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ON THE NONEXISTENCE OF CONTINUOUS CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS

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Abstract. A real linear dynamical system $\dot{x} = Fx+Gu$, $y = Hx$ is thought of as being represented by the triple of matrices (F, G, H). A base change in state space changes the triple to (SFS^{-1}, SG, HS^{-1}) for a certain $S \epsilon GL_n(\mathbf{R})$. In this paper we discuss existence and nonexistence of canonical forms for this action of $GL_n(\mathbf{R})$.

1. Introduction and statement of results. A real linear, constant, finite dimensional dynamical system is thought of as being represented by a triple of real matrices (F, G, H) where F is an $n \times n$ matrix, G an $n \times m$ matrix and H an $p \times n$ matrix; i.e., there are m inputs, p outputs and the state space dimension is n. The dynamical system itself is then

$$x = Fx+Gu, \; y = Hx \tag{1.1}$$

in the continuous case, or

$$x(t+1) = Fx(t)+Gu(t), \; y(t) = Hx(t) \tag{1.2}$$

in the discrete case. A change of coordinates in state space changes the triple of matrices (F, G, H) into the triple (SFS^{-1}, SG, HS^{-1}). We are interested in continuous canonical forms for this action of $GL_n(\mathbf{R})$, the group of real invertible $n \times n$ matrices. Cf. **3.3** below for a precise definition of what a canonical form is.

The triple (F, G, H) is completely reachable if the matrix

$$R(F, G) = (G \, FG \ldots F^nG) \tag{1.3}$$

consisting of all the columns of the matrices F^iG, $i = 0, \ldots, n$ has rank n. The triple (F, G, H) is completely observable if the matrix

$$Q(F, H) = (H^T F^T H^T \; \ldots (F^T)^n H^T)$$

where the upper T denotes transposes, has rank n. Cf. (Kalman 1969) for these notations.

Let $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})$ denote the space of all triples of matrices (F, G, H), $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr}$ the subspace of all completely reachable triples and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr, co}$ the subspace of all triples which are completely observable and completely reachable.

In (Hazewinkel and Kalman 1975a, b) we studied pairs of completely reachable matrices $(F, G) \epsilon \mathcal{F}\mathcal{G}_{cr}$ (over arbitrary fields) by algebraic geometric methods and proved that there are no algebraic continuous canonical forms on $\mathcal{F}\mathcal{G}_{cr}$ if $m \geqslant 2$. We can embed $\mathcal{F}\mathcal{G}_{cr}$ into $\mathcal{F}\mathcal{G}\mathcal{H}_{cr}$ by means of the GL_n invariant map.

$$(F, G) \to (F, G, 0) \tag{1.4}$$

So this result implies the nonexistence of algebraic continuous canonical forms on $\mathcal{F}\mathcal{G}\mathcal{H}_{cr}$ (and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{C})_{cr}$) but gives at first sight no information on the existence of canonical forms on $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$. Firstly, because the results of (Hazewinkel and Kalman 1975a, b) as stated there do not rule out the existence of nonalgebraic continuous canonical forms on $\mathcal{F}\mathcal{G}(\mathbf{R}_{cr})$ and $\mathcal{F}\mathcal{G}(\mathbf{C}_{cr})$, and secondly because there seems to be no $GL_n(\mathbf{R})$ invariant embedding $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$.

So that the results of Hazewinkel and Kalman (1975a, b) leave it open whether $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$ admits a canonical form or not. In fact, this had better be the case because there does exist an (algebraic) continuous canonical form on $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$ if $p = 1$.

We have :

1.5. THEOREM. *There does not exits a continuous canonical form on* $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$ *if and only if* $m \geqslant 2$ *and* $p \geqslant 2$. *A fortiori there are no continuous canonical forms an* $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})$, $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr}$, $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{co}$ *if* $m \geqslant 2$ *and* $p \geqslant 2$.

In this paper we show how one can use results on the nonexistence of canonical forms on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ to deduce results on the nonexistence of canonical forms on $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$ for suitable p. We are thus able to prove theorem 1.5 for the case $m \geqslant 2$, $p \geqslant 2n$ and we indicate a similar proof for the cases $p \geqslant 2$, $m \geqslant 2n$ and $p, m > n$. For the general case cf. (Hazewinkel). The basic idea of the proof presented here is very simple. The Gramm-Schmidt orthonormalization process shows that there exists a continuous $GL_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ (resp. $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$) if and only if there exists an $O_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$ (resp. $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}^{ortho}$) where the superscript "ortho" means that we consider only those pairs (resp. triples) such that $R(F, G)$ has orthonormal row vectors, and where $O_n(\mathbf{R})$ is the group of orthogonal $n \times n$ matrices.

This trick is useful because there does exist an $O_n(\mathbf{R})$ invariant embedding $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho} \to \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}^{ortho}$ for suitable p, viz.

$$(F, G) \to (F, G, \bar{R}(F, G)^T) \tag{1.6}$$

where $\bar{R}(F, G)$ is the matrix

$$\bar{R}(F, G) = (G\; FG \ldots F^{n-1}(G) \tag{1.7}$$

and the upper T indicates transposes.

However, Gramm-Schmidt orthonormalization is essentially nonalgebraic which is one more reason why we cannot use the results of Hazewinkel and Kalman (1975a, b) as they stand, but have to extend them to prove nonexistence of continuous (possibly non-algebraic) canonical forms on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$. This is done in section 2 below. The methods are the same as those of (Hazewinkel and Kalman, 1975a, b) : the quotient $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}/GL_n(\mathbf{R})$ is shown to exist and to admit a universal family of completely reachable pairs over it. Then we need a new proof that the underlying bundle of the universal family is nontrivial if $m \geqslant 2$, because a priori there is no reason why the bundle of $\mathbf{R}$-points $E(\mathbf{R}) \to B(\mathbf{R})$ of a nontrivial algebraic bundle $E \to B$ should be nontrivial. The rest of the nonexistence proof is then as in (Hazewinkel and Kalman, 1975b), 6.1., Section 3 contains the orthonormalization trick alluded to above and in section 4 theorem 1.5 is proved for suitable m, n, p.

2. A fine moduli space for continuous families of real linear dynamical systems. In this section we consider completely reachable pairs of real matrices (F, G) of size $n\times n$ and $m\times n$ respectively. As usual $R(F, G)$ is the matrix $(G\ FG\ F^2G \ldots F^nG)$ with columns F^ig_j, $j = 1, \ldots, m$; $i = 0, \ldots, n$ where g_j is the j-th column of G. We number the columns of $R(F, G)$ by means of the pairs (i, j) ordered lexicographically. Let J be this set of indices.

2.1. Nice Selections and Sucessor Indices. A *nice selection* is a subset α of J with the property that $(i, j) \in \alpha \Longrightarrow (i', j) \in \alpha$ for all $i' \leqslant i$. A *successor index* $k = (i, j)$ of a nice selection α is an element $(i, j) \in J$ such that $(i', j) \in \alpha$ for all $i' \leqslant i$. Note that there is precisely one successor index of the form (i, j) for α for every $j = 1, \ldots, m$. This successor index is denoted $s(\alpha, j)$.

2.2 Construction of the Differentiable Manifold $\mathbf{M}_{m,n}(\mathbf{R})$. For each nice selection α let $U_\alpha = \mathbf{R}^{mn}$. For $x \in U_\alpha$ with components x_k, $k = 1, \ldots, mn$, let $x(i)$, $i = 1, \ldots, m$ denote the columnvector with entries $x(i)_j = x_{(i-1)n+j}$, $j = 1, \ldots, m$. (I.e. we write x as an $n\times m$ array). For each $x \in U_\alpha = \mathbf{R}^{nm}$ there is a unique pair of real matrices $(F, G) \in FG(\mathbf{R})_{cr}$ such that

$$R(F, G)_\alpha = I_n \tag{2.2.1}$$

where $R(F, G)_\alpha$ is the matrix consisting of the columns of $R(F, G)$ with indices in α (in their original order), and such that

$$R(F, G)_{s(\alpha, j)} = x(j), \quad j = 1, \ldots, m \tag{2.2.2}$$

where $R(F, G)_{s(\alpha, j)}$ is the column of $R(F, G)$ with index $s(\alpha, j)$, the j-th successor index of α. For a proof cf. (Hazewinkel and Kalman 1975b) sections 3.4, 3.5. This pair of matrices is denoted $\psi_\alpha(x)$.

For each ordered pair of nice selections α and β we define

$$U_{\alpha\beta} = \{x \in U_\alpha \mid (R\psi_\alpha(x))_\beta \text{ is nonsingular}\} \tag{2.2.3}$$

and we identify $U_{\alpha\beta}$ and $U_{\beta\alpha}$ by means of the correspondence

$$x \longleftrightarrow y \Longleftrightarrow (R\psi_\alpha(x))_\beta^{-1}(R\psi_\alpha(x)) = R\psi_\beta(y) \tag{2.2.4}$$

These identifications define a differentiable manifold denoted $\mathcal{M}_{mn}(\mathbf{R})$ which is covered by the coordinate patches $U_\alpha = \mathbf{R}^{mn}$, α a nice selection.

There is a natural map

$$\pi : \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{M}_{m\,n}(\mathbf{R}) \tag{2.2.5}$$

which is defined as follows. For each $(F, G) \in \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ there is a nice selection α such that $R(F, G)_\alpha$ is nonsingular (Hazewinkel and Kalman 1975a, lemma 2.4.1). We now map (F, G) to the point $x \in U_\alpha \subset \mathcal{M}_{m,n}(\mathbf{R})$ determined by

$$\pi(F, G) = x \in U_\alpha \subset \mathcal{M}_{m,n}(\mathbf{R}) \Longleftrightarrow \psi_\alpha(x) = R(F, G)_\alpha^{-1}R(F, G) \tag{2.2.6}$$

This is independent of the choice of α because of the identifications (2.2.4). The map π is surjective because $\pi\psi_\alpha(x) = x$ for $x \in U_\alpha$, and we have for $x \in U_\alpha$

$$\pi^{-1}(x) = \{(SFS^{-1}, SG \mid S \in GL_n(\mathbf{R})\} \quad \text{if } (F, G) = \psi_\alpha(x). \tag{2.2.7}$$

In other words $\mathcal{M}_{m,n}(\mathbf{R})$ is the quotient of $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ under the action of $GL_n(\mathbf{R})$. Cf. (Hazewinkel and Kalman (1975a), 3.3 and (Hazewinkel and Kalman 1975b), 3.5–3.7 for proofs.

2.3. Continuous Families of Completely Reachable Pairs. Let X be a topological space. A *continuous family of pairs* over X is an n-dimensional real vector bundle E over X together with a vectorbundle endomorphism $F : E \to E$ and m sections $g_1, \ldots, g_m : X \to E$. For each $x \in X$ we have an endomorphism $F(x) : E(x) = \mathbf{R}^n \to E(x)$ and m vectors $g_1(x), \ldots, g_m(x) \in E(x) = \mathbf{R}^n$. After a choice of basis in $E(x)$ these vectors and this endomorphism define a pair of matrices, i.e., an element of $\mathcal{F}\mathcal{G}(\mathbf{R})$. Note that the element so defined is welldefined up to the action of $GL_n(\mathbf{R})$ (= change of basis). The family $(E, F, g_1, \ldots, g_m)$ is said to be completely reachable if all these elements of $\mathcal{F}\mathcal{G}(\mathbf{R})$ are in fact in $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$.

Two continuous families over X, $(E, F, g_1, \ldots, g_m)$, $(E', F', g'_1, \ldots, g'_m)$ are said to be isomorphic if there is a vectorbundle isomorphism $\phi : E \to E'$ such that

$$\phi F = F'\phi \tag{2.3.1}$$

$$\phi g_i = g'_i \qquad i = 1, \ldots, m. \tag{2.3.2}$$

For every space X let $A(X)$ be the set of isomorphism classes of continuous families of completely reachable pairs over X. By means of the pullback construction which associates to a continuous map $f : Y \to X$ and a family $(E, F, g_1, \ldots, g_m)$ over X, the family $(f^!E, f^!g_1, \ldots, f^!g_m)$ over Y, we can turn A into a contravariant functor from the category of topological spaces to the category of sets. Cf. (Husemoller 1966) for background material on vectorbundles and pullback.

2.4. The Canonical Map Associated to a Completely Reachable Family. Let $\Sigma = (E, F, g_1, \ldots, g_m)$ be a family of completely reachable pairs over X. For each $x \in X$ we then have a completely reachable pair $F(x)$, $G(x)$ over x (cf. 2.3 above) which is determined up to a choice of basis in $E(x)$. This means that $\pi(F(x), g(x))$ is welldefined. (Cf. (2.2.5), (2.2.6) above for the definition of π). Associated to Σ we have thus defined a continuous map $f(\Sigma) : X \to \mathcal{M}_{m,n}(\mathbf{R})$. Note that isomorphic families give rise to the same maps $X \to \mathcal{M}_{m,n}(\mathbf{R})$.

2.5. Definition of the Uuiversal Family. For each nice selection α let $E_\alpha = U_\alpha \times \mathbf{R}^n$ be the trivial vectorbundle over U_α. We define the bundle endomorphism $F_\alpha : E_\alpha \to E_\alpha$ and the sections $g_{1\alpha}, \ldots, g_{m\alpha} : U_\alpha \to E_\alpha$ as follows. For $x \in U_\alpha$ write

$$\psi_\alpha(x) = (F_\alpha(x), G_\alpha(x)) \tag{2.5.1}$$

We then define

$$F_\alpha(x, v) = (x, F_\alpha(x)v) \tag{2.5.2}$$

$$g_{i\alpha}(x) = (x, G_\alpha(x)_i) \quad i = 1, \ldots, m \tag{2.5.3}$$

where $G_\alpha(x)_i$ is the i-th column of $G_\alpha(x)$.

We now construct a family $\Sigma^u = (E^u, F^u, g_1{}^u, \ldots, g_m{}^u)$ over $\mathcal{M}_{m,n}(\mathbf{R})$ by patching together the partial families $(E_\alpha, F_\alpha, g_{1\alpha}, \ldots, g_{m\alpha})$. This is done as follows. Let $E_{\alpha\beta} = \{(x, v) \in E_\alpha \mid x \in U_{\alpha\beta}\}$ and let $\phi_{\alpha\beta} : U_{\beta\alpha} \to U_{\beta\alpha}$ be the diffeomorphism defined in (2.2.4) above. We now define the isomorphism.

$$\tilde{\phi}_{\alpha\beta} : E_{\alpha\beta} \to E_{\beta\alpha} \tag{2.5.4}$$

by the formula

$$\tilde{\phi}_{\alpha\beta}(x, v) = (\phi_{\alpha\beta}\ (x), (R\psi_\alpha(x))_\beta{}^{-1}v) \tag{2.5.5}$$

It is easy to check that these isomorphisms are compatible with the endomorphisms F_α, F_β and the sections $g_{i\alpha}, g_{\alpha\beta}$, $i = 1, \ldots, m$, so that these identifications yield a family Σ^u such that the restriction of Σ^u to U_α is isomorphic to the family $(E_\alpha, F_\alpha, g_{1\alpha}, \ldots, g_{m\alpha})$ for all nice selections α.

It follows that

$$f(\Sigma^u) = \text{identity on } \mathcal{M}_{m,n}(\mathbf{R}) \tag{2.5.6}$$

(Cf. 2.4 and (2.2.7)).

2.6. THEOREM *$\mathcal{M}_{m,n}(\mathbf{R})$ is a fine moduli space for the punctor A.*

This means the following. Let Top(X, Y) be the set of continuous maps from the topological space X to the topological space Y. Then theorem 2.6 says that the map $\Sigma \to f(\Sigma)$ of section 2.4 above induces a bijection from A(X) to Top $(X, \mathcal{M}_{m,n}(\mathbf{R}))$ for all topological spaces X. More precisely theorem 2.6 says that: (i) For every $f \in$ Top$(X, \mathcal{M}_{m,n}(\mathbf{R}))$ there is a family Σ^f such that $f(\Sigma^f) = f$. (N.B. The family $f^! \Sigma^u$ is such a family), and (ii) for every family of completely reachable pairs Σ over a space X there is a unique map $f : X \to \mathcal{M}_{m,n}(\mathbf{R})$ such that $f^!\Sigma^u$ is isomorphic to Σ. This map is of course $f(\Sigma) : X \to \mathcal{M}_{m,n}(\mathbf{R})$ and what is left to prove is that $f(\Sigma)^! \Sigma^u$ and Σ are isomorphic families. This is done exactly as in (Hazewinkel and Kalman 1975a), 3.6.

2.7. An Embedding $S^1 \to \mathcal{M}_{m,n}(\mathbf{R})$. The next thing we want to do is to show that the bundle E^u underlying the universal family Σ^u over $\mathcal{M}_{m,n}(\mathbf{R})$ is not the trivial bundle if $m \geqslant 2$. (If $m = 1$ there is only one nice selection and it follows that the bundle is trivial in that case). To this end we first construct an explicit embedding $\phi : S^1 = \mathbf{P}^1(\mathbf{R}) \to$

$\mathcal{M}_{m,n}(\mathbf{R})$ for $m, n \geqslant 2$ where S^1 is the circle and $\mathbf{P}^1(\mathbf{R})$ is one-dimensional real projective space. This is done as follows. Define a continuous map

$$\phi_1 : \mathbf{R} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}, \quad t \to (F(t, 1), G(t, 1)) \tag{2.7.1}$$

where $F(t, 1)$ is equal to the matrix consisting of the columnvectors

$$F(t, 1)_1 = e_1, \ F(t, 1)_2 = e_1 + e_2, \ F(t, 1)_i = e_{i+1} \quad \text{for } i = 3, \ldots, n-1 \tag{2.7.2}$$
$$F(t, 1)_n = 2e_3 \quad \text{if } n \geqslant 3,$$

where e_j is the j-th unit columnvector. The matrix $G(t, 1)$ consists of the columnvectors

$$G(t, 1)_1 = te_1, \ G(t, 1)_2 = e_1 + e_2, \ G(t, 1)_i = 0 \quad \text{if } i \geqslant 3 \quad \text{if } n = 2, \tag{2.7.3}$$

and

$$G(t, 1)_1 = te_1 + e_3, \ G(t, 1)_2 = e_1 + e_2, \ G(t, 1)_i = 0 \quad \text{if } i \geqslant 3 \quad \text{if } n \geqslant 3. \tag{2.7.4}$$

Note that $R(\phi_1(t))_\alpha$ is nonsingular for all t for the nice selection

$$\alpha = \{(0, 1), \ldots, (n-3, 1), (0, 2), (1, 2)\} \tag{2.7.5}$$

We also define a continuous map

$$\phi_2 : \mathbf{R} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}, \ s \to (F(s, 2), G(s, 2) \tag{2.7.6}$$

with

$$F(s, 2)_1 = e_1, F(s, 2)_2 = se_1 + e_2, F(s, 2)_i = e_{i+1} \text{ for } i = 3, \ldots, n-1 \tag{2.7.7}$$
$$F(s, 2)_n = 2e_3 \quad \text{if } n \geqslant 3,$$

and with

$$G(s, 2)_1 = e_1, \ G(s, 2)_2 = se_1 + e_2, \ G(s, 2)_i = 0 \quad \text{for } i \geqslant 3 \tag{2.7.8}$$
if $n = 2$, and

$$G(s, 2)_1 = e_1 + e_3, \ G(s, 2)_2 = se_1 + e_2, \ G(s, 2)_i = 0 \quad \text{for } i \geqslant 3 \tag{2.7.9}$$
if $n \geqslant 3$.

Note that $R(\phi_2(s))_\beta$ is nonsingular for all s for the nice selection

$$\beta = \{(0, 1), \ldots, (n-2, 1), (0, 2)\} \tag{2.7.10}$$

The pairs of matrices $\phi_1(t)$ and $\phi_2(s)$ are equivalent pairs if $t \neq 0$, $s \neq 0$ and $ts = 1$. The matrix transforming the pair $\phi_1(t)$ into $\phi_2(s)$ is then equal to

$$\left(\begin{array}{cc:c} t^{-1} & 0 & 0 \\ 0 & 1 & \\ \hdashline 0 & & I_{n-2} \end{array}\right) = \left(\begin{array}{cc:c} s & 0 & 0 \\ 0 & 1 & \\ \hdashline 0 & & I_{n-2} \end{array}\right)$$

This means that the composed maps

$$\pi\phi_1 : \mathbf{R} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{M}_{m,n}(\mathbf{R})$$

and

$$\pi\phi_2 : \mathbf{R} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{M}_{m,n}(\mathbf{R})$$

combine to define a continuous map

$$\phi : S^1 = \mathbf{P}^1(\mathbf{R}) \to \mathrm{M}_{m,n}(\mathbf{R}) \tag{2.7.11}$$

Let $(t:s)$ be homogeneous coordinates for $\mathbf{P}^1(\mathbf{R})$. Then

$$\begin{aligned} &\phi(t:s) \in U_\alpha \quad \text{if } s \neq 0 \\ &\phi(t:s) \in U_\beta \quad \text{if } t \neq 0 \end{aligned} \tag{2.7.12}$$

where α and β are the nice selections given by (2.7.5) and (2.7.10) above. It remains to construct an embedding $\mathbf{p}^1(\mathbf{R}) \to \mathcal{M}_{m,n}(\mathbf{R})$ in the case $n = 1$. This is done as follows. We define

$$\phi_1 : \mathbf{R} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}, \quad t \to (F(t,1), G(t,1))$$

where $G(t,1)_1 = t$, $G(t,1)_2 = 1$, $G(t,1)_i = 0$ $i \geqslant 3$ and $F(t,1) = 0$ and

$$\phi_2 : \mathbf{R} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}, \; s \to (F(s,2),\; G(s,2))$$

where $G(s,2)_1 = 1$, $G(s,2)_2 = s$, $G(s,2)_i = 0$, $i \geqslant 3$ and $F(s,2) = 0$.

As above these two applications combine to define a continuous map

$$\phi : \mathbf{P}^1(\mathbf{R}) \to \mathcal{M}_{m,1}(\mathbf{R}).$$

2.8. Proposition. *The underlying vector bundle of the universal family* Σ^u *over* $\mathcal{M}_{m,n}(\mathbf{R})$ *is nontrivial iff* $m \geqslant 2$.

Proof. The only if part is trivial as there is only one nice selection if $m = 1$. There are several ways to prove the if part. One is by algebraic geometry as follows: $\mathcal{M}_{m,n}(\mathbf{C})$ embeds naturally into the Grassmann variety of complex n-planes in complexe $(n+1)m$ space which in turn is a closed subvariety of projective space of (complex) dimension N with $N+1$ equal to the binomial coefficient $\binom{(n+1)m}{n}$. Cf. (Hazewinkel and Kalman 1975a) for details. The underlying bundle E^u of Σ^u is the restriction to $\mathcal{M}_{m,n}(\mathbf{C})$ of the canonical bundle over the Grassmann variety. The n-th exterior product of this bundle is the restriction of the canonical line bundle ξ_1 over $\mathbf{P}^N(\mathbf{C})$ which is very ample. Now the map ϕ defined above is defined by polynomials and defines an algebraic geometric embedding $\mathbf{P}^1(\mathbf{C}) \xrightarrow{\phi} \mathcal{M}_{m,n}(\mathbf{C}) \xrightarrow{i}$ Grassmann $\xrightarrow{j} \mathbf{P}^N(\mathbf{C})$. It follows that $(j \circ i \circ \phi)^! \xi_1$ is very ample and its real restriction to $\mathbf{P}^1(\mathbf{C})$ is then also nontrivial. I.e., the n-th exterior product of $\phi^! E^u$ is nontrivial which proves that E^u is nontrivial.

Alternatively one simply calculates the bundle $\overset{n}{\Lambda}\phi^! E^u$ explicitly. This line bundle over $\mathbf{P}^1(\mathbf{R})$ is trivial over the pieces $\{(t:s) \mid s \neq 0\} \subset \mathbf{P}^1(\mathbf{R})$ and $\{(t, s) \ t \neq 0\} \subset \mathbf{P}^1(\mathbf{R})$ by (2.7.12) And if $n \geqslant 2$ these trivial pieces are identified on the intersection $\{(t, s) \mid t \neq 0, s \neq 0\}$ by means of multiplication with the number

$$\det \left(\begin{array}{cc:c} t^{-1}s & 0 & \\ & & 0 \\ 0 & 1 & \\ \hdashline 0 & & I_{n-2} \end{array} \right) = t^{-1}s$$

Similarly if $n = 1$ these pieces are also identified by multiplication with the number $t^{-1}s$.

This defines a nontrivial bundle over $\mathbf{P}^n(\mathbf{R})$, which proves that the bundle E^u was also nontrivial.

3. The Gramm-Schmidt orthonormalization process and canonical forms.

In this section we discuss the equivalence given by the Gramm-Schmidt orthonormalization process between the existence of $GL_n(\mathbf{R})$ canonical forms for all pairs and triples of matrices and the existence of $O_n(\mathbf{R})$ canonical forms for orthonormal pairs and triples of matrices.

3.1. The Space $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$, $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}$ and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}^{ortho}$.

We define $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$ as the space of all pairs of matrices (F, G) such that the rows of $R(F, G)$ are a set of orthonormal vectors (in $\mathbf{R}^{(n+1)m}$).

Note that $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho} \subset \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$.

We define $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})$ as the space of all triples of real matrices F, G, H of sizes $n\times n$, $n\times m$, $p\times n$, and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}$ is the subspace of all completely observable and completely reachable triples. I.e. $(F, G, H) \in \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}$ iff the matrices $R(F, G) = (G\ FG \dots F^nG)$ and $Q(F, H) = (H^TF^TH^T \ \dots\ (F^T)^nH^T)$ are both of rank n. Here H^T, F^T are the transposes of H, F. Cf. (Kalman 1969) for more details about these notions. Finally we define $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{crco}^{Ortho}$ as the subspace of $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}$ consisting of the triples of matrices in $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}$ such that moreover the rows of $R(F, G)$ are orthnormal.

3.2. LEMMA. *Let A, B be two $n\times r$ matrices of rank n, where $r \geqslant n$. Then we have*

(*i*) *If the rows of A are orthonormal and $U \in O_n(\mathbf{R})$ is an orthogonal $n\times n$ matrix, then the rows of UA are orthonormal,*

(*ii*) *If the rows of A and the rows of B are both orthonormal and if the rows of A and the rows of B span the same subspace of $\mathbf{R}^r$, then there is an orthonormal $n\times n$ matrix $U \in O_n(\mathbf{R})$ such that* $B = UA$.

Proof. Easy.

3.3. Canonical Forms. The group $GL_n(\mathbf{R})$ acts on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\,co}$ respectively as follows :

$$(F, G)^S = (SFS^{-1}, SG), (F, G, H)^S = (SFS^{-1}, SG, HS^{-1}),\ S \in GL_n(\mathbf{R}) \tag{3.3.1}$$

Two pairs (resp. triples) of matrices (F, G) and (F', G') (resp. (F, G, H) and (F', G', H')) are equivalent under $GL_n(\mathbf{R})$ if there is an $S \in GL_n(\mathbf{R})$ such that $(F, G)^S = (F', G')$ (resp. $(F, G, H)^S = (F', G', H')$). We now define a canonical form for the action of $GL_n(\mathbf{R})$ on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ as a continuous map

$$\gamma : \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{F}\mathcal{G}(\mathbf{R}) \tag{3.3.2}$$

such that for every two pairs (F, G), $(F', G') \in \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ we have

$$(F, G) \text{ is equivalent under } GL_n(\mathbf{R}) \text{ to } \gamma(F, G) \tag{3.3.3}$$

and

$$(F, G), (F', G') \text{ are equivalent under } GL_n(\mathbf{R}) \text{ iff } \gamma(F, G) = \gamma(F', G'). \tag{3.3.4}$$

A canonical form on $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\, co}$ under $GL_n(\mathbf{R})$ is defined similarly. The group $O_n(\mathbf{R})$ of real orthogonal $n \times n$ matrices acts on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$ and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\, co}^{ortho}$ as follows

$$(F, G)^U = (UFU^{-1}, UG), (F, G, H)^U = (UFU^{-1}, UG, HU^{-1}). \tag{3.3.5}$$

This follows from lemma 3.2(i). We now define a canonical form on $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}^{ortho}$ under $O_n(\mathbf{R})$ as a continuous map

$$\gamma : \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\, co}^{ortho} \to \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R}) \tag{3.3.6}$$

such that for every two triples (F, G, H), (F', G', H') we have

$$\gamma(F, G, H) \text{ is equivalent under } O_n(\mathbf{R}) \text{ to } (F, G, H) \tag{3.3.7}$$

and

$$(F, G, H) \text{ and } (F', G', H') \text{ are equivalent under } O_n(\mathbf{R}) \text{ iff } \gamma(F, G, H) = \gamma(F', G', H'). \tag{3.3.8}$$

A canonical form on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$ under $O_n(\mathbf{R})$ is defined similarly.

3.4. Gramm-Schmidt Orthonormalization. Let $(F, G) \in \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$. The matrix $R(F, G)$ is then of rank n. Applying the Gramm-Schmidt Orthonormalization to the rows of $R(F, G)$ we find an $n \times (n+1)m$ matrix R' with orthonormal rows, whose rows span the same subspace of $\mathbf{R}^{(n+1)m}$ as the rows of $R(F, G)$. It follows that $R' = SR(F, G)$ for a certain unique $S \in GL_n(\mathbf{R})$. It follows that $R' = R(SFS^{-1}, SG)$. Orthonormalization is also continuous. It follows that orthonormalization defines continuous (well defined) maps

$$\mu : \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho} \tag{3.4.1}$$

$$\nu : \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,\, co} \to \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}^{ortho} \tag{3.4.2}$$

Note that $\mu(F, G)$ and (F, G) and $\nu(F, G, H)$ and (F, G, H), are equivalent under $GL_n(\mathbf{R})$. The maps μ, ν take $GL_n(\mathbf{R})$ equivalent elements into $O_n(\mathbf{R})$ equivalent elements. More precisely we have

3.5. Lemma. *Two pairs (F, G), (F', G') in $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ (resp. two triples (F, G, H), (F', G', H')) in $\mathcal{F}\mathcal{G}\mathcal{H}((\mathbf{R})_{cr,\, co})$ are equivalent under $GL_n(\mathbf{R})$ iff the pairs $\mu(F, G), \mu(F', G')$ (resp. the triples $\nu(F, G, H)$, $\{(F', G', H'))$ are equivalent under $O_n(\mathbf{R})$.*

Proof. This follows from lemma 3.2.

3.6. Proposition. (*i*) *There exists a canonical form on* $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ *under* $GL_n(\mathbf{R})$ *iff there exists a canonical form on* $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$ *under* $O_n(\mathbf{R})$,

(*ii*) *There exists a canonical form on* $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$ *under* $GL_n(\mathbf{R})$ *iff there exists a canonical form on* $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}^{ortho}$ *under* $O_n(\mathbf{R})$.

Proof. Let $\gamma : \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \to \mathcal{F}\mathcal{G}(\mathbf{R})$ be a $GL_n(\mathbf{R})$ canonical form. Then

$$\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho} \xrightarrow{i} \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \xrightarrow{\gamma} \mathcal{F}\mathcal{G}(\mathbf{R})_{cr} \xrightarrow{\mu} \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}.$$

where i is the inclusion, is an $O_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$. (Note that two elements of $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho}$ are $O_n(\mathbf{R})$ equivalent iff they are $GL_n(\mathbf{R})$ equivalent; this follows from lemma 3.2). Inversely if $\gamma : \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho} \to \mathcal{F}\mathcal{G}(\mathbf{R})$ is an $O_n(\mathbf{R})$ canonical form then $\gamma \circ \mu$ is a $GL_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$. Part (ii) of the lemma is proved in the same way.

4. On the nonexistence of canonical forms. We have now enough material to prove the nonexistence of $GL_n(\mathbf{R})$ canonical forms on $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}$ for those dimensions (m, n, p) for which $p \geqslant 2n$, $m \geqslant 2$. The first step is the following theorem.

4.1. THEOREM. *There exists a* $GL_n(\mathbf{R})$ *canonical form on* $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ *iff the underlying bundle* E^u *of the universal family* Σ^u *is trivial.*

Proof. This is proved exactly as the algebraic geometric case in (Hazewinkel and Kalman 1975b), 6.1.

4.2. COROLLARY. There does not exist a $GL_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ if $m \geqslant 2$, this follows from 4.1 together with 2.8.

4.3. An $O_n(\mathbf{R})$—invariant embedding

For each $(F, G) \in \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}$ let $\bar{R}(F, G)$ be the matrix

$$\bar{R}(F, G) = G\ FG \ldots F^{-1}G)$$

For all m, n we can now define an $O_n(\mathbf{R})$ invariant embedding

$$\rho : \mathcal{F}\mathcal{G}(\mathbf{R})_{cr}^{ortho} \to \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr,co}^{ortho} \tag{4.3.1}$$

as follows

$$(F, G) \to (F, G, \bar{R}(F, G)^T). \tag{4.3.2}$$

This is $O_n(\mathbf{R})$ invariant because $U^T = U^{-1}$ for $U \in O_n(\mathbf{R})$ and $\bar{R}((F, G)^U) = U\bar{R}(F, G)$. The triple $(F, G, \bar{R}(F, G)^T)$ is completely observable because $\bar{R}(F, G)^T$ has rank n.

4.4. THEOREM. *There does not exist a continuous canonical form under* $GL_n(\mathbf{R})$ *for completely reachable and completely observable linear dynamical systems of dimension* n *with* m *impacts and* p *outputs in the cases*

(*i*) $m \geqslant 2, p \geqslant 2n$

(*ii*) $p \geqslant 2, m \geqslant 2n$

(*iii*) $p, m > n$.

Proof. (i) We have $O_n(\mathbf{R})$ invariant embeddings

$$\mathcal{F}\mathcal{G}_{2,n}(\mathbf{R})_{cr}^{ortho} \to \mathcal{F}\mathcal{G}\mathcal{H}_{2,n,2n}(\mathbf{R})_{cr,co}^{ortho} \to \mathcal{F}\mathcal{G}\mathcal{H}_{m,n,p}(\mathbf{R})_{cr,co}^{ortho} \quad (4.4.1)$$

where the first embedding is the one defined in 4.3 above and the second one consists of adding some zero columns to G (if $m > 2$) and some zero rows to H (if $p > 2n$). Now suppose there existed a $GL_n(\mathbf{R})$ canonical form for $\mathcal{F}\mathcal{G}\mathcal{H}_{m,n,p}(\mathbf{R})_{cr,co}$ then there would be an $O_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}\mathcal{H}_{m,n,p}(\mathbf{R})_{cr,co}^{ortho}$ by 3.6 and by the $O_n(\mathbf{R})$ invariant inclusions (4.4.1) above an $O_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}_{2,n}(\mathbf{R})_{cr}^{ortho}$ which in turn would imply the existence of an $GL_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}_{2,n}(\mathbf{R})_{cr}$ (again by 3.6), which contradicts 4.2.

Part (ii) of the theorem is proved by dualizing this whole paper. I.e., instead of completely reachable pairs (F, G) one studies completely observable pairs (F, H) etc. etc.

Part (iii) of the theorem uses : 1°) the nonexisoence of a $GL_n(\mathbf{R})$ canonical form on $\mathcal{A}_{n,m}$, the space of all $n\times m$ matrices of rank n under the action $A^S = SA$, if $m > n$, and 2°) the $O_n(\mathbf{R})$-invariant embedding

$$\mathcal{A}_{n,m}^{ortho} \to \mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr.co}^{ortho}$$

$$A \to (0, A, A^T).$$

4.5. As was already stated in the introduction theorem 4.4 holds in greater generality : there exists a continuous $GL_n(\mathbf{R})$ canonical form for completely observable and completely reachable linear dynamical systems of dimension n with in inputs and p outputs if and only if $m = 1$ or $p = 1$. Cf. Hazewinkel.

Of course the nonexistence of a $GL_n(\mathbf{R})$ canonical form on $\mathcal{F}\mathcal{G}\mathcal{H}_{m,n,p}(\mathbf{R})_{cr,co}$ implies a fortiori the nonexistence of such a form on the larger spaces $\mathcal{F}\mathcal{G}\mathcal{H}_{n,m,p}(\mathbf{R})$, $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{cr}$ and $\mathcal{F}\mathcal{G}\mathcal{H}(\mathbf{R})_{co}$.

References

Grothendieck, A. and Dieudonné, J. (1961) : *Elements de la Géometrie Algébrique Ch. II Publ.. Math. J.H.E.S.*, 8.

Hazewinkel, M. and Kalman, R. E. (1975a) : *Moduli and Canonical Forms for Linear Dynamical Systems*. Report 7504, Econometric Inst., Erasmus Univ. Rotterdam.

Hazewinkel, M. and Kalman, R. E. (1975b) : *On Invariants, Canonical Forms and Moduli for Linear, Constant, Finite Dimensional Dynamical Systems*. Proc. CNR-CISM Conference on Algebraic System Theory, Udine. (To appear in Lecture Notes in Math., Springer).

Hazewinkel M. (In preparation).

Husemoller, D. (1966) : *Theory of Fibre Bundles*, McGraw-Hill.

Kalman, R. E. (1969) : *Controllability and Observability*. In : CIME Symposium 1968 on Controllability and Observability, Edizione Cremonese, Roma.

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