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# ON FORMAL GROUPS, NORM MAPS AND $\mathbb{Z}_p$-EXTENSIONS

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## 1. INTRODUCTION.

The main purpose of the present note is to give a more elementary and conceptual and less computational proof of the main theorem of [4]. At the same time we generalize the theorem.

Let K be a local field, L/K a finite galois extension. Let A be the ring of integers of A and F(X,Y) a (commutative one dimensional formal group (law) over A, i.e. F(X,Y) is a formal power series in two variables over A of the form $F(X,Y) = X + Y + \sum_{i,j\geq 1} a_{ij}X^iY^j$ such that $a_{ij} = a_{ji}$ and $F(F(X,Y),Z) = F(X,F(Y,Z))$. Let $\mathfrak{m}_L$ be the maximal ideal of A(L), the ring of integers of L. The group recipe F(X,Y) can be used to define a new abelian group structure on the set $\mathfrak{m}_L$, viz. $x +_F y = F(x,y)$ where $x, y \in \mathfrak{m}_L$. This group is denoted F(L). There is a natural norm map

$$\text{(1.1.)} \qquad \text{F-Norm}_{L/K}\colon F(L) \to F(K),\ x \mapsto \sigma_1 x +_F \sigma_2 x +_F +\dots +_F \sigma_n x$$

where $\{\sigma_1,\dots,\sigma_n\} = \mathrm{Gal}(L/K)$. The general problem is to describe the image (or the cokernel) of the maps $\text{F-Norm}_{L/K}$. For example if F is the multiplicative group $G_m(X,Y) = X + Y + XY$, then F-Norm becomes the ordinary norm map

$$\text{(1.2)} \qquad N_{L/K} : U^1(L) \to U^1(K)$$

where $U^1(L) = \{x \in U(L) = A(L)^* \mid x \equiv 1 \bmod \mathfrak{m}_L\}$. The study of Coker $N_{L/K}$ is what a not inconsiderable part of local class field theory is about.

Let $k = A/\mathfrak{m}_K$. By reducing the coefficients of the formal group F(X,Y) mod $\mathfrak{m}_K$ we obtain a formal group over k, denoted $F^*(X,Y)$. A general conjecture now states that if K is of mixed characteristic and absolutely unramified i.e. $\mathfrak{m}_K = pA(K)$, then $\mathrm{Coker}(\text{F-Norm}_{L/K})$ depends only on $F^*(X,Y)$ and L/K.

Let $K_\infty/K$ be an infinite galois extension of Galois group isomorphic to $\mathbb{Z}_p$, the p-adic integers. Such an extension is called a $\mathbb{Z}_p$-extension

or a $\Gamma$-extension. Let $K_n$ be the invariant field of the closed subgroup $p^n\mathbb{Z}_p$.
The abelian group $F(K)$ carries a natural filtration $F(K) = F^1(K) \supset F^2(K) \supset \ldots \supset F^n(K) \supset \ldots$ where $F^n(K) = \{x \in F(K) \mid x \in \mathfrak{m}_K^n\}$.

We write F-Norm$_{n/o}$ for F-Norm$_{K_n/K}$. The main theorem of this paper is now.

1.3. Theorem.

Let $K_\infty/K$ be a totally ramified $\mathbb{Z}_p$-extension of an absolutely unramified mixed characteristic local field $K$ with perfect residue field $k$ of characteristic $p > 2$. Let $F(X))$ be a formal group over $A$ of height $h \geq 2$. Then we have

$$\text{(1.3.1)} \qquad \text{Im}(\text{F-Norm}_{n/o}) = F^{\alpha_n}(K)$$

with $\alpha_n$ given by $\alpha_n = n-[h^{-1}(n-1)]$, where $[r]$ for $r \in \mathbb{R}$ denotes the entier of $r$.
If $h = \infty$ then the theorem holds with $\alpha_n = n$, which fits naturally.
If $h = 1$ then the statement of the theorem holds if $k$ is algebraically closed and is false if $k$ is finite.

The main theorem of [4] dealt with the case: $K$ local field with finite residue field and $K_\infty/K$ the cyclotomic $\mathbb{Z}_p$-extension.
For a definition of the height of $F(X,Y)$ cf. [1], cf also 2.1 below.
For some motivation as to why one would want to study cokernels of norm maps of formal groups and, more especially, why one is interested in this problem in the case of $\mathbb{Z}_p$-extensions, cf [7], cf. also [10].
All formal groups in this paper will be commutative and one dimensional.
The notation introduced above will remain in force throughout this paper.
In addition we use $A_n$ for the ring of integers of $K_n$, $v_n$ for the normalized exponential valuation of $K_n$, $\pi_n$ for a uniformizing element of $K_n$, i.e. $v_n(\pi_n) = 1$ and $Tr_{n/o}$ is the trace map from $K_n$ to $K = K_o$.
Finally $\mathbb{N} = \{1,2,3,\ldots\}$ denotes the natural numbers, $\mathbb{Q}$ the rational numbers, $\mathbb{Q}_p$- the p-adic numbers, and $\mathbb{R}$ the real numbers.

## 2. PREREQUISITES FROM FORMAL GROUP THEORY.

In this section we discuss the material from formal group theory needed for the proof of theorem 1.3.

### 2.1. The Formal Group $F_V(X,Y)$.

Choose a prime number p. Let $\mathbb{Q}[V]$, $\mathbb{Z}[V]$ be short for $\mathbb{Q}[V_1,V_2,\ldots]$, $\mathbb{Z}[V_1,V_2,\ldots]$. We define polynomials $a_i(V) \in \mathbb{Q}[V]$ by the following recursion formula

$$(2.1.1) \qquad pa_n(V) = \sum_{k=1}^{n} a_{n-k}(V)V_k^{p^{n-k}} \quad , \quad a_o(V) = 1$$

Further define

$$(2.1.2) \qquad f_V(X) = \sum_{i=o}^{\infty} a_i(V)X^{p^i} \quad , \quad F_V(X,Y) = f_V^{-1}(f_V(X) + f_V(Y))$$

where $f_V^{-1}(X)$ is the inverse power series of $f_V(X)$, i.e. $f_V^{-1}(f_V(X)) = X$. Then we have according to [6] that $F_V(X,Y)$ is a power series in X,Y with its coefficients in $\mathbb{Z}[V]$. Therefore $F_V(X,Y)$ is a formal group over $\mathbb{Z}[V]$.

Now let A be the ring of integers of a local field K of residue characteristic p (same p as above). Two formal groups F(X,Y), G(X,Y) over A are said to be strictly isomorphic over A if there exists a power series $\alpha(X) = X + c_2X^2 + \ldots$, $c_i \in A$ such that $\alpha(F(X,Y)) = G(\alpha(X),\alpha(Y))$
Let $t_1$, $t_2$, ... be a sequence of elements of A. We denote with $F_t(X,Y)$ the formal group over A obtained by substituting $t_i$ for $V_i$, i = 1, 2, ... in $F_V(X,Y)$. Let h be the smallest number n in $\mathbb{N}$ such that $t_n \in U(A)$, the units of A, then h is the <u>height</u> of $F_t(X,Y)$, cf [6]. If $t_n \in \mathfrak{m}_K$ for all n then the height of $F_t(X,Y)$ is $\infty$.
According to [6] we have

### 2.2. Proposition.

Every formal group over the ring of integers A of a local field K of residue characteristic p is strictly isomorphic to a formal group of the form $F_t(X,Y)$.

2.3. Let K be an totally unramified mixed characteristic local field of residue characteristic p. Let $t_1, \ldots, t_{h-1} \in \mathfrak{m}_K$, $t_h \in U(K)$, $t_{h+1}, t_{h+2}, \ldots \in A$. Let $a_i \in K$ be the element obtained from $a_i(V)$ by substituting $t_n$ for $V_n$, $n = 1, 2, \ldots$

We then have

Lemma.

(i) $v(a_{rh}) = -r$ for all $r \in \mathbb{N} \cup \{0\}$.

(ii) $v(a_n) \geq -r+1$ if $(r-1)h \leq n < rh$, $r, n \in \mathbb{N}$.

Proof. By induction. For $n = 0$ we have $a_o = 1$, and therefore $v(a_o) = 0$. Now let $0 < n < h$. By induction we can assume $v(a_s) \geq 0$ for $s \leq n-1$. According to (2.1.1) we have

$$(2.3.1) \qquad a_n = a_{n-1}(p^{-1}t_1^{p^{n-1}}) + \ldots + a_1(p^{-1}t_{n-1}^{p}) + p^{-1}t_n$$

Because $n < h$ and because $K$ is absolutely unramified, i.e. $v(p) = 1$,

we have that $v(p^{-1}t_i^{p^{n-i}}) \geq 0$, as $v(t_i) \geq 1$ for $1 \leq i < h$. Hence $v(a_n) \geq 0$. Now let $n \geq h$. According to (2.1.1) we have

$$(2.3.2) \quad a_n = a_{n-1}(p^{-1}t_1^{p^{n-1}}) + \ldots + a_{n-h}(p^{-1}t_h^{p^{n-h}}) + \ldots + a_1(p^{-1}t_{n-1}^{p}) + p^{-1}t_n$$

Suppose that

$$(2.3.3) \qquad (r-1)h \leq n < rh, \quad r > 1$$

We have for $i = 1, \ldots, h-1$, that

$$(2.3.4) \qquad v(a_{n-i}p^{-1}t_i^{p^{n-i}}) \geq v(a_{n-i}) - 1 + p \geq -r + 2$$

by induction. For $i = h$ we have

$$(2.3.5) \qquad v(a_{n-h}p^{-1}t_h^{p^{n-h}}) = v(a_{n-h}) - 1$$

And for $i = h+1, \ldots, n$ we have

$$(2.3.5) \quad v(a_{n-i}p^{-1}t_i^{p^{n-i}}) \geq v(a_{n-i}) - 1 \geq \begin{cases} -r+1 & \text{if } (r-1)h < n < rh \\ -r+2 & \text{if } n = (r-1)h \end{cases}$$

The lemma now follows by induction from (2.3.2) - (2.3.5).

2.4. Lemma.

Let K be as above in 2.3. Let $t_1, t_2, \ldots \in \mathfrak{m}_K$ and let $a_i \in K$ be the element obtained from $a_i(V)$ by substituting $t_n$ for $V_n$, n = 1, 2, ... Then $a_i \in A$ for all i = 0, 1, 2, ...

Proof. This follows by induction from 2.3.1 because $p^{-1}t_i^{p^{n-i}} \in A$ for all i = 1, 2, ... in this case.

## 3. PREREQUISITES FROM LOCAL CLASS FIELD THEORY.

In this section we discuss the information we need from local field theory and local class field theory.

3.1. The Integer m(L/K).

Let L/K be a totally ramified galois extension of prime degree p = char(k). Then there is a natural number m(L/K) such that

$$\mathrm{Tr}_{L/K}(\mathfrak{m}_L^r) = \mathfrak{m}_K^s \qquad (3.1.1)$$

where s is equal to

$$s = \left[\frac{(m(L/K)+1)(p-1)+t}{p}\right] \qquad (3.1.2)$$

The number m(L/K) is uniquely determined by (3.1.1), (3.1.2). Cf. [8] , Ch V, §3 or [2] section 2.6.

3.2. Upper Ramification Groups.

Let L/K be a finite galois extension. Then one can define upper ramification subgroups $\mathrm{Gal}(L/K)^i \subset \mathrm{Gal}(L/K)$, $i = -1,0,1,2,\ldots$ . For a definition, cf [8], Ch. IV. These upper ramification subgroups behave nicely with respect to quotients. If M/K is a subgalois extension of L/K then $\mathrm{Gal}(M/K)^i$ is equal to the image of $\mathrm{Gal}(L/K)^i$ in Gal(M/K) under the natural projection $\mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K)$.

3.3. Proposition.

Let $K_\infty/K$ be a totally ramified $\mathbb{Z}_p$-extension of a mixed characteristic totally unramified local field K of residue characteristic p. Then the upper ramification groups are equal to

(3.3.1) $\quad \mathrm{Gal}(K_\infty/K)^i = p^{i-1}\mathbb{Z}_p \subset \mathbb{Z}_p = \mathrm{Gal}(K_\infty/K),\ i = 1, 2, \ldots$

Proof. Let $K_{ab}$ be the maximal abelian extension of K. Because $K_\infty/K$ is totally ramified and because $\mathrm{Gal}(K_\infty/K)$ is a pro-p-group we have that $\mathrm{Gal}(K_\infty/K) = \mathrm{Gal}(K_\infty/K)^o = \mathrm{Gal}(K_\infty/K)^1$. We have therefore a natural epimorphism

(3.3.2) $\quad \phi : \mathrm{Gal}(K_\infty/K)^1 \to \mathrm{Gal}(K_\infty/K) = \mathbb{Z}_p$

Now by local class field theory $\mathrm{Gal}(K_{ab}/K)^1 = \textcircled{p}(U^1_K)$ where $U^1_K$ is the proalgebraic group associated (via the Greenberg construction) to the group of 1-units of A and where $\textcircled{p}(U^1_K)$ is the maximal constant quotient of $\pi_1(U^1_K)$ the first homotopy group of $U^1_K$. Furthermore $\mathrm{Gal}(K_{ab}/K)^n$ = image of $\textcircled{p}(U^n_K)$ in $\textcircled{p}(U^1_K)$ under the map induced by the inclusion $U^n_K \to U^1_K$. Cf. [2] Ch. II for all this. Cf. also [9] in the case that K is algebraically closed. Now, because K is totally unramified the map $p^i$ = raising to the power $p^i$: $U^1_K \xrightarrow{p^i} U^1_K$ induces an isomorphism $U^1_K \simeq U^{i+1}_K$. Cf. [9] or [2], [3]. It follows that the image of $p^i$: $\textcircled{p}(U^1_K) \to \textcircled{p}(U^1_K)$ is equal to the image of $\textcircled{p}(U^{i+1}_K)$ in $\textcircled{p}(U^1_K)$. We therefore have

$$\mathrm{Gal}(K_\infty/K)^i = \phi(\mathrm{Gal}(K_{ab}/K)^i) = \phi(\mathrm{Im}(\textcircled{p}(U^i_K)) \to \textcircled{p}(U^1_K)) =$$

$$= \phi(p^{i-1}\textcircled{p}(U^1_K)) = p^{i-1}\phi(\textcircled{p}(U^1_K)) = p^{i-1}\mathbb{Z}_p$$

Remark. If k is finite one has $\textcircled{p}(U^i_K) = U^i(K)$ canonically, which somewhat simplifies the proof in this case.

3.4. Corollary.

Let $K_\infty/K$ be as above in 3.3. Let $m_n = m(K_n/K_{n-1})$, $n = 1, 2, \ldots$ then we have that $m_n = 1 + p + \ldots + p^{n-1}$.

Proof. This follows from the relation $\mathrm{Gal}(L/K)^{\psi(i)} = \mathrm{Gal}(L/K)_i$, where $\psi$ is the Herbrand $\psi$-function and the definition of the $\mathrm{Gal}(L/K)_i$. Cf. [8], Ch.IV, Ch. V.

Finally we need one not difficult result from local field theory, viz.

3.4. Trace Lemma ([5] Proposition 4.1).

Let $L/K$ be a totally ramified galois extension of degree $p = \mathrm{char}(k)$; let $m = m(L/K)$ and $r = [p^{-1}((m+1)(p-1)+1]$. Let $\pi_L$ be a uniformizing element of $L$ and $\pi_K = (-1)^{p-1}N_{L/K}(\pi_L)$. Then we have for all $\ell \in \mathbb{N}$.

$$(3.4.1) \qquad \mathrm{Tr}_{L/K}(\pi_L^{\ell p}) \equiv p\pi_K^{\ell} \mod \pi_K^{2r+\ell-1}$$

## 4. PROOF OF THEOREM 1.3.

We have now all we need to prove theorem 1.3. Because of proposition 2 we can assume that $F(X,Y)$ is a formal group of the type $F_t(X,Y)$ for certain $t_1, t_2, \ldots \in A$. Let $K_\infty/K$ be as in theorem 1.3.

4.1. Let $f_t(X)$ be the power series obtained from $f_V(X)$ by substituting $t_i$ for $V_i$, $i = 1, 2, \ldots$ I.e. $f_t(X) = \sum_{i=o}^{\infty} a_i X^{p^i}$. Now because $K$ is totally unramified we have (because of lemma 2.3, 2.4) that $a_i x^{p^i} \in \pi A$ for all $x \in \pi A$ and that the series $\Sigma a_i x^{p^i}$ converges in $\pi A$. (NB This is not true for $x \in \pi_n A_n$). It follows that

$$(4.1.1) \qquad f_t\colon F_t(K) \to \pi A, \; x \mapsto f_t(x)$$

is an isomorphism of $F_t(K)$ with (the additive group) $\pi A$, which takes $F_t^n(K)$ isomorphically into $\pi^n A$.

Because $F_t(X,Y) = f_t^{-1}(f_t(X) + f_t(Y))$ we have a commutative diagram

$$(4.1.2) \qquad \begin{array}{ccc} F_t(K_n) & \xrightarrow{f_t} & K_n \\ \downarrow{\scriptstyle F_t\text{-Norm}_{n/o}} & & \downarrow{\scriptstyle \mathrm{Tr}_{n/o}} \\ F_t(K) & \underset{f_t}{\xrightarrow{\sim}} & \pi A \subset K \end{array}$$

4.2. The Functions $\lambda_{n/o}(t)$.

For each $n \in \mathbb{N}$ define the function $\lambda_{n/n-1}$ by

(4.2.1) $$\lambda_{n/n-1}(t) = [p^{-1}((m_n+1)(p-1)+t)]$$

where $m_n = m(K_n/K_{n-1}) = 1 + p + \ldots + p^{n-1}$. Now define $\lambda_{n/o}$ inductively by

(4.2.2) $$\lambda_{n/o}(t) = \lambda_{n-1/o}(\lambda_{n/n-1}(t))$$

It is then clear from 3.1 that

(4.2.3) $$Tr_{n/o}(\pi_n^t A_n) = \pi^{\lambda_{n/o}(t)} A$$

It is not difficult to calculate $\lambda_{n/o}(t)$. The result is

(4.2.4) $$\lambda_{n/o}(t) = n \quad \text{for} \quad 1 \leq t \leq (p-1)^{-1}(p^n-1)$$

$$\lambda_{n/o}(t) = 1 + \ell + n \quad \text{for} \quad (p-1)^{-1}(p^n-1)+\ell p^n+1 \leq t \leq (p-1)^{-1}(p^n-1)+(\ell+1)p^n$$

4.3. Proof of $\mathrm{Im}(F\text{-}Norm_{n/o}) \subset F^{\alpha_n}(K)$ in the case $h = \infty$.

Let $h = \infty$. From now on we write $F$ for $F_t$. Because of 4.1 it suffices to prove that $Tr_{n/o}(f_t(x)) \subset \pi^n A$ for all $x \in \pi_n A_n$. According to lemma 2.4 we have that $a_i \in A$ for $i = 1, 2, \ldots$ in the case $h = \infty$. Hence by (4.2.3) and (4.2.4) $Tr_{n/o}(a_i x^{p^i}) \in \pi^n A$ for all $i = 1, 2, \ldots$ One easily checks (using (4.2.3) and (4.2.4) again) that the series $\sum_i Tr_{n/o}(a_i x^{p^i})$ converges. It follows that $Tr_{n/o}(f_t(x)) \in \pi^n A$ for all $x \in \pi_n A_n$.

4.4. Proof of $\mathrm{Im}(F\text{-}Norm_{n/o}) \supset F^n(K)$ in the case $h = \infty$.

To prove this we have to show that $\pi^n A \supset \mathrm{Im}(Tr_{n/o} \circ f_t)$. Let $t \in \mathbb{N}$ be such that $\lambda_{n/o}(t+1) > \lambda_{n/o}(t)$. It follows from (4.2.3) that then

(4.4.1) $$v(Tr_{n/o}(\pi_n^t)) = \lambda_{n/o}(t)$$

For each $\ell = 0, 1, 2, \ldots$ let $s_\ell = (p-1)^{-1}(p^n-1) + \ell p^n$. Then we have using (4.4.1), (4.2.3) and (4.2.4)

(4.4.2) $\quad Tr_{n/o}(f_t(y\pi_n^{s_\ell})) \equiv Tr_{n/o}(y\pi_n^{s_\ell}) = y\ Tr_{n/o}(\pi_n^{s_\ell}) \mod \pi^{n+\ell+1}$

Using this and (4.4.2) we see that the maps

$$Tr_{n/o} \circ f_t\colon F^{s_\ell}(K_n) \to \pi^{n+\ell}A/\pi^{n+\ell+1}A$$

are surjective for all $\ell = 0,1,2,\ldots$ Because the groups $F(K_n)$ and $\pi^n A$ are complete and Hausdorff it follows that

$$Tr_{n/o} \circ f_t\colon F(K_n) \to \pi^n A$$

is surjective. Cf. [4] lemma 3.2 or [8] Ch. V, §1, lemma 2.

4.5. Proof of $Im(F\text{-}Norm_{n/o}) \subset F^{\alpha_n}(K)$ in case $h < \infty$.

Now let $2 \leq h < \infty$. Because of 4.1 it suffices to prove that

(4.5.1) $\quad Tr_{n/o}(f_t(x)) \subset \pi^{\alpha_n}A,\ x \in \pi_n A_n$

We have that $\alpha_n = n - [h^{-1}(n-1)]$, and that

(4.5.2) $\quad v(Tr_{n/o}(a_i x^{p^i})) \geq \lambda_{n/o}(p^i v_n(x)) + v(a_i)$

Write $n = \ell h + r$, with $1 \leq r \leq h$. Then $\alpha_n = n - \ell$. If $i < n$ then $v(a_i) \geq -\ell$ by lemma 2.3 and $\lambda_{n/o}(p^i v_n(x)) \geq n$. Hence

(4.5.3) $\quad v(Tr_{n/o}(a_i x^{p^i})) \geq \alpha_n$ for $i < n$

If $i = n$, then $\lambda_{n/o}(p^i v_n(x)) \geq n+1$ by (4.2.4) and $v(a_i) \geq -\ell-1$ so that also

(4.5.4) $\quad v(Tr_{n/o}(a_n x^{p^n}) \geq \alpha_n$

Finally if $i > n$, then $p^i \geq (p-1)^{-1}(p^n-1) + (p^{i-n}-1)p^n + 1$ and $v(a_i) \geq -\ell-1-(i-n)$, hence

(4.5.5) $\quad v(Tr_{n/o}(a_i x^{p^i}) \geq \alpha_n$ for $i > n$

because $\lambda_{n/o}(p^i v_n(x)) \geq \lambda_{n/o}(p^i) \geq n+p^{i-n}$ and $p^{i-n} \geq (i-n) + 1$ if $i > n$

The series $\Sigma\,Tr_{n/o}(a_i x^{p^i})$ converges. The inclusion (4.5.1) now follows from (4.5.3) - (4.5.5).

4.6. Proof of $F^{\alpha_n}(K) \subset Im(F\text{-}Norm_{n/o})$ in case $h < \infty$.

Choose uniformizing elements $\pi_\ell \in A_\ell$ such that $N_{\ell/\ell-1}(\pi_\ell) = (-1)^{p-1}\pi_{\ell-1}$ for $\ell = 1, \ldots, n$.

Let $n = \ell h+r$, $1 \le r \le h$. For each s such that $n-\ell \le s < n$ let

$$t_s = (p-1)^{-1}(p^{n-(n-s)h}-1) \tag{4.6.1}$$

We try to calculate $Tr_{n/o}(a_i(\pi_n^{t_s})^{p^i}) \bmod (\pi^{\alpha_{n+1}})$. To this end we first prove that for $a \le r < n$

$$Tr_{n/n-a}(\pi_n^{p^r\ell}) \equiv \pi_{n-a}^{p^{r-a}\ell}p^a \mod p^{a+1}\pi_{n-a}^{p^{r-a}\ell-1} \tag{4.6.2}$$

This is done by induction. The case a = 1 is the trace lemma 3.4 above. Assuming the lemma for all b < a. We have

$$Tr_{n/n-a+1}(\pi_n^{p^r\ell}) \equiv \pi_{n-a+1}^{p^{r-a+1}\ell}p^{a-1} \mod p^a\pi_{n-a+1}^{p^{r-a+1}\ell-1}$$

(4.6.3)

$$Tr_{n-a+1/n-a}(\pi_{n-a+1}^{p^{r-a+1}\ell}p^{a+1}) \equiv \pi_{n-a}^{p^{r-a}\ell}p^a \mod p^{a+1}\pi_{n-a}^{p^{r-a}\ell-1}$$

It therefore suffices to show that $\lambda_{n-a+1/n-a}(ap^{(n-a+1)}+p^{r-a+1}\ell-1) \ge (a+1)p^{n-a} + p^{r-a}\ell-1$ which is easily checked. Cf. (4.2.4).

Now write $j_s = (n-s)h$. Then by (4.6.2)

$$Tr_{n/n-j_s}((\pi_n^{t_s})^{p^{j_s}}) \equiv \pi_{n-j_s}^{t_s}p^{j_s} \mod p^{j_s+1}\pi_{n-j_s}^{t_s-1}$$

But

$$\lambda_{n-j_s/o}(t_s+1) > \lambda_{n-j_s/o}(t_s) \text{ and } (p^{n-j_s})(j_s+1) + t_s - 1 > p^{n-j_s}j_s + t_s$$

It follows that

$$(4.6.4)\qquad v(Tr_{n/o}(\pi_n^{t_s p^{j_s}})) = j_s + \lambda_{n-j_s/o}(t_s) = n$$

and as

$$v(a_{j_s}) = v(a_{(n-s)h}) = -(n-s)$$

by lemma 2,3, we have that

$$(4.6.5)\qquad v(Tr_{n/o}(a_{j_s}\pi_n^{t_s p^{j_s}})) = s \qquad n-\ell \leq s < n$$

If $i < j_s$, then $\lambda_{n/o}(\pi_n^{t_s p^i}) \geq n$ and $v(a_i) > -(n-s)$, hence

$$(4.6.6)\qquad v(Tr_{n/o}(a_i\pi_n^{t_s p^i})) > s \quad n-\ell \leq s < n,\ i < j_s$$

If $i = j_s + 1$, then because $h \geq 2$ we have that

$$(4.6.7)\qquad v(a_i) \geq s-n \qquad i = j_s + 1$$

and because $p^{j_s+1} t_s \geq (p-1)^{-1}(p^n-1) + 1$ we have that

$$(4.6.8)\qquad v(Tr_{n/o}(a_{j_s+1}\pi_n^{t_s p^{j_s+1}}) > s$$

Finally if $i \geq j_s + 2$, then because $h \geq 2$ we have that

$$(4.6.9)\qquad v(a_i) \geq s - n - \tfrac{1}{2}(i-j_s)$$

and

$$(4.6.10)\qquad p^i t_s \geq (p-1)^{-1}(p^n-1) + \tfrac{1}{2}(i-j_s)p^n + 1$$

(To see that (4.6.10) holds, use $j_s < n$). It follows that

$$(4.6.11)\qquad v(Tr_{n/o}(a_i\pi_n^{t_s p^i}) > s \quad \text{for } i \geq j_s+2$$

The series $\Sigma\, Tr_{n/o}(a_i\pi_n^{t_s p^i} y^{p^i})$ converges for all $y \in A$.
It then follows from (4.6.5), (4.6.6), (4.6.8) and (4.6.11) that

(4.6.12) $$\mathrm{Tr}_{n/o}(f_t(y\pi_n^{t_s})) \equiv y^{p^{j_s}} b_s \mod \pi^{s+1}$$

where $b_s \in A$ is an element of valuation s.
Because k is perfect it follows that

(4.6.13) $$\mathrm{Tr}_{n/o} \circ f_t : F^{t_s}(K_n) \to \pi^s A/\pi^{s+1} A$$

is surjective for $\alpha_n = n-\ell \leq s < n$

Now suppose that $s \geq n$. For these s let

(4.6.14) $$t_s = (p-1)^{-1}(p^n-1) + (s-n)p^n$$

Then $\lambda_{n/o}(t_s) = s$ and $\lambda_{n/o}(t_s+1) = s+1$. It follows that

(4.6.15) $$v(\mathrm{Tr}_{n/o}(a_o\pi_n^{t_s})) = s \qquad s \geq n$$

Because $h \geq 2$, $v(a_1) \geq 0$, hence

(4.6.16) $$v(\mathrm{Tr}_{n/o}(a_1\pi_n^{pt_s})) > s \qquad s \geq n$$

because $pt_s \geq t_s + 1$

Finally for $i \geq 2$, we have $v(a_i) \geq -\frac{1}{2}i$ if i is even and $v(a_i) \geq -\frac{1}{2}i+\frac{1}{2}$ if i is odd and $p^i t_s \geq t_s + (\frac{1}{2}i)p^n + 1$.
Hence

(4.6.17) $$v(\mathrm{Tr}_{n/o}(a_i\pi_n^{p^i t_s})) > s \qquad s \geq n,\ i \geq 2$$

It follows from (4.6.15) - (4.6.17) that

(4.6.18) $$\mathrm{Tr}_{n/o}(f_t(y\pi_n^{t_s})) \equiv b_s y \mod \pi^{s+1},\ s \geq n$$

where $b_s$ is an element of A of valuation s. Hence

(4.6.19) $$\mathrm{Tr}_{n/o} \circ f_t : F^{t_s}(K_n) \to \pi^s A/\pi^{s+1} A$$

is surjective for $s \geq n$. Combining (4.6.13), (4.5) and (4.6.19) and using that $F(K_n)$ and $\pi^{\alpha_n}A$ are complete Hausdorf filtered (topological) groups we see that the image of $Tr_{n/o} \circ f_t$ is equal to $\pi^{\alpha_n}A$. According to (4.1) this implies that the image of $F\text{-}Norm_{n/o}$ is equal to $F^{\alpha_n}(K)$ which is what we set out to prove.

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# LIST OF SYMBOLS

Latin lower case a,x,y,n,k,p,h,r,v,f,c,t,s,i,j,m,o,l,b,d,e

Latin upper case K,L,F,X,Y,Z,A,U,V,G,N,T,I,C,M

Latin lower case as sub or superscript p,i,j,n,m,k,t,h,r,s,a,b,l

Latin upper case as sub or superscript L,K,F,V

Latin upper case boldface $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{N}$, G

Latin lower case script p

Greek lower case $\sigma,\alpha,\phi,\psi,\pi,\lambda$

Greek upper case $\Gamma$

Greek lower case as sub or superscript $\alpha,\pi,\psi,\lambda$

Greek upper case as sub or superscript

German lower case

Numerals 0,1,2,3,4,5,6,7,8,9

Numerals as sub or superscript 0,1,2

Special symbols $/,(,),\Sigma,+,\in,\rightarrow,\equiv,-,\infty,\supset,>,[,],\{,\},\cup,<,\leq,\geq,\subset,\simeq,\rightarrow,\tilde{\rightarrow}$ ,o,

Special symbols as sub or superscript $\geq,/,*,\infty,-,=,+$

Groups of symbols: Norm, mod, Gal, Coker, Im, Tr

The latin lower case letter O does not occur in the formulas except as part of the groups Norm, mod, Coker

The latin uppercase letter O does not occur

Greek letters: straight underline in red

German Letters: straight underline in green

Boldface: wiggly underline in black

Script: encircled in blue