MATHEMATICS

ON A FORMULA OF C. S. MEIJER

BY

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Some years ago C. S. MEIJER ([1], p. 127, formula (G); [3], p. 355, formula (113)) published a formula for generalized hypergeometric functions, which contains many known formulae on special functions. Meijer's formula is

$$(1)\quad \begin{cases} {}_{p+k}\Phi_{q+l}\begin{pmatrix}\gamma_1, \ldots, \gamma_k, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q, \delta_1, \ldots, \delta_l; \lambda\zeta\end{pmatrix} = \\ \sum_{r=0}^{\infty} \frac{1}{r!}\, {}_{k+1}\Phi_l\begin{pmatrix}-r, \gamma_1, \ldots, \gamma_k; \\ \delta_1, \ldots, \delta_l; \lambda\end{pmatrix} (\alpha_1)_r \ldots (\alpha_p)_r (-\zeta)^r\, {}_p\Phi_q\begin{pmatrix}\alpha_1+r, \ldots, \alpha_p+r; \\ \beta_1+r, \ldots, \beta_q+r; \zeta\end{pmatrix}. \end{cases}$$

Here we use the following notation:

$$(\alpha)_r \doteq \begin{cases} \alpha(\alpha+1)\ldots(\alpha+r-1) & \text{if } r \text{ is a positive integer,} \\ 1 & \text{if } r=0. \end{cases}$$

If p and q are non-negative integers, and $p \leqslant q+1$ or for some $i (1 \leqslant i \leqslant p)$ α_i is a non-positive integer, then

$$ {}_p\Phi_q\begin{pmatrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \zeta\end{pmatrix}$$

is the analytic function of ζ defined in a neighbourhood of $\zeta=0$ by

$$(2)\qquad {}_p\Phi_q\begin{pmatrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \zeta\end{pmatrix} = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{n!\, \Gamma(\beta_1+n) \ldots \Gamma(\beta_q+n)} \zeta^n.$$

The series on the right of (2) has a finite radius of convergence only in the case that $p=q+1$ and no $\alpha_i (i=1, \ldots, p)$ is equal to a non-positive integer. The analytic continuation for this case will not be described here. It can be found in [2], § 2 and in [4]. For our purpose it is sufficient to know, that 0, 1, ∞ are the only singularities (branchpoints in general). Hence, if C is any simple curve connecting 1 and ∞ and $0 \notin C$, there exists a unique analytic function on the complement of C, which has the power series representation (2) in a neighbourhood of $\zeta=0$. The curve C will not be mentioned explicitly in the sequel, but is assumed to be suitably chosen. (Meijer uses the rays $(1, 1+i\infty)$ and $(1, 1-i\infty)$).

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In this paper (1) is proved in a new and simple way. The conditions for the validity of (1) given by MEIJER ([1], p. 127; [3], p. 355) will be deduced anew. Finally a relation for generalized Heine series is given, which is analogous to (1).

Formula (1) is valid in each of the following eight cases Ia, ..., Ie, IIa, IIb, III:

I. None of the numbers $\gamma_1, \ldots, \gamma_k, \alpha_1, \ldots, \alpha_p$ is equal to $0, -1, -2, \ldots$, and

a. $p<q+1$ and $p+k<q+l+1$, for all values of λ and ζ.

b. $p<q+1$, $p+k=q+l+1$, for $|\lambda\zeta|<1$.

c. $p=q+1$, $k<l$, for $\operatorname{Re}\zeta<\frac{1}{2}$ and all values of λ.

d. $p=q+1$, $k=l=0$, for $\zeta\neq 1$ and $|(\lambda-1)\zeta|<|\zeta-1|$.

e. $p=q+1$, $k=l>0$, for $\operatorname{Re}\zeta<\frac{1}{2}$ and $|(\lambda-1)\zeta|<|\zeta-1|$.

II. $k\geqslant 1$ and at least one of the numbers $\gamma_1, \ldots, \gamma_k$, but none of the numbers $\alpha_1, \ldots, \alpha_p$ is equal to $0, -1, -2, \ldots$, and

a. $p<q+1$, for all values of λ and ζ.

b. $p=q+1$, for $\operatorname{Re}\zeta<\frac{1}{2}$ and all values of λ.

III. $p\geqslant 1$ and at least one of the numbers $\alpha_1, \ldots, \alpha_p$ is equal to $0, -1, -2, \ldots$, for all values of λ and ζ.

Proof. By (2) the right-hand member of (1) can be formally written as

$$\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{n=0}^{r}\frac{(-r)_n(\gamma_1)_n\ldots(\gamma_k)_n}{n!\,\Gamma(\delta_1+n)\ldots\Gamma(\delta_l+n)}\lambda^n\sum_{m=0}^{\infty}\frac{(\alpha_1)_{r+m}\ldots(\alpha_p)_{r+m}(-1)^r}{m!\,\Gamma(\beta_1+r+m)\ldots\Gamma(\beta_q+r+m)}\zeta^{m+r}=$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=n}^{\infty}\frac{(\gamma_1)_n\ldots(\gamma_k)_n(\alpha_1)_{r+m}\ldots(\alpha_p)_{r+m}}{n!\,\Gamma(\delta_1+n)\ldots\Gamma(\delta_l+n)\,\Gamma(\beta_1+r+m)\ldots\Gamma(\beta_q+r+m)}\frac{(-1)^{r+n}\lambda^n\zeta^{m+r}}{m!\,(r-n)!}=$$

$$\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}\frac{(\gamma_1)_n\ldots(\gamma_k)_n(\alpha_1)_j\ldots(\alpha_p)_j\lambda^n\zeta^j}{n!\,\Gamma(\delta_1+n)\ldots\Gamma(\delta_l+n)\,\Gamma(\beta_1+j)\ldots\Gamma(\beta_q+j)}\sum_{r=n}^{j}\frac{(-1)^{r+n}}{(r-n)!\,(j-r)!}. \tag{3}$$

Now using

$$\sum_{r=n}^{j}\frac{(-1)^{r+n}}{(r-n)!\,(j-r)!}=\frac{(1-1)^{j-n}}{(j-n)!}=\begin{cases}0 & \text{if } j>n\\ 1 & \text{if } j=n,\end{cases}$$

we see that (3) equals the left-hand member of (1).

In each of the cases Ia, b, IIa and III the absolute convergence of (3) can be shown by estimates of the type

$$\left|\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{\Gamma(\beta_1+n)\ldots\Gamma(\beta_p+n)}\right|\leqslant C\,n^{\operatorname{Re}\{(\alpha_1+\ldots+\alpha_p)-(\beta_1+\ldots+\beta_p)\}}. \tag{4}$$

Hence, in the following we may restrict ourselves to the case $p=q+1$, where none of the numbers $\alpha_1, \ldots, \alpha_p$ is equal to $0, -1, -2, \ldots$. In this case we can again prove by (4) the absolute convergence of (3), but only for small values of $|\zeta|$ and $|\lambda|$, provided that $k\leqslant l$ or that one of the numbers $\gamma_1, \ldots, \gamma_k$ is equal to $0, -1, -2, \ldots$ Next we shall show that (1) has a larger region of validity. We need two lemmas.

Lemma 1. If $\zeta \neq 1$, then

$$\limsup_{r\to\infty} \left| \frac{(-\zeta)^r (\alpha_1)_r \ldots (\alpha_{q+1})_r}{r!} \, {}_{q+1}\Phi_q \begin{pmatrix} \alpha_1+r, \ldots, \alpha_{q+1}+r; \\ \beta_1+r, \ldots, \beta_q+r; \zeta \end{pmatrix} \right|^{\frac{1}{r}} = \left| \frac{\zeta}{\zeta-1} \right|.$$

Proof. The function

$$f(w) = {}_{q+1}\Phi_q \begin{pmatrix} \alpha_1, \ldots, \alpha_{q+1}; \\ \beta_1, \ldots, \beta_q; \zeta(1-w) \end{pmatrix}$$

is analytic in w for $|w| < |1-\zeta^{-1}|$. Using (2) we can easily derive that

$$\left[\frac{d^r}{dw^r} f(w)\right]_{w=0} = (\alpha_1)_r \ldots (\alpha_{q+1})_r \, (-\zeta)^r \, {}_{q+1}\Phi_q \begin{pmatrix} \alpha_1+r, \ldots, \alpha_{q+1}+r; \\ \beta_1+r, \ldots, \beta_q+r; \zeta \end{pmatrix}$$

for $|\zeta|<1$. Both members being analytic in ζ (if $\zeta \neq 1$), the equality is valid in the cut ζ-plane. Hence, the Taylor expansion of $f(w)$ in powers of w is

$$f(w) = \sum_{r=0}^{\infty} w^r \frac{(-\zeta)^r (\alpha_1)_r \ldots (\alpha_{q+1})_r}{r!} \, {}_{q+1}\Phi_q \begin{pmatrix} \alpha_1+r, \ldots, \alpha_{q+1}+r; \\ \beta_1+r, \ldots, \beta_q+r; \zeta \end{pmatrix}.$$

As $f(w)$ is analytic for $|w| < |1-\zeta^{-1}|$, the radius of convergence of the Taylor series is equal to $|1-\zeta^{-1}|$. Lemma 1 expresses this fact in a different way.

Lemma 2. If none of the numbers $\gamma_1, \ldots, \gamma_k$ is equal to $0, -1, -2, \ldots$, then

$$\limsup_{r\to\infty} \left| {}_{k+1}\Phi_l \begin{pmatrix} -r, \gamma_1, \ldots, \gamma_k; \\ \delta_1, \ldots, \delta_l; \lambda \end{pmatrix} \right|^{\frac{1}{r}} = \begin{cases} 1 & \text{if } k<l, \\ \max(1, |1-\lambda|) & \text{if } k=l>0, \\ |1-\lambda| & \text{if } k=l=0. \end{cases}$$

However, if one of the numbers $\gamma_1, \ldots, \gamma_k$ is equal to $0, -1, -2, \ldots$, the lim sup equals 1 in all cases.

Proof. The proof runs along the same lines as that of lemma 1. The starting point is now

$$(5) \quad \frac{1}{1-w} \, {}_{k+1}\Phi_l \begin{pmatrix} 1, \gamma_1, \ldots, \gamma_k; \\ \delta_1, \ldots, \delta_l; \frac{\lambda w}{w-1} \end{pmatrix} = \sum_{r=0}^{\infty} w^r \, {}_{k+1}\Phi_l \begin{pmatrix} -r, \gamma_1, \ldots, \gamma_k; \\ \delta_1, \ldots, \delta_l; \lambda \end{pmatrix}.$$

This formula is the special case $p=1, q=0, \alpha_1=0, \zeta=w(w-1)^{-1}$ of (1). Hence, it is valid for small values of $|\lambda|$ and $|w|$. If λ is fixed, the function $g(w)$ on the left of (5) has a Taylor expansion in powers of w. It is easily seen that the coefficient of w^r in that expansion is an analytic function of λ. From these considerations it is clear that for each λ the expansion (5) holds in a certain neighbourhood of $w=0$. Now $g(w)$ has a singularity in $w=1$ if $k<l$, in 1 and $(1-\lambda)^{-1}$ if $k=l>0$, and in $(1-\lambda)^{-1}$ if $k=l=0$.

This yields the first part of lemma 2. If one of the numbers $\gamma_1, \ldots, \gamma_k$ is equal to $0, -1, -2, \ldots$, then $g(w)$ has a singularity at $w=1$. This completes the proof.

From lemma 1 and lemma 2 it follows that the series on the right of (1) converges absolutely in the cases Ic, d, e and IIb. Moreover, the convergence is uniform, if λ and ζ are restricted to compact sets. Hence, this series represents a function which is analytic in λ and in ζ. As (1) holds for small values of $|\lambda|$ and $|\zeta|$, the validity of (1) is also proved in the cases Ic, d, e and IIb.

The above-mentioned generalization of (1) to Heine series (for definition and properties of Heine or basic series see [5], ch. VIII) is

$$(6)\quad \left\{\begin{aligned} &{}_{k+p}\Psi_{l+s}\begin{pmatrix}\gamma_1, \ldots, \gamma_k, \alpha_1, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_l, \beta_l, \ldots, \beta_s;\ \lambda\zeta\end{pmatrix} = \sum_{r=0}^{\infty} \frac{[\alpha_1]_r \ldots [\alpha_p]_r \zeta^r}{[q^{-r}]_r [\beta_1]_r \ldots [\beta_s]_r} \cdot \\ &\qquad \cdot {}_{k+1}\Psi_l\begin{pmatrix}q^{-r}, \gamma_1, \ldots, \gamma_k; \\ \delta_1, \ldots, \delta_l;\ \lambda\end{pmatrix} {}_p\Psi_s\begin{pmatrix}\alpha_1 q^r, \ldots, \alpha_p q^r; \\ \beta_1 q^r, \ldots, \beta_s q^r;\ \zeta\end{pmatrix},\end{aligned}\right.$$

where $0<q<1$,

$$[\alpha]_r = \begin{cases}(1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{r-1}) & \text{if } r \geqslant 1, \\ 1 & \text{if } r=0,\end{cases}$$

and

$${}_p\Psi_s\begin{pmatrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_s;\ \zeta\end{pmatrix} = \sum_{r=0}^{\infty} \frac{[\alpha_1]_r \ldots [\alpha_p]_r}{[q]_r [\beta_1]_r \ldots [\beta_s]_r} \zeta^r.$$

(6) is always valid if $|\zeta|<1$ and $|\lambda|<1$. A proof and a more precise discussion of the validity of (6) can be given in a similar way as was done for formula (1).

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