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ON NORM MAPS FOR ONE DIMENSIONAL FORMAL GROUPS III

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1. Introduction

The main purpose of the present note is to give a more elementary and conceptual and less computational proof of the main theorem of [4]. At the same time we generalize the theorem.

Let K be a local field, L/K a finite galois extension. Let A be the ring of integers of K and $F(X, Y)$ a (commutative one dimensional) formal group (law) over A, i.e. $F(X, Y)$ is a formal power series in two variables over A of the form $F(X, Y) = X + Y + \sum_{i,j\geq 1} a_{ij}X^iY^j$ such that $a_{ij} = a_{ji}$ and $F(F(X, Y), Z) = F(X, F(Y, Z))$. Let $\mathfrak{m}_L$ be the maximal ideal of $A(L)$, the ring of integers of L. The group recipe $F(X, Y)$ can be used to define a new abelian group structure on the set $\mathfrak{m}_L$, viz. $x +_F y = F(x, y)$ where $x, y \in \mathfrak{m}_L$. This group is denoted $F(L)$. There is a natural norm map

$$(1.1.)\quad F - \mathrm{Norm}_{L/K} : F(L) \to F(K), \qquad x \mapsto \sigma_1 x +_F \sigma_2 x +_F + \cdots +_F \sigma_n x$$

where $\{\sigma_1, \cdots, \sigma_n\} = \mathrm{Gal}\,(L/K)$. The general problem is to describe the image (or the cokernel) of the maps $F - \mathrm{Norm}_{L/K}$. For example if F is the multiplicative group $\hat{\mathbf{G}}_m(X, Y) = X + Y + XY$, then $F - \mathrm{Norm}$ becomes the ordinary norm map

$$(1.2)\qquad N_{L/K} : U^1(L) \to U^1(K)$$

where $U^1(L) = \{x \in U(L) = A(L)^* \mid x \equiv 1 \bmod \mathfrak{m}_L\}$. The study of Coker $N_{L/K}$ is what a not inconsiderable part of local class field theory is about.

Let K_∞/K be an infinite galois extension of Galois group isomorphic to $\mathbf{Z}_p$, the p-adic integers. Such an extension is called a $\mathbf{Z}_p$-extension or a Γ-extension. Let K_n be the invariant field of the closed subgroup $p^n\mathbf{Z}_p$, $n = 0, 1, 2, \cdots$, where we write $K_0 = K$.

The abelian group $F(K)$ carries a natural filtration $F(K) = F^1(K) \supset F^2(K) \supset \cdots \supset F^n(K) \supset \cdots$ where $F^n(K) = \{x \in F(K) \mid x \in \mathfrak{m}_K{}^n\}$.

We write $F - \mathrm{Norm}_{n/0}$ for $F - \mathrm{Norm}_{K_n/K}$. The main theorem of this paper is now.

1.3. THEOREM. *Let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension of an absolutely unramified mixed characteristic local field K with perfect residue field k of characteristic $p > 2$. Let $F(X, Y)$ be a formal group over A of height $h \geq 2$. Then we*

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have

$$\text{(1.3.1)} \qquad \operatorname{Im}(F - \operatorname{Norm}_{n/0}) = F^{\alpha_n}(K)$$

with α_n *given by* $\alpha_n = n - [h^{-1}(n - 1)]$, *where* $[r]$ *for* $r \in \mathbf{R}$ *denotes the integral part of* r, i.e. $[r]$ *is the smallest integer* $\leq r$.

For a definition of the height of a formal group (law) $F(X, Y)$ cf. [1], cf. also 2.1. below.

If $h = \infty$, the theorem holds with $\alpha_n = n$ which fits naturally. If $h = 1$ then the statement of the theorem holds if k is algebraically closed but is false if k is finite. Indeed, taking $F(X, Y) = \hat{\mathbf{G}}_m(X, Y) = X + Y + XY$ and taking K_∞/K to be the cyclotomic $\mathbf{Z}_p$-extension the statement of the theorem says that $N_{n/0} : U^1(K_n) \to U^1(K)$ is surjective which is false by local field theory which says that Coker $(N_{n/0})$ is isomorphic to Gal (K_n/K) in this case.

The main theorem of [4] dealt with the case: K local field of characteristic zero with finite residue field and K_∞/K the cyclotomic $\mathbf{Z}_p$-extension.

As isomorphic formal group laws have the same height the result above is really one about isomorphism classes of formal group laws i.e. about formal groups. For that matter the whole problem of studying F-$\operatorname{Norm}_{L/K}$ is a problem about formal groups rather than formal group laws as an isomorphism $\beta(X)$: $F(X, Y) \to G(X, Y)$ of formal group laws over A induces filtration preserving isomorphisms $F(L) \to G(L)$ compatible with norm maps.

Let $k = A/\mathfrak{m}_K$. By reducing the coefficients of the formal group law $F(X, Y)$ modulo $\mathfrak{m}_K$ we obtain a formal group law over k, denoted $F^*(X, Y)$. It is reasonable to expect (as suggested to me by B. Mazur) that if K is of mixed characteristic and absolutely unramified, i.e. $\mathfrak{m}_K = pA(K)$, then Coker $(F$-$\operatorname{Norm}_{L/K})$ depends (up to isomorphism) only on the isomorphism class of $F^*(X, Y)$ and L/K.

The result above gives some positive evidence for this. Indeed, provided one restricts attention to extensions L/K of the form K_n/K where K_n is a finite level of some $\mathbf{Z}_p$-extension, the result 1.3 above says that Coker $(F$-$\operatorname{Norm}_{L/K})$ depends only on the height of $F^*(X, Y)$ and L/K.

As to the motivation why one would want to study cokernels of norm maps for formal groups: basically the goal is to look for a class field type theory for other algebraic groups than just $\mathbf{G}_m$, the multiplicative group. In [7] §4, such a theory is developed for A an abelian variety with non degenerate reduction and invertible Hasse matrix, and the results obtained play an important role in the remainder of [7]. The development of the theory goes via $\hat{A}$ the formal group obtained by completing A along the identity and relies heavily (as does local class field theory) on the fact that $\hat{A}(L) \xrightarrow{\text{Norm}} \hat{A}(K)$ is surjective if L/K is a finite extension of a local field K with algebraically closed residue field. One consequence of theorem 1.3. is that this fails if $h(\hat{E}) \geq 2$, i.e. it fails for elliptic curves in the supersingular case (cf. also [7], §1, d1).

In local class field theory, in the theory developed in [7] and also in [10] the $\mathbf{Z}_p$-extensions play an especially distinguished role which may be seen as motivation for paying particular attention to $\mathbf{Z}_p$-extensions.

Of course, from the point of a class field theory associated to an algebraic group in general, a weak consequence of theorem 1.3 is an analogue of that well known theorem of class field theory which says that the subgroup of universal norms is trivial. We have: if height $(F(X, Y)) \geq 2$ then

$$\bigcap_{L/K} F\text{-Norm}_{L/K}(F(L)) = \{0\}.$$

All formal groups in this paper will be commutative and one dimensional. The notation introduced above will remain in force throughout this paper. In addition we use A_n for the ring of integers of K_n, v_n for the normalized exponential valuation of K_n, π_n for a uniformizing element of K_n, i.e. $v_n(\pi_n) = 1$ and $Tr_{n/0}$ is the trace map from K_n to $K = K_0$. We write π and v instead of π_0 and v_0. If K is absolutely unramified one can of course take $\pi = p$. (Cf., however, 4.6 below). Finally $\mathbf{N} = \{1, 2, 3, \cdots\}$ denotes the natural numbers, $\mathbf{Q}$ the rational numbers, $\mathbf{Q}_p$ the p-adic numbers, and $\mathbf{R}$ the real numbers.

2. Prerequisites from formal group theory

In this section we discuss the material form formal group theory needed for the proof of theorem 1.3.

2.1. *Generalities.* Let A be the ring of integers of an absolutely unramified mixed characteristic local field K with residue field k.

Let $F(X, Y)$ be a formal group law over A and let p be the characteristic of k. Inductively one defines for all $n \in \mathbf{N} = \{1, 2, 3, \cdots\}$, $[n]_F(X) = F(X, [n-1]_F(X))$, $[1]_F(X) = X$. Now consider $[p]_F(X)$. If $[p]_F(X) \equiv 0 \bmod \mathfrak{m}_K$ than $F(X, Y)$ is said to be of infinite height. If $[p]_F(X) \not\equiv 0 \bmod \mathfrak{m}_K$ then the first power of X whose coefficient is not $\equiv 0 \bmod \mathfrak{m}_K$ is necessarily of the form X^{p^h} for some $h \in \mathbf{N}$. Then $F(X, Y)$ is said to have height h.

Two formal groups $F(X, Y)$, $G(X, Y)$ over A are said to be strictly isomorphic if there exists a power series $\beta(X) = X + c_2X^2 + \cdots$ $c_i \in A$ such that $\beta(F(X, Y)) = G(\beta(X), \beta(Y))$. Note that the power series $\beta(X)$ induces isomorphisms $F(L) \to G(L)$ which preserve the filtration and which are compatible with the norm maps.

2.2. *Honda's method of constructing formal group laws over rings of integers of absolutely unramified local fields.* Now let K be absolutely unramified. Choose elements $t_1, t_2, \cdots \in A$ and let $f_t(X)$ be the power series

$$f_t(X) = \sum_{i=0}^{\infty} a_i X^{p^i}$$

(2.2.1.) $\quad a_0 = 1, \quad pa_n = a_{n-1}t_1^{\sigma^{n-1}} + \cdots + a_1 t_{n-1}^{\sigma} + t_n, \quad n \in \mathbf{N}$

where σ is the Frobenius endomorphism of K (characterized by $\sigma(a) \equiv a^p \bmod \mathfrak{m}_K$ for all $a \in A$).

Now let

(2.2.2.) $\quad F_t(X, Y) = f_t^{-1}(f_t(X) + f_t(Y))$

where $f_t^{-1}(X)$ is the inverse power series of $f_t(X)$ i.e. $f_t^{-1}(f_t(X)) = X$. Then according to [6], theorem 4, and proposition 3.5, one has

2.3. PROPOSITION. *Every formal group $F(X, Y)$ over A is strictly isomorphic to a formal group of the form $F_t(X, Y)$. If height $(F(X, Y)) = \infty$ then $t_i \in m_K = pA$ for all i and if height $(F(X, Y)) = h \in \mathbf{N}$ then $t_1, \cdots, t_{h-1} \in m_K = pA$ and t_h is a unit of A.*

We remark that the hypothesis "K is absolutely unramified" is necessary for this proposition.

2.4. Let A, K be as before and let

$$t_1 \cdots, t_{h-1} \in m_K, t_h \in U(K), t_{h+1}, t_{h+2}, \cdots \in A$$

and let $f_t(X)$ be as in (2.2.1.). We then have

LEMMA.
(i) $v(a_{rh}) = -r$ for all $r \in \mathbf{N} \cup \{0\}$
(ii) $v(a_n) \geq -r + 1$ for all $(r-1)h < n < rh$, $r, n \in \mathbf{N}$.

Proof. By induction. For $n = 0$ we have $a_0 = 1$ and $v(a_0) = 0$. Now let $0 < n < h$. We have

$$a_n = p^{-1}a_{n-1}t_1^{\sigma^{n-1}} + \cdots + p^{-1}a_1t_{n-1}^{\sigma} + p^{-1}t_n$$

and assuming with induction that $v(a_i) \geq 0$ for $0 \leq i < n < h$ we find $v(a_n) \geq$ (as $v(t_i) \geq 1$ for $i \leq h-1$. Now take $n = h$. Then

$$a_h = p^{-1}a_{h-1}t_1^{\sigma^{h-1}} + \cdots + p^{-1}a_1t_{h-1}^{\sigma} + p^{-1}t_h$$

All terms in this expression have valuation ≥ 0 except $p^{-1}t_h$ which has valuatio -1. Thus $v(a_h) = -1$. Now let $n = rh + s$, $1 \leq s \leq h-1$, $r \geq 1$. The

$$a_n = p^{-1}a_{n-1}t_1^{\sigma^{n-1}} + \cdots + p^{-1}a_{n-s}t_s^{\sigma^{n-s}}$$
$$+ p^{-1}a_{n-s-1}t_{s+1}^{\sigma^{n-s-1}} + \cdots + p^{-1}a_1t_{n-1}^{\sigma} + p^{-1}$$

Now by induction $v(a_{n-i}) \geq -r$ and $v(t_i) \geq 1$ for $i = 1, \cdots, s$ and $v(a_{n-j})$: $-r+1$ (and $v(t_j) \geq 0$) for $j = s+1, \cdots, n$. So that $v(a_n) \geq -r$.
Finally let $n = rh$, $r \geq 2$. Then

$$a_n = p^{-1}a_{n-1}t_1^{\sigma^{n-1}} + \cdots + p^{-1}a_{n-h+1}t_{h+1}^{\sigma^{n-h+1}}$$
$$+ p^{-1}a_{n-h}t_h^{\sigma^{n-h}} + p^{-1}a_{n-h-1}t_{h+1}^{\sigma^{n-h-1}} + \cdots + p^{-}$$

By induction hypothesis we have (using $v(t_i) \geq 1$ for $i = 1, \cdots, h-1, v(t_h) =$ $v(t_j) \geq 0$ for $j > h$) that $v(p^{-1}a_{n-i}t_i^{\sigma^{n-i}}) \geq -r + 1$ for $i = 1, \cdots, h -$ $v(p^{-1}a_{n-h}t_h^{\sigma^{n-h}}) = -r$, $v(p^{-1}a_{n-j}t_j^{\sigma^{n-j}}) \geq -r + 1$ for $j \geq h+1$. Hen $v(a_n) = -r$ if $n = rh$.

2.5. LEMMA. *Let A, K be as before $t_i \in pA = m_K$ for all i. Then $a_i \in$ for all $i \in \mathbf{N}$.*

Proof. This follows from 2.2.1. because $p^{-1}t_i \in A$ for all i in this case.

3. Prerequisites from local class field theory

In this section we discuss the information we need from local field theory and local class field theory.

3.1. *The integer* $m(L/K)$. Let L/K be a totally ramified galois extension of prime degree $p = \text{char}(k)$. Then there is a natural number $m(L/K)$ such that

$$Tr_{L/K}(\mathfrak{m}_L^r) = \mathfrak{m}_K^s \tag{3.1.1}$$

where s depends on r according to the formula.

$$s = \left[\frac{(m(L/K)+1)(p-1)+r}{p}\right] \tag{3.1.2}$$

The number $m(L/K)$ is uniquely determined by (3.1.1), (3.1.2). Cf. [8], Ch V, §3 or [2] section 2.6.

3.2. *Upper ramification groups.* Let L/K be a finite galois extension. Then one can define upper ramification subgroups $\text{Gal}(L/K)^i \subset \text{Gal}(L/K)$, $i = -1, 0, 1, 2, \cdots$. For a definition, cf [8], Ch. IV. These upper ramification subgroups behave nicely with respect to quotients. If M/K is a subgalois extension of L/K then $\text{Gal}(M/K)^i$ is equal to the image of $\text{Gal}(L/K)^i$ in $\text{Gal}(M/K)$ under the natural projection $\text{Gal}(L/K) \to \text{Gal}(M/K)$.

3.3. PROPOSITION. *Let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension of a mixed characteristic totally unramified local field K of residue characteristic p. Then the upper ramification groups are equal to*

$$\text{Gal}(K_\infty/K)^i = p^{i-1}\mathbf{Z}_p \subset \mathbf{Z}_p = \text{Gal}(K_\infty/K), \qquad i = 1, 2, \cdots \tag{3.3.1}$$

Proof. Let K_{ab} be the maximal abelian extension of K. Because K_∞/K is totally ramified and because $\text{Gal}(K_\infty/K)$ is a pro-p-group we have that $\text{Gal}(K_\infty/K) = \text{Gal}(K_\infty/K)^0 = \text{Gal}(K_\infty/K)^1$. We have therefore a natural epimorphism

$$\phi : \text{Gal}(K_{ab}/K)^1 \to \text{Gal}(K_\infty/K) = \mathbf{Z}_p \tag{3.3.2}$$

Now by local class field theory $\text{Gal}(K_{ab}/K)^1 = \mathsf{p}(U_K^1)$ where U_K^1 is the pro-algebraic group associated (via the Greenberg construction) to the group of 1-units of A and where $\mathsf{p}(U_K^1)$ is the maximal constant quotient of $\pi_1(U_K^1)$ the first homotopy group of U_K^1. Furthermore $\text{Gal}(K_{ab}/K)^n =$ image of $\mathsf{p}(U_K^n)$ in $\mathsf{p}(U_K^1)$ under the map induced by the inclusion $U_K^n \to U_K^1$. Cf. [2] Ch. II for all this. Cf. also [9] in the case that k is algebraically closed. Now, because K is absolutely unramified and $p > 2$ the map $p^i : U_K^1 \to U_K^1$ (= raising to the power p^i) induces an isomorphism $U_K^1 \xrightarrow{\sim} U_K^{i+1}$. Cf. [9] n^0 1.7, Cor 1 or [2] section 2.3, cf. also [3] lemma 5.7. It follows that the image of $p^i : \mathsf{p}(U_K^1) \to \mathsf{p}(U_K^1)$ is equal to the image of $\mathsf{p}(U_K^{i+1})$ in $\mathsf{p}(U_K^1)$. We therefore have

$$\begin{aligned}\text{Gal}(K_\infty/K)^i &= \phi(\text{Gal}(K_{ab}/K)^i) = \phi(\text{Im}(\mathsf{p}(U_K^i)) \to \mathsf{p}(U_K^1))\\ &= \phi(p^{i-1}\mathsf{p}(U_K^1)) = p^{i-1}\phi(\mathsf{p}(U_K^1)) = p^{i-1}\mathbf{Z}_p\end{aligned}$$

Remarks.

1. If k is finite one has $\mathfrak{p}(U_K{}^i) = U^i(K)$ canonically, cf. [3] no. 7.4 or [2] no. 9.2 which somewhat simplifies the proof in this case.
2. The proof above is the only place where the hypothesis $p > 2$ is used.

3.4. COROLLARY. *Let K_∞/K be as above in* 3.3. *Let* $m_n = m(K_n/K_{n-1})$, $n = 1, 2, \cdots$ *then we have that* $m_n = 1 + p + \cdots + p^{n-1}$.

Proof. This follows from the relation $\mathrm{Gal}\,(L/K)^{\psi(i)} = \mathrm{Gal}\,(L/K)_i$, where ψ is the Herbrand ψ-function, and the definition of the $\mathrm{Gal}\,(L/K)_i$. Cf. [8], Ch. IV, Ch. V.

Finally we need one not difficult result from local field theory, viz.

3.5. TRACE LEMMA ([5] *Proposition* 4.1). *Let L/K be a totally ramified galois extension of degree* $p = \mathrm{char}\,(k)$; *let*

$$m = m(L/K) \quad \textit{and} \quad r = [p^{-1}((m + 1)(p - 1) + 1)].$$

Let π_L be a uniformizing element of L and $\pi_K = (-1)^{p-1}N_{L/K}(\pi_L)$. *Then we have for all* $l \in \mathbf{N}$,

$$\mathrm{Tr}_{L/K}\,(\pi_L{}^{lp}) \equiv p\pi_K{}^l \bmod \pi_K{}^{2r+l-1}. \tag{3.5.1}$$

4. Proof of theorem 1.3

4.1. We have now all we need to prove theorem 1.3. Because of proposition 2.3 we can assume that $F(X, Y)$ is a formal group (law) of the type $F_t(X, Y) = f_t^{-1}(f_t(X) + f_t(Y))$ where $f_t(X)$ is as in 2.2.1. Now because K is absolutely unramified we have by lemmas 2.4 and 2.5 that $a_ix^{p^i} \in \pi A$ $(= pA)$ for all $x \in \pi A$ and that the series $\sum a_ix^{p^i}$ converges in πA.

(*NB* the series $\sum a_ix^{p^i}$ is also convergent for $x \in \pi_nA_n$ but the values of these series will in general not be in π_nA_n). It follows that

$$f_t : F_t(K) \to \pi A, \qquad x \mapsto f_t(x) \tag{4.1.1}$$

is an isomorphism of $F_t(K)$ with (the additive group) πA, which takes $F_t{}^n(K)$ isomorphically onto π^nA.

Because $F_t(X, Y) = f_t^{-1}(f_t(X) + f_t(Y))$ we have a commutative diagram

$$\begin{array}{ccc} F_t(K_n) & \xrightarrow{f_t} & K_n \\ \big\downarrow{\scriptstyle F_t\text{-Norm}_{n/0}} & & \big\downarrow{\scriptstyle \mathrm{Tr}_{n/0}} \\ F_t(K) & \underset{f_t}{\xrightarrow{\sim}} & \pi A \subset K \end{array} \tag{4.1.2}$$

4.2. *The functions* $\lambda_{n/0}(t)$. For each $n \in \mathbf{N}$ define the function $\lambda_{n/n-1}$ by

$$\lambda_{n/n-1}(t) = [p^{-1}((m_n + 1)(p - 1) + t)] \tag{4.2.1}$$

where $m_n = m(K_n/K_{n-1}) = 1 + p + \cdots + p^{n-1}$. Now define $\lambda_n/_0$ inductively by

(4.2.2) $$\lambda_{n/0}(t) = \lambda_{n-1/0}(\lambda_{n/n-1}(t))$$

It is then clear from 3.1 that

(4.2.3) $$\mathrm{Tr}_{n/0}(\pi_n{}^t A_n) = \pi^{\lambda_{n/0}(t)} A.$$

It is not difficult to calculate $\lambda_{n/0}(t)$. First observe that if $a_1, \cdots, a_n \in \mathbf{Z}$, $t \in \mathbf{Z}$ and the numbers $b_0, b_1, \cdots b_{n-1}$ are obtained by $b_{n-1} = [p^{-1}(a_n + t)]$, $b_{n-2} = [p^{-1}(a_{n-1} + b_{n-1})], \cdots, b_0 = [p^{-1}(a_1 + b_1)]$, then

$$b_0 = [p^{-n}t + p^{-n}a_n + \cdots + p^{-1}a_1]$$

Substituting $a_n = (p-1)(m_n + 1) = p^n - 1 + p - 1$ one finds that

$$\lambda_{n/0}(t) = [p^{-n}t + n + 1 - p^{-n} - p^{-n}(p-1)^{-1}(p^n - 1)]$$

so that

(4.2.4.) $$\begin{aligned} &\lambda_{n/0}(t) = n \quad \text{for} \quad 1 \le t \le (p-1)^{-1}(p^n - 1) \\ &\lambda_{n/0}(t) = 1 + l + n \quad \text{if} \quad \begin{cases} t \ge 1 + lp^n + (p-1)^{-1}(p^n - 1) \quad \text{and} \\ t \le (l+1)p^n + (p-1)^{-1}(p^n - 1) \end{cases} \end{aligned}$$

4.3. *Proof that* Im $(F\text{-Norm}_{n/0}) \subset F^{\alpha^n}(K)$ *holds in the case* $h = \infty$. Let $h = \infty$. From now on we write F for F_t. Because of 4.1 it suffices to prove that $\mathrm{Tr}_{n/0}(f_t(x)) \subset \pi^n A$ for all $x \in \pi_n A_n$. According to lemma 2.5 we have that $a_i \in A$ for $i = 1, 2, \cdots$ in the case $h = \infty$. Hence by (4.2.3) and (4.2.4) $\mathrm{Tr}_{n/0}(a_i x^{p^i}) \in \pi^n A$ for all $i = 1, 2, \cdots$ One easily checks (using (4.2.3) and (4.2.4) again) that the series $\sum_i \mathrm{Tr}_{n/0}(a_i x^{p^i})$ converges. It follows that $\mathrm{Tr}_{n/0}(f_t(x)) \in \pi^n A$ for all $x \in \pi_n A_n$.

4.4. *Proof that* Im $(F\text{-Norm}_{n/0}) \supset F^{\alpha^n}(K)$ *holds in the case* $h = \infty$. To prove this we have to show that $\pi^n A \subset \mathrm{Im}\,(\mathrm{Tr}_{n/0} \circ f_t)$. Let $t \in \mathbf{N}$ be such that $\lambda_{n/0}(t+1) > \lambda_{n/0}(t)$. It follows from (4.2.3) that then

(4.4.1) $$v(\mathrm{Tr}_{n/0}(\pi_n{}^t)) = \lambda_{n/0}(t)$$

For each $l = 0, 1, 2, \cdots$ let $s_l = (p-1)^{-1}(p^n - 1) + lp^n$. Then we have using (4.4.1), (4.2.3) and (4.2.4)

(4.4.2) $$\mathrm{Tr}_{n/0}(f_t(y\pi_n{}^{s_l})) \equiv \mathrm{Tr}_{n/0}(y\pi_n{}^{s_l}) = y\,\mathrm{Tr}_{n/0}(\pi_n{}^{s_l}) \bmod \pi^{n+l+1}$$

Using this and (4.4.1) we see that the maps

$$\mathrm{Tr}_{n/0} \circ f_t : F^{s_l}(K_n) \to \pi^{n+l} A / \pi^{n+l+1} A$$

are surjective for all $l = 0, 1, 2, \cdots$ Because the groups $F(K_n)$ and $\pi^n A$ are complete and Hausdorff it follows that

$$\mathrm{Tr}_{n/0} \circ f_t : F(K_n) \to \pi^n A$$

is surjective. Cf. [4] lemma 3.2 or [8] Ch. V. §1, lemma 2.

4.5. *Proof that* Im $(F\text{-Norm}_{n/0}) \subset F^{\alpha_n}(K)$ *holds in case* $h < \infty$. Now let $2 \le h < \infty$. Because of 4.1 it suffices to prove that

$$\mathrm{Tr}_{n/0}(f_t(x)) \subset \pi^{\alpha_n}A, \qquad x \in \pi_n A_n \tag{4.5.1}$$

We have that $\alpha_n = n - [h^{-1}(n-1)]$, and that

$$v(\mathrm{Tr}|_{n/0}(a_i x^{p^i})) \ge \lambda_{n/0}(p^i v_n(x)) + v(a_i) \tag{4.5.2}$$

Write $n = lh + r$, with $1 \le r \le h$. Then $\alpha_n = n - l$. If $i < n$ then $v(a_i) \ge -l$ by lemma 2.4 and $\lambda_{n/0}(p^i v_n(x)) \ge n$. Hence

$$v(\mathrm{Tr}_{n/0}(a_i x^{p^i})) \ge \alpha_n \quad \text{for} \quad i < n \tag{4.5.3}$$

If $i = n$, then $\lambda_{n/0}(p^i v_n(x)) \ge n + 1$ by (4.2.4) and $v(a_i) \ge -l - 1$ so that also

$$v(\mathrm{Tr}_{n/0}(a_n x^{p^n}) \ge \alpha_n \tag{4.5.4}$$

Finally if $i > n$, then $p^i \ge (p-1)^{-1}(p^n - 1) + (p^{i-n} - 1)p^n + 1$ and $v(a_i) \ge -l - 1 - (i - n)$, hence

$$v(\mathrm{Tr}_{n/0}(a_i x^{p^i})) \ge \alpha_n \quad \text{for} \quad i > n \tag{4.5.5}$$

because $\lambda_{n/0}(p^i v_n(x)) \ge \lambda_{n/0}(p^i) \ge n + p^{i-n}$ and $p^{i-n} \ge (i-n) + 1$ if $i > n$ The series $\sum \mathrm{Tr}_{n/0}(a_i x^{p^i})$ converges. The inclusion (4.5.1) now follows from (4.5.3)–(4.5.5).

4.6. *Proof that* $F^{\alpha_n}(K) \subset$ Im $(F\text{-Norm}_{n/0})$ *holds in case* $h < \infty$. Choose uniformizing elements $\pi_l \in A_l$ such that $N_{l/l-1}(\pi_l) = (-1)^{p-1}\pi_{l-1}$ for $l = 1, \cdots, n$.

Let $n = lh + r$, $1 \le r \le h$. For each s such that $n - l \le s < n$ let

$$t_s = (p-1)^{-1}(p^{n-(n-s)h} - 1) \tag{4.6.1}$$

We try to calculate $\mathrm{Tr}_{n/0}(a_i(\pi_n^{t_s})^{p^i})$. To this end we first prove that for $a \le r < n$.

$$\mathrm{Tr}_{n/n-a}(\pi_n^{p^r l}) \equiv \pi_{n-a}^{p^{r-a}l} p^a \bmod p^{a+1}\pi_{n-a}^{p^{r-a}l-1} \tag{4.6.2}$$

This is done by induction. The case $a = 1$ is the trace lemma 3.5 above. Assuming the result for all $b < a$, we have

$$\begin{aligned} &\mathrm{Tr}_{n/n-a+1}(\pi_n^{p^r l}) \equiv \pi_{n-a+1}^{p^{r-a+1}l} p^{a-1} \bmod p^a \pi_{n-a+1}^{p^{r-a+1}l-1} \\ &\mathrm{Tr}_{n-a+1/n-a}(\pi_{n-a+1}^{p^{r-a+1}l} p^{a-1}) \equiv \pi_{n-a}^{p^{r-a}l} p^a \bmod p^{a+1}\pi_{n-a}^{p^{r-a}l-1} \end{aligned} \tag{4.6.3}$$

It therefore suffices to show that

$$\lambda_{n-a+1/n-a}(ap^{(n-a+1)} + p^{r-a+1}l - 1) \ge (a+1)p^{n-a} + p^{r-a}l - 1$$

which is easily checked. Cf. (4.2.1). Now write $j_s = (n-s)h$. Then by (4.6.2)

$$\mathrm{Tr}_{n/n-j_s}((\pi_n^{t_s})^{p^{j_s}}) \equiv \pi_{n-j_s}^{t_s} p^{j_s} \bmod p^{j_s+1}\pi_{n-j_s}^{t_s-1}$$

But

$$\lambda_{n-j_s/0}(t_s + 1) > \lambda_{n-j_s/0}(t_s) \quad \text{and} \quad (p^{n-j_s})(j_s + 1) + t_s - 1 > p^{n-j_s}j_s + t_s$$

It follows that

(4.6.4) $$v(\mathrm{Tr}_{n/0}(\pi_n^{t_s p^{j_s}})) = j_s + \lambda_{n-j_s/0}(t_s) = n$$

and as

$$v(a_{j_s}) = v(a_{(n-s)h}) = -(n-s)$$

by lemma 2.4, we have that

(4.6.5) $$v(\mathrm{Tr}_{n/0}(a_{j_s}\pi_n^{t_s p^{j_s}})) = s \qquad n - l \leq s < n$$

If $i < j_s$, then $\lambda_{n/0}(\pi_n^{t_s p^i}) \geq n$ and $v(a_i) > -(n-s)$, hence

(4.6.6) $$v(\mathrm{Tr}_{n/0}(a_i\pi_n^{t_s p^i})) > s \qquad n - l \leq s < n, \qquad i < j_s$$

If $i = j_s + 1$, then because $h \geq 2$ we have that

(4.6.7) $$v(a_i) \geq s - n \qquad i = j_s + 1$$

and because $p^{j_s+1}t_s \geq (p-1)^{-1}(p^n - 1) + 1$ we have that

(4.6.8) $$v(\mathrm{Tr}_{n/0}(a_{j_s+1}\pi_n^{t_s p^{j_s+1}})) > s$$

Finally if $i \geq j_s + 2$, then because $h \geq 2$ we have that

(4.6.9) $$v(a_i) \geq s - n - \tfrac{1}{2}(i - j_s)$$

and

(4.6.10) $$p^i t_s \geq (p-1)^{-1}(p^n - 1) + \tfrac{1}{2}(i - j_s)p^n + 1$$

(To see that (4.6.10) holds, use $j_s < n$). It follows that

(4.6.11) $$v(\mathrm{Tr}_{n/0}(a_i\pi_n^{t_s p^i})) > s \quad \text{for} \quad i \geq j_s + 2$$

The series $\sum \mathrm{Tr}_{n/0}(a_i\pi_n^{t_s p^i}y^{p^i})$ converges for all $y \in A$. It then follows from (4.6.5), (4.6.6), (4.6.8) and (4.6.11) that

(4.6.12) $$\mathrm{Tr}_{n/0}(f_t(y\pi_n^{t_s})) \equiv y^{p^{j_s}}b_s \bmod \pi^{s+1}$$

where $b_s \in A$ is an element of valuation s. Because k is perfect it follows that

(4.6.13) $$\mathrm{Tr}_{n/0} \circ f_t : F^{t_s}(K_n) \to \pi^s A/\pi^{s+1}A$$

is surjective for $\alpha_n = n - l \leq s < n$. Now suppose that $s \geq n$. For these s let

(4.6.14) $$t_s = (p-1)^{-1}(p^n - 1) + (s - n)p^n$$

Then $\lambda_{n/0}(t_s) = s$ and $\lambda_{n/0}(t_s + 1) = s + 1$. It follows that

(4.6.15) $$v(\mathrm{Tr}_{n/0}(a_0\pi_n^{t_s})) = s \qquad s \geq n$$

Because $h \geq 2$, $v(a_1) \geq 0$, hence

(4.6.16) $$v(\mathrm{Tr}_{n/0}(a_1\pi_n^{pt_s})) > s \qquad s \geq n$$

because $pt_s \geq t_s + 1$.

Finally for $i \geq 2$, we have $v(a_i) \geq -\frac{1}{2}i$ if i is even and $v(a_i) \geq -\frac{1}{2}i + \frac{1}{2}$ if i is odd and $p^i t_s \geq t_s + (\frac{1}{2}i)p^n + 1$. Hence

(4.6.17) $$v(\mathrm{Tr}_{n/0}\,(a_i \pi_n^{p^i t_s})) > s \qquad s \geq n, \qquad i \geq 2$$

It follows from (4.6.15)–(4.6.17) that

(4.6.18) $$\mathrm{Tr}_{n/0}(f_t(y\pi_n^{t_s})) \equiv b_s y \bmod \pi^{s+1}, \qquad s \geq n$$

where b_s is an element of A of valuation s. Hence

(4.6.19) $$\mathrm{Tr}_{n/0} \circ f_t : F^{t_s}(K_n) \to \pi^s A / \pi^{s+1} A$$

is surjective for $s \geq n$. Combining (4.6.13), (4.5) and (4.6.19) and using that $F(K_n)$ and $\pi^{\alpha_n} A$ are complete Hausdorff filtered (topological) groups we see that the image of $\mathrm{Tr}_{n/0} \circ f_t$ is equal to $\pi^{\alpha_n} A$. According to (4.1) this implies that the image of F-$\mathrm{Norm}_{n/0}$ is equal to $F^{\alpha_n}(K)$ which is what we set out to prove.

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