $p = 1$, equation (2.7.11) defines $K_{m,n,p} \cap V_\alpha$ (because
if rank $Q(F,G) = n$ then there is a nice "selection" β
from the row indices of $Q(F,H)$ such that $\det(Q(F,H)_\beta) \neq 0$
by the transposed version of lemma 2.1.1). Equation
(2.7.12) which is obtained from (2.7.11) by a suitable
specification of some of the variables shows that (2.7.5)
is nontrivial, so that the codimension of $K_{m,n,p} \cap V_\alpha$ in
V_α is one for each nice selection α proving that the
codimension of $K_{m,n,1}$ in $M_{m,n,p}^{cr}$ is one. Now suppose
that $p > 1$. And consider the selections

$$\beta_i = '\{(0,i),(1,i),\ldots,(n-1,i)\} \quad i = 1,\ldots,p$$

Specifying the y and z as before (NB the specification
to be used depends on α!), the equations

$$\det(Q(F_\alpha(x), H_\alpha(x))_{\beta_i}) = 0 \quad i = 1, \ldots, p \qquad (2.7.7)$$

specify to

$$\pm z^n_{i\ell} = 0 \quad i = 1, \ldots, p \qquad (2.7.8)$$

The equations (2.7.14) are independent, hence so are the
equations (2.7.13) proving that the codimension of $K_{m,n,p}$
$\cap V_\alpha$ in V_α is $\geq p$. This holds for all nice selections α
so that the codimension of $K_{m,n,p}$ in $M^{cr}_{m,n,p}$ is always $\geq p$.
We have now proved assertion 2.7.2.

2.7.12. <u>Remark</u>

     To prove 2.7.2 all one really needs is the existence
of a triple $(F,G,H) \in W_\alpha$ for each α such that F' is a
cyclic matrix. This can be seen as follows: U is a non-
empty open subvariety of $L_{m,n,p}$. Let $L' = \{(F,G,H) \in$
$L_{m,n,p} | F'$ is cyclic} this also defined a nonempty open
subvariety of $L_{m,n,p}$. Because $L_{m,n,p}$ is irreducible
$L' \cap U_\alpha \neq \phi$. Let $(F,G,H) \in L' \cap U$ and let $(F,G,H) =$
$(F,G,H)^T$ where $T = R(F,G)^{-1}_\alpha$. Then $(F,G,H) \in W_\alpha$ and F'
is cyclic.

3. THE FINE MODULI VARIETY $M^{cr}_{m,n,p}$.

     We now proceed to study families of linear dynamical
systems. Some motivation as to why one would like to

study families is given in section 1.8 of [3]. Moreover,
in this paper we shall use families to investigate whether
there exist continuous canonical forms or not. This is
not necessary; one can also use the principal algebraic
GL_n bundle $L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$. Cf. also 2.6 above. This
part of the theory in the algebraic geometric case is
practically completely analogous to the corresponding
part of the topological case which was treated in sec-
tion 4 of [3].

3.1. Families of Linear Dynamical Systems

3.1.1. Definition

A family of linear dynamical systems over a variety
S of dimensions (n,m,p) consists of

(i)   an algebraic n-dimensional vectorbundle $p:E \to S$

(ii)   an algebraic vectorbundle endomorphism $F:E \to E$

(iii) an algebraic vectorbundle homomorphism $G:SxA^m \to E$

(iv)   an algebraic vectorbundle homomorphism $H:E \to SxA^p$.

Let $s \in S$, then F,G,H induce homomorphisms $F_s:E_s \to E_s$,
$G_s:sx\underline{A}^m \to E_s$, $H_s:E_s \to sx\underline{A}^p$; $E_s = p^{-1}(s)$ is the fibre
over s. (Cf. Appendix 3 of [1]). Choosing a basis
$e_1(s),\ldots,e_n(s)$ of E_s and taking the obvious bases in
$sx\underline{A}^m$ and $sx\underline{A}^p$ we calculate the matrices of F_s,G_s,H_s
relative these bases. Let the result be

$(F(s,e),G(s,e),H(s,e))$. This triple depends on $e_1(s),\ldots,$ $e_n(s)$ only to the extent that a different choice of $e_1(s),\ldots,e_n(s)$ gives a triple in the same orbit (under GL_n) as $(F(s,e),G(s,e),H(s,e))$.

The family Σ is said to be <u>completely reachable</u> if $(F(s,e),G(s,e)H(s,e)) \in L^{cr}_{m,n,p}$ for all s. (This is well defined because $L^{cr}_{m,n,p}$ is GL_n invariant).

3.1.2. <u>The Canonical Morphism Associated to Completely Reachable Family</u>

Now let Σ be a completely reachable family. Then F_s,G_s,H_s define a unique orbit in $L^{cr}_{m,n,p}$ and thus a unique point in $M^{cr}_{m,n,p}$ which we shall denote $f_\Sigma(s)$. Thus we have a map $f_\Sigma : S \to M^{cr}_{m,n,p}$. Using the local triviality of the bundle E one shows by means of the algebraic analogues of the constructions in 4.1.2 - 4.1.8 of [3] that f_Σ is a morphism in the category of varieties.

3.1.3

In the topological case we associated a continuous map $f_\Sigma : X \to M_{m,n,p}(\mathbb{R})$ to every family Σ, and used this map to define complete reachability of families. This cannot be done in the algebraic geometric case because the variety $M_{m,n,p}$ does not exist.

3. 2 .   <u>The Universal Family</u> Σ^u over $M_{m,n,p}$.

Let α be a nice selection. Let $E_\alpha = V_\alpha \times \underline{A}^n$, p_α: $E_\alpha \to V_\alpha$ the obvious projection. We define families Σ_α of linear dynaminal systems with underlying bundles E_α by the formulas

$$F_\alpha(x,v) = (x,F_\alpha(x)v), \quad G_\alpha(x,u) = (x,G_\alpha(x)u),$$
$$H_\alpha(x,v) = (x,H_\alpha(x)v) \qquad\qquad (3.2.1)$$

where for $x \in V_\alpha$, $\psi_\alpha(x) = (F_\alpha(x),G_\alpha(x),H_\alpha(x))$, cf. [3].

2. 4 . 5

Now let $E_{\alpha\beta} = V_{\alpha\beta} \times \underline{A}^n$ and define the isomorphisms $\phi_{\alpha\beta}$: $E_{\alpha\beta} \to E_{\beta\alpha}$ by formula (4.3.6) of [3]. Then glueing together the E_α by means of the $\phi_{\alpha\beta}$ we obtain an algebraic vectorbundle E^u. The $F_\alpha,G_\alpha,H_\alpha$ are compatible with the $\phi_{\alpha\beta}$ in the sense of (4.3.9) - (4.3.11) of [3] and thus define homomorphisms F^u: $E^u \to E^u$, g^u: $M_{m,n,p}^{cr} \times \underline{A}^m \to E^u$, H^u: $E^u \to M_{m,n,p}^{cr} \times \underline{A}^p$. This defines the family Σ^u. The family Σ^u is completely reachable (because this is true for the families Σ_α), and the associated map f_{Σ^u}: $M_{m,n,p}^{cr} \to M_{m,n,p}^{cr}$ is the identity map (because the triple $(F_\alpha(x),$ $G_\alpha(x),H_\alpha(x))$ maps to $x \in V_\alpha \subset M_{m,n,p}$ under π: $L_{m,n,p}^{cr} \to M_{m,n,p}^{cr}$).

3. 3 .   <u>The Fine Moduli Variety</u> $M_{m,n,p}^{cr}$

3.3.1

Two families Σ, $\bar{\Sigma}$ are isomorphic if there is an
algebraic vectorbundle isomorphism $\phi:E \rightarrow \bar{E}$ such that
$\bar{F}\phi = \phi F$, $\phi G = \bar{G}$, $H = \bar{H}\phi$. For each $S \varepsilon \underline{Sch}_k$, the cate-
gory of algebraic varieties over k, let $\Phi_{m,n,p}(S)$ be the
set of isomorphism classes of completely reachable fami-
lies of linear dynamical systems over S. By means of
the pullback construction we turn $\Phi_{m,n,p}(S)$ into a func-
tor $\Phi_{m,n,p}:\underline{Sch}_k^{opp} \rightarrow \underline{Set}$.

3.3.2. Theorem

The variety $M^{cr}_{m,n,p}$ is a fine moduli variety for
$\Phi_{m,n,p}$ or, in other words, the functor $\Phi_{m,n,p}$ is repre-
sentable by $M^{cr}_{m,n,p}$. More precisely, the assignment
$\Sigma \rightarrow f_\Sigma$ induces a functorial isomorphism $\Phi_{m,n,p}(S) \rightarrow$
$\underline{Sch}_k(S,M^{cr}_{m,n,p})$; the inverse isomorphism assigns the iso-
morphism class of $f^!\Sigma^u$ to f: $S \rightarrow M^{cr}_{m,n,p}$.

Proof. Identical with the proof of the correspond-
ing theorem 4.5.2 of [3].

4. EXISTENCE AND NONEXISTENCE OF ALGEBRAIC CONTINUOUS
   CANONICAL FORMS

In [1] we used the fact that $M_{m,n}$ admits an embedding
$\underline{\mathbb{P}}^1 \rightarrow M_{m,n}$ if $m \geq 2$ to show that there is no algebraic

continuous form for completely reachable pairs of matrices. This cannot be used to prove e.g. part (iii) of theorem 1.5 because as we have seen $M^{cr,co}_{m,n,p}$ is a quasi-affine algebraic variety. Further the example we used in [3] to prove nonexistence of continuous canonical forms for real linear dynamical systems if $m \geq 2$ and $p \geq 2$ is essentially nonalgebraic. There is, however, a three (instead of one) dimensional version of it which is algebraic and that is the example we shall use in this paper. We proceed via moduli varieties as in [2].

4.1. Triviality of E^u and Existence of Continuous Algebraic Canonical Forms

4.1.1. Theorem

Let $L \subset L^{cr}_{m,n,p}$ be a GL_n-invariant subvariety of $L^{cr}_{m,n,p}$ and let $M = \pi(L)$. Then there exists a continuous algebraic canonical form on L if and only if the algebraic vector bundle $E^u|M$ is trivial.

Proof. Let $\Phi^L_{m,n,p}$ be the subfunctor of $\Phi_{m,n,p}$ defined by considering only isomorphism classes of families Σ over S such that f_Σ maps S into $M = \pi(L)$. It follows directly from theorem 3.3.2 that $\Sigma \to f_\Sigma$ then defines a functorial isomorphism $\Phi^L_{m,n,p}(S) \overset{\sim}{\to} Sch_k(S,M)$ and that the inverse isomorphism is given by $f \to f^{!}(\Sigma^u|M)$ where $\Sigma^u|M = (E^u|M, F^u|M, G^u|M, H^u|M)$ is the restriction of Σ^u to

M. Now suppose that there exists a continuous algebraic

canonical form $c: L \to L$. Because c kills the action of

GL_n there is a unique morphism $\bar{c}: M \to L$ such that $c = \bar{c}\pi$.

For each $x \in M$ we write $\bar{c}(x) = (F_c(x), G_c(x), H_c(x))$. Note

that $\pi\bar{c} = id$, by condition (1.3.1) of the definition of

canonical form.

We now define a family Σ^c over M as follows:

$\Sigma^c = (C^c, F^c, G^c, H^c)$ with $E^c = M \times A^n$, $F^c(x,v) = (x, F_c(x)v)$,

$G^c(x,u) = (x, G_c(x)u)$, $H^c(x,v) = (x, H_c(x)v)$. Because $\pi c =$

id and $c(x) = (F_c(x), G_c(x), H_c(x))$ we have that $f_{\Sigma^c}: M \to M$

is the identity morphism, cf. 3.1.2. But, according to

theorem 3.3.2, or rather the relative version discussed

in the beginning of this proof, we have that $(F_{\Sigma^c})^!(E^u|M)$

is isomorphic to Σ^c. which in particular means that

$(f_{\Sigma^c})^!(E^u|M) \simeq E^c = M \times \underline{A}^n$; but $f_{\Sigma^c} = id$, hence $E^u|M$ is

trivial.

Inversely suppose that $E^u|M$ is trivial. Then we can

find n algebraic sections $e_1, \ldots, e_n: M \to E^u|M$ such that

$e_1(x), \ldots, e_n(x)$ is a basis for E^u_x for all $x \in M$. Let

$F(x,e), G(x,e), H(x,e)$ be the matrices of $F_x: E^u_x \to E^u_x$, $G_x:$

$\{x\} \times A^m \to E^u_x$, $H_x: E^u_x \to x \times A^p$ relative the obvious bases

in $x \times \underline{A}^m$ and $x \times \underline{A}^p$ and the basis $\{e_1(x), \ldots, e_n(x)\}$

of E^u_x. We now define a morphism $c: L \to L$ as follows

$c(F,G,H) = (F(x,e), G(x,e), H(x,e))$ where $x = \pi(F,G,H)$

One easily checks that this is a continuous algebraic
canonical form.

4.1.2. The Local Canonical Froms $c_{\#\alpha}$.

Let α be a nice selection. The bundle $E^u|U_\alpha$ is triv-
ial (by the definition of E^u cf. 3.2) hence by theorem
4.1.1 there exist continuous algebraic canonical forms
on U_α. Such canonical forms are well known. An example
is the canonical form $c_{\#\alpha}$ defined by

$$c_{\#\alpha}(F,G,H) = (F,G,H)^T, T = R(F,G)_\alpha^{-1} \qquad (4.1.3)$$

4.1.3 Corollary

If $m = 1$ there is a continuous algebraic canonical
form on $L_{m,n,p}^{cr}$.

Proof. If $m = 1$ there is only one nice selection α,
and hence $L_{m,n,p}^{cr} = U_\alpha$ by lemma 2.1.1.

4.2. Duality

The assignment $\delta: (F,G,H) \to (F',H',G')$ defines an
isomorphism of algebraic varieties $L_{m,n,p} \to L_{p,n,m}$. If
$L \subset L_{m,n,p}$ is GL_n-invariant then so is $\delta(L) \subset L_{p,n,m}$ (but
δ is not GL_n-invariant). As in [3], 3.1.6 one now easily
shows that there is a continuous canonical form on $\delta(L)$.

4.2.1. Corollary

There is an algebraic continuous canonical form on $L^{cr}_{m,n,p}$ if $p = 1$.

4.3. Example of a Nontrivial Algebraic Line Bundle

Let $U_1 = \underline{A}^1 \times (\underline{A}^2 \setminus (0,0))$, $U_2 = \underline{A}^1 \times (\underline{A}^2 \setminus (0,0))$. We give U_1 coordinates (t, y_1, y_2) and U_2 coordinates (s, x_1, x_2). Let $U_{12} = '\{(t, y_1, y_2) \in U_1 \mid t \neq 0\}$, $U_{21} = '\{(s, x_1, x_2) \in U_2 \mid s \neq 0\}$. We define an isomorphism $\phi: U_{12} \to U_{21}$ by $(t, y_1, y_2) \to (t^{-1}, y_1, t, y_2 t)$. Let X be the prevariety obtained by glueing U_1 and U_2 together by means of ϕ. In fact X is a variety viz. the quasi-affine subvariety of $\underline{A}^4 = \{(z_1, z_2, z_3, z_4)\}$ given by $z_1 z_4 = z_2 z_3$ and ($z_1 \neq 0$ or $z_2 \neq 0$ or $z_3 \neq 0$ or $z_4 \neq 0$). The embeddings of U_1 and U_2 is this subvariety are given by $(t, y_1, y_2) \to (y_1 t, y_2 t, y_2)$, $(s, x_1, x_2) \to (x_1, x_1 s, x_2, x_2 s)$. It is easy to check that this respects the identification ϕ given above.

We now define an algebraic line bundle V over X by glueing $U_1 \times \underline{A}^1$ and $U_2 \times \underline{A}^1$ together by means of the isomorphism

$\overset{\curlyvee}{\phi}: U_{12} \times \underline{A}^1 \to U_{21} \times \underline{A}^1$, $(t, y_1, y_2, u) \to (s, x_1, x_2, v)$ iff

$$ts = 1, \quad x_1 = ty, \quad x_2 = ty_2, \quad v = t^{-1}u \qquad (4.3.1)$$

Now suppose that this line bundle is trivial. Then

there must be everywhere nonzero sections $U_1 \to U_1 \times \underline{A}^1$,
$(t,y_1,y_2) \to ((t,y_1,y_2), g_1(t,y_1,y_2))$; $U_2 \to U_2 \times \underline{A}^1$,
$(s,x_1,x_2) \to ((s,x_1,x_2), g_2(s,x_1,x_2))$ compatible with the
identification $\overset{\vee}{\phi}$. Now g_1 and g_2 are morphisms $\underline{A}_1 \times$
$(\underline{A}_2 \times (0,0)) \to \underline{A}_1$. Because $A_1 \times (\ )$ is of codimension 2
$\underline{A}^1 \times \underline{A}^2 = \underline{A}^3$ this means that g_1 and g_2 extend to morphisms
on all of $\underline{A}^3$, i.e. g_1 and g_2 are polynomials. Putting
everything together we therefore have that C is a trivial
line bundle iff there are polynomials $g_1(t,y_1,y_2)$,
$g_2(s,x_1,x_2)$ such that $g_1(t,y_1,y_2) \neq 0$ if $y_1 \neq 0$ or $y_2 \neq 0$
and $g_2(s,x_1,x_2) \neq 0$ if $x_1 \neq 0$ or $x_2 \neq 0$ and such that
moreover

$$tg_1(t,y_1,y_2) = g_2(t^{-1},ty_1,ty_2) \qquad (4.3.2)$$

for all points (t,y_1,y_2) such that $t \neq 0$ and $y_1 \neq 0$ or
$y_2 \neq 0$. One easily checks that the only polynomials
$g_1(t,y_1,y_2)$ such that $g_1(t,y_1,y_2) \neq 0$ for all (t,y_1,y_2)
for which $y_1 \neq 0$ or $y_2 \neq 0$ are constants. Similarly
$g_2(s,x_1,x_2)$ is a constant. But then (4.3.2) is a contra-
diction. So we have proved

4.3.2 Lemma

The line bundle V defined by 4.3.1 is nontrivial.

4.4. Examples

Let $p \geq 2$ and $m \geq 2$. We write down a number of G,F

and H matrices as follows

If $n = 1$, $m \geq 2$ $G_{1,m}(t,s) = (t \ s \ 0 \ \ldots \ 0)$

$$(4.4.1)$$

If $n < 2 < m < n$ $G_{n,m}(t,s) = \begin{pmatrix} t & s & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \hline a & 1 & & & \\ \vdots & \vdots & & B & \\ a & 1 & & & \end{pmatrix}$

$$(4.4.2)$$

where a is a nonzero element of k different from 1, and where B is an $(n-2) \times (m-2)$ matrix with coefficients in k such that the columns of B and the column vector $(1,\ldots,1)'$ together span an $m-1$ dimensional subspace of k^{n-2}. Such a B exists because $2 < m < n$.

If $n > 2 = m$ $G_{n,2}(t,s) = \begin{pmatrix} t & s \\ 1 & 1 \\ a & 1 \\ \vdots & \vdots \\ a & 1 \end{pmatrix}$ $(4.4.3)$

If $m > n \geq 2$ $G_{n,m}(t,s) =$

$$
\begin{pmatrix}
t & s & 0 & \ldots & & 0 & 0 & \ldots & 0 \\
1 & 1 & 1 & 0 & \ldots & 0 & & & \cdot \\
0 & 0 & 0 & \ldots & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & 0 & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & & & & & \\
0 & 0 & \underbrace{0 \ldots 0}_{n-1} & & 1 & \underbrace{0 \ldots 0}_{m-n-1} &
\end{pmatrix}
$$

$$(4.4.4)$$

If $m = n \geq 2$ $G_{n,n}(t,s) =$

$$
\left.
\begin{pmatrix}
t & s & 0 & \ldots & & 0 \\
1 & 1 & \cdot & & & \cdot \\
a & 1 & \cdot & & & \cdot \\
\cdot & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & & & \\
a & 1 & \underbrace{0 \ldots 0}_{n-2} &
\end{pmatrix}
\right\} n-2
$$

$$(4.4.5)$$

$$
F_n =
\begin{pmatrix}
a_1 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & a_2 & \cdot & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & & & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & \cdot & 0 & a_n
\end{pmatrix}
$$

$$(4.4.6)$$

where $a_1,\ldots,a_n$ are n different elements of k which are all different from zero

$$H_{p,n}(y_1, y_2) = G_{n,p}(y_1, y_2)', \qquad (4.4.7)$$

4.5 An Embedding $X \to M^{cr}_{m,n,p}$

Let U_1, U_2 be as in 4.3 above. We define for all n,m,p with $m \geq 2$ and $p \geq 2$

$$\sigma_{n,m,p}: U_1 \to L^{cr}_{m,n,p}, (t, y_1, y_2) \to (F_n, G_{n,m}(t,1), H_{p,n}(y_1, y_2))$$

$$(4.5.1)$$

$$\bar{\sigma}_{n,m,p}: U_2 \to L^{cr}_{m,n,p}, (s, x_1, x_2) \to (F_n, G_{n,m}(1,s), H_{p,n}(x_1, x_2))$$

We now note that if $ts = 1$, $x_1 = y_1 t$, $x_2 = y_2 t$

$$(F_n, G_{n,m}(t,1), H_{p,n}(y_1, y_2))^{T(t)} = (F_n, G_{n,m}(1,s), H_{p,n}(x_1, x_2))$$

$$(4.5.2)$$

where

$$T(t) = \begin{pmatrix} t^{-1} & 0 & . & . & . & 0 \\ 0 & 1 & . & & & . \\ . & . & . & . & & . \\ . & & . & . & . & . \\ . & & & . & . & 0 \\ 0 & . & . & . & 0 & 1 \end{pmatrix}$$

This means that the morphisms $U_1 \to M^{cr}_{m,n,p}$, $U_2 \to M^{cr}_{m,n,p}$,

obtained from the morphisms $\sigma_{n,m,p}$ and $\overline{\sigma}_{n,m,p}$ of (4.5.1) be composing with $\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$, combine to define a morhpism

$$\tau_{m,n,p}: X \to M^{cr}_{m,n,p} \qquad\qquad (4.5.3)$$

where X is the variety defined in 4.3 above.

4.5.4.

Let α be the nice selection′ $\{(0,2),(1,2),\ldots,(n-1,2)\}$ then we see from 4.4 that $\sigma_{m,n,p}(U_1) \subset U_\alpha$ and hence $\tau_{m,n,p}(U_1) \subset V_\alpha$. Let β be the nice selection′ $\{(0,1),(1,1),\ldots,(n-1,1)\}$ then we see from 4.4 that $\overline{\sigma}_{m,n,p}(U_2) \subset U_\beta$ and hence $\tau_{m,n,p}(U_2) \subset V$. It follows that the pullback of E^u by means of $\tau_{m,n,p}$ is an algebraic vectorbundle over X whose restrictions to U_1 and U_2 are trivial, and the gluing data of this bundle are given by (Cf. [1] Appendix 3.6).

$$\tilde{\psi}: U_{12} \times \underline{A}^n \to U_{21} \times \underline{A}^n$$

$$\qquad\qquad\qquad\qquad (4.5.5)$$

$$((t,y_1,y_2),u) \to ((t^{-1},ty_1,ty_2),T(t,y_1,y_2)u)$$

where $T(t,y_1,y_2)$ is equal to the matrix

$$R(F_n,G_{n,m}(t,1))^{-1}_\beta R(F_n,G_{n,m}(t,1))_\alpha \qquad (4.5.6)$$

where α and β are the nice selections′ $\{(0,2),(1,2),\ldots,(n-1,2)\}$ and′ $\{(0,1),(1,1),\ldots,(n-1,1)\}$. Let $E \to X$ be

this bundle. The exterior product bundle $\overset{n}{\wedge} E \to X$ is then

the line bundle obtained by gluing together $U_1 \times \underline{A}^1$ and

$U_2 \times \underline{A}^1$ by means of the isomorphism

$$\psi \colon U_{12} \times \underline{A}^1 \to U_{21} \times \underline{A}^1$$

$$(4.5.7)$$

$$((t,y_1,y_2),u) \qquad ((t^{-1},ty_1,ty_2), \ \det(T(t,y_1,y_2))u)$$

and from (4.5.6) we see that

$$\det(T(t,y_1,y_2)) = \begin{cases} t^{-1} & \text{if } n \leq 2 \\ t^{-1}a^{n-2} & \text{if } n \leq 2 \end{cases} \qquad (4.5.8)$$

It follows that the line bundle defined by $\hat{\psi}$ is nontrivial.
Cf. 4.3 above.

4.5.9. Proposition

The algebraic vectorbundle $\tau^!_{n,m,p}F^u$ is nontrivial
if $p \geq 2$, $m \geq 2$.

Proof. This follows from the above because if
$E \to X$ is a trivial algebraic n-dimensional vector bundle
then $\overset{n}{\wedge} E \to X$ is a trivial line bundle.

4.5.10. Corollary

Let M be a subvariety of $M^{cr}_{m,n,p}$ such that $\tau_{n,m,p}(x)$
$\subset M$. Then $E^u|M$ is a nontrivial algebraic vectorbundle.

4.6. Nonexistence of Continuous Algebraic Canonical Forms

We can now prove theorem 1.5.

4.6.1. Proof of Theorem 1.5

First let $m \geq 2$ and $p \geq 2$. Let $M^W = \pi(L^W_{m,n,p})$ where $L^W_{m,n,p}$ runs through the subvarieties listed in 1.4. Then we see from 4.4

$$\tau_{m,n,p}(X) \subset M^{\rho,\mu} \qquad\qquad (4.6.1)$$

if $m \neq n$ and $p \neq n$, and that in any case (still assuming $p \geq 2$ and $m \geq 2$)

$$\tau_{m,n,p}(X) \subset M^{cr,co,\mu} \qquad\qquad (4.6.2)$$

By corollary 4.5.10 and theorem 4.4.1 this takes care of the only if parts of statements (iii), (iv), (v), (vi) of theorem 1.5. (Because $L^{\rho,\mu}_{m,n,p} \subset L^{\rho}_{m,n,p}$ and $L^{cr,co}_{m,n,p} \subset L^{cr,co,\rho}_{m,n,p}$). On the other hand if $m = 1$ in cases (iii) and (iv) and $m = 1$ or n in cases (v) and (vi) then the respective subvarities are contained in one U_α for a certain nice selection α. By 4.1.2 there are therefore continuous algebraic canonical forms in these cases. The corresponding fact for $p = 1$ in cases (iii), (iv) and $p = 1$ or n in cases (v), (vi) follows by duality. Cf. 4.2. This proves (iii) - (vi) of theorem 1.5.

The if part of (i) is corollary 4.1.3; the if part
of (ii) follows by duality. Cf. 4.2. To prove the only
if part of (i) observe that if $m \geq 2$ $\pi((F_n, G_{n,m}(t,s)),0)$,
where $t \neq 0$ or $s \neq 0$, depends only on the point $(t:s) \in \underline{P}^1$
and not on the actual t and s. Thus

$$\tau:\ (t:s) \to \pi((F_n, G_{n,m}(t,s)),0)$$

defines a morphism $\underline{P}^1 \to M^{cr}_{m,n,p}$ for all (m,n,p) such that
$m \geq 2$. As in 4.5 one now proves that $\tau^! E^u$ is nontrivial.
By 4.5.10 and 4.4.1 this proves the only if part of (i).
The only if part of (ii) follows by duality. Cf. 4.2.
This concludes the proof of theorem 4.5.

4.7. On Relations between Various Local Canonical Forms

Let $U \subset L^{cr}_{m,n,p}$ be a GL_n invariant subvariety of
$L^{cr}_{m,n,p}$, and suppose that there is a continuous algebraic
canonical form c: $U \to U$. Let $\kappa: U \to \underline{A}^1$ be a morphism,
e.g. a "coordinate function." Then $\kappa c: U \to A^1$ is GL_n
invariant, showing that "the coordinate functions of a
canonical form are invariants."

4.7.1

Now let a: $U \to GL_n$ be a morphism which kills the
action of GL_n on U. Then if c: $U \to U$ is a continuous
algebraic canonical form so is c^a: $U \to U$ which is

defined by $(F,G,H) \rightarrow c(F,G,H)^{a(F,G,H)}$.   Inversely if c'

is a second continuous algebraic canonical form on U

then $c' = c^a$ for some morphism $a: U \rightarrow GL_n$ which kills

the action of GL_n on U.   All this is proved as in section

3.6 of [1].

4.7.3

      The situation becomes slightly more complicated if

we take $U = U_\alpha^{co}$.   We still have the canonical forms

$c_{\#\alpha}$ and all other canonical forms are obtained by means

of a morphism $\hat{a}: V_\alpha^{co} \rightarrow GL_n$.   Now if p = 1 then $\det(\hat{a}(x))$

need not be a constant independent of $x \in V_\alpha^{co}$, because

the codimension of $V_\alpha \setminus V_\alpha^{co}$ in V_α is one if p = 1.   An

example of this is found by taking m = 1 = p and compar-

ing the canonical form $c_{\#\alpha}$ and its dual on $M_{1,n,1}^{cr,co}$.   How-

ever if $p \geq 2$, then the codimension of $V_\alpha \setminus V_\alpha^{co}$ in V_α

is ≥ 2 (cf. section 2.7 above), which means that in

this case we again have that $\hat{a}: V_\alpha^{co} \rightarrow GL_n$ is given by

n^2 polynomials such that $\det(\hat{a}(x))$ is a constant inde-

pendent of $x \in V_\alpha^{co}$.

<div align="center">BIBLIOGRAPHY</div>

[1] M. Hazewinkel, R. E. Kalman. Moduli and Canonical
     Forms for Linear Dynamical Systems. (To appear; a
     preliminary version of this is available in preprint
     form as report 7504, Econometric Inst., Erasmus
     Univ. Rotterdam).

[2] M. Hazewinkel, R. E. Kalman. On Invariants, Canoni-
     cal Forms and Moduli for Linear Constant, Finite
     Dimensional, Dynamical Systems. In: Proc. CNR-CISM
     Symposium on "Algebraic System Theory," Udine, 1975.
     To appear in Lecture Notes in Mathematics, Springer.

[3] M. Hazewinkel. Moduli and Canonical Forms for
     Linear Dynamical Systems, II: The Topological Case.
     Report 7609, Econometric Inst., Erasmus Univ. Rotter-
     dam, 1976.

[4] R. E. Kalman, P. L. Falb, M. A. Arbib. Topics in
     Mathematical Systems Theory. McGraw-Hill, 1969.

[5] D. Mumford. Introduction to Algebraic Geometry.
     Preliminary Version of the First Three Chapters.
     Lecture Notes, Harvard Univ.

[6] D. Mumford. Geometric Invariant Theory. Springer,
     1965.

[7] C. J. Byrnes, N. E. Hurt. On the moduli of linear
     dynamical systems. (To appear in Adv. Math.)