CONSTRUCTING FORMAL GROUPS III: APPLICATIONS TO COMPLEX COBORDISM AND BROWN-PETERSON COHOMOLOGY

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1. Introduction

In parts I and II, cf. [5], [6] and also [7], of this series of papers we constructed a universal \( p \)-typical one dimensional commutative formal group and a universal one dimensional commutative formal group. The extraordinary cohomology theories \( \text{BP} \) (Brown-Peterson cohomology) and \( \text{MU} \) (complex cobordism cohomology) are complex oriented and hence define one dimensional formal groups over \( \text{BP}_*(pt) \) and \( \text{MU}_*(pt) \) respectively. Cf. [1]. These formal groups are respectively \( p \)-typically universal and universal. Cf. [1], [3], [4] and [18]. Let \( \mu_\text{BP} \) and \( \mu_\text{MU} \) denote these formal groups. The logarithms of the formal groups \( \mu_\text{BP} \) and \( \mu_\text{MU} \) are known, cf. [17], and have a very simple expression in terms of the cobordism classes of the complex projective spaces. Using the formulas for the logarithms of the universal formal groups of [5] and [6] one then obtains a free polynomial basis for \( \text{BP}_*(pt) \) and \( \text{MU}_*(pt) \) in terms of the classes of the complex projective spaces. This is the subject matter of Sections 2, 3 below.

In [5] we also constructed a universal isomorphism between \( p \)-typical formal groups. The associated map \( \mathbb{Z}[V_1, V_2, \ldots] \to \mathbb{Z}[V_1, V_2, \ldots; T_1, T_2, \ldots] \) (when localized at \( p \)) can be identified with the right unit map \( \mu_\text{R} : \text{BP}_*(pt) \to \text{BP}_*(\text{BP}) \) of the Hopf algebra \( \text{BP}_*(\text{BP}) \).

In Sections 4, 5 below, we use the universal isomorphism of [5] to obtain a recursive description of the homomorphism \( \eta_\text{R} \). This description is useful in the calculation of various \( \text{BP} \) cohomology operations, cf. Sections 6, 7 below. To obtain this recursive description of \( \eta_\text{R} \) we need an isomorphism formula (Section 5 below) which is also useful in the theory of formal groups itself, cf. part III of [8].
Finally in Section 8, we use the universal isomorphism and the functional equation lemma of [5] to derive the main theorem of [20]. All formal groups in this paper will be commutative and one dimensional. Some of the results of this paper were announced in [7], [11].

Acknowledgements. Luilevicius [16] was the first to write down a formula similar to (3.1.3) and to prove that it gives generators for \( \text{BP}_*(pt) \) in the case \( p = 2 \).

Once one has the various universal \( p \)-typical formal groups (which are more or less canonical) they can be fitted together in various ways (all noncanonical). One way to do this is described in part II of [8] and gives the generators for \( \text{MU}_*(pt) \) described in [8, part II] and [7]. Subsequently Kozma [14] wrote down a different set of polynomial generators for \( \text{MU}_*(pt) \), which satisfy more elegant recursion formulas. These generators correspond to a different way of fitting the various universal \( p \)-typical formal groups together, which, however, does not generalize to more dimensional formal groups, but does generalize if one restricts attention to more dimensional curvilinear formal groups. Cf. the introduction of [6] and [12] for more details.

2. The formal groups of complex cobordism and Brown–Peterson cohomology

2.1. Complex oriented cohomology theories. Let \( h^* \) be a complex oriented cohomology theory (defined on finite CW complexes); and let \( e^*(L) \) denote the Euler class in \( h^*(X) \) of a complex line bundle \( L \) over \( X \). Cf. [3, part I, §5], [1, part II, §2], or [19] for a definition of “complex oriented”.

For complex line bundles \( L_1, L_2 \) one has

\[
e^*(L_1 \otimes L_2) = \sum_{i+j} a_i e^*(L_1)^i e^*(L_2)^j
\]

with \( a_i \in h_* (pt) \), and by naturality the coefficients \( a_i \) do not depend on \( L_1 \) and \( L_2 \). So we have a well-defined formal power series

\[
F_n (X, Y) = \sum a_i X^i Y^j
\]

which in fact defines a (one dimensional commutative) formal group over \( h_* (pt) \) by commutativity and associativity of tensor products and naturality of Euler classes.

2.2. The formal groups of \( \text{MU} \) and \( \text{BP} \). Choose a prime number \( p \). Let \( \text{MU} \) stand for the complex cobordism spectrum and \( \text{BP} \) for the Brown–Peterson spectrum associated to the prime number \( p \). These theories are complex oriented. Let \( \mu_{\text{MU}} \) and \( \mu_{\text{BP}} \) be the associated formal groups. Cf. [1], [3], [19]. Let \( \log_{\text{MU}} \) and \( \log_{\text{BP}} \) be their logarithmic series, i.e.

\[
\mu_{\text{MU}} (X, Y) = \log^{-1}_{\text{MU}} (\log_{\text{MU}} (X) + \log_{\text{MU}} (Y))
\]
(2.2.2) \[ \mu_{\text{sp}}(X, Y) = \log_{\text{sp}}^{-1}(\log_{\text{sp}}(X) + \log_{\text{sp}}(Y)). \]

One then has (Miščenko's theorem, cf. [17])

(2.2.3) \[ \log_{\text{MU}}(X) = \sum_{n=0}^\infty m_n X^{n+1}, \]

(2.2.4) \[ \log_{\text{BP}}(X) = \sum_{n=0}^\infty m_n x^{x^n} \]

with \( m_0 = 1 \) and \( m_n = (n + 1)^{-1}[\text{CP}^n] \), where \([\text{CP}^n] \) is the cobordism class of complex projective space of (complex) dimension \( n \). Cf. [1], [3], [17], [18].

The formal group \( \mu_{\text{MU}} \) is universal by a theorem of Quillen [18] and it follows immediately that \( \mu_{\text{BP}} \) is \( p \)-typically universal. Cf. also [3].

3.1. Generators for \( \text{BP}_*(pt) \). Choose a prime number \( p \). Let \( f_v(X) \) be the power series defined by formula (2.2.1) in [5] (cf. also [7]) and let \( F_v(X, Y) = f_v^\circ(f_v(X) + f_v(Y)) \). According to Theorems 2.3 and 2.8 of [5] \( F_v(X, Y) \) is a \( p \)-typically universal formal group over \( \mathbb{Z}[V] = \mathbb{Z}[V_1, V_2, V_3, \ldots] \). Write

(3.1.1) \[ f_v(X) = \sum_{n=0}^\infty a_n(V) X^n, \quad a_0(V) = 1 \]

then we have according to formula (4.3.1) of [5]:

(3.1.2) \[ p a_n(V) = a_{n-1}(V) V^{p-1} + \cdots + a_1(V) V + n. \]

Because \( F_v(X, Y) \) over \( \mathbb{Z}_p[V] \) and \( \mu_{\text{sp}}(X, Y) \) over \( \text{BP}_*(pt) \) are both \( p \)-typically universal formal groups (for \( p \)-typical formal groups over \( \mathbb{Z}_p \)-algebras) there exist (cf. [5, Definition 2.4]) mutually inverse isomorphisms \( \phi : \mathbb{Z}_p[V] \to \text{BP}_*(pt) \), \( \psi : \text{BP}_*(pt) \to \mathbb{Z}_p[V] \) such that \( \phi \) applied to the coefficients of \( f_v(X) \) gives the coefficients of \( \mu_{\text{sp}}(X) \). Applying \( \phi \) to (3.1.2) and writing \( v \) for \( \psi(V) \) we therefore find elements \( v_1, v_2, v_3, \ldots \) of \( \text{BP}_*(pt) \) which constitute a free polynomial basis for \( \text{BP}_*(pt) \) and which are related to the \( m_n = (n + 1)^{-1}[\text{CP}^n] \) of (2.2.4) above by the relations

(3.1.3) \[ p l_n = l_{n-1} v_1^{p-1} + l_{n-2} v_2^{p-2} + \cdots + l_1 v_{n-1} + v_n \]

where we have written \( l_n \) for \( m_{p^n-1} \).

3.2. Generators for \( \text{MU}_*(pt) \). Let \( f_U(X) \) be the power series defined by formulas (2.2.1) and (2.2.4) of [6], and let \( F_U(X, Y) = f_U^\circ(f_U(X) + f_U(Y)) \). According to Theorems 2.3 and 2.4 of [6] \( F_U(X, Y) \) is a universal formal group over \( \mathbb{Z}[U] = \mathbb{Z}[U_2, U_3, U_4, \ldots] \). Write

(3.2.1) \[ f_U(X) = \sum_{n=0}^\infty b_n(U) X^n, \quad b_1 = 1. \]

Then if we specify the coefficients \( n(i_1, \ldots, i_r) \) occurring in the definition of \( f_U(X) \) according to [6, Section 7] we have the following recursion formula:
where the integers \( \nu(n) \) and \( \mu(n, d) \) are defined as follows:

\[
\nu(n) = 1 \text{ if } n \text{ is not a power of a prime number}
\]

\[
\nu(p') = p \text{ for all prime numbers } p \text{ and } r \in \mathbb{N} = \{1, 2, 3, \ldots\},
\]

\[
\mu(n, d) = \prod_{p \mid n} c(p, d)
\]

where the product is defined over all prime numbers \( p \) dividing \( n \) and the \( c(p, d) \) are integers which can be chosen arbitrarily subject to

\[
c(p, d) = 1 \text{ if } \nu(d) = 1, p,
\]

\[
c(p, d) = \begin{cases} 1 \mod p & \text{if } \nu(d) = q \neq p, \\ 0 \mod q & \end{cases}
\]

More precisely: first one chooses \( c(p, d) \in \mathbb{Z} \) for all prime numbers \( p \) and \( d \in \mathbb{N} \) such that (3.2.5) holds: then one constructs \( f_U(X) \) and \( F_U(X, Y) \) according to the formulas (7.1.2), (7.1.3), (2.2.1), (2.2.4) and (2.2.7) of [6]; the result is then a universal formal group \( F_U(X, Y) \) over \( \mathbb{Z}[U] \) with logarithm \( f_U(X) \) satisfying (3.2.2) with \( \nu(n) \) and \( \mu(n, d) \) given by (3.2.3) and (3.2.4). Different choices for the \( c(p, d) \) result in different universal formal groups \( F_U(X, Y) \). Because \( F_U(X, Y) \) over \( \mathbb{Z}[U] \) and \( \mu_{MU}(X, Y) \) over \( MU_q(pt) \) are both universal formal groups there are mutually inverse isomorphisms \( \phi : \mathbb{Z}[U] \rightarrow MU_q(pt) \), \( \psi : MU_q(pt) \rightarrow \mathbb{Z}[U] \) such that \( \phi \) applied to the coefficients of \( f_U(X) \) gives the coefficients of \( \mu_{MU}(X) \). Applying \( \phi \) to (3.2.2) and writing \( u_i, i = 1, 2, \ldots \) for \( \phi(f_U) \) we find elements \( u_1, u_2, \ldots \) in \( MU_q(pt) \) which constitute a free polynomial basis for \( MU_q(pt) \) and which are related to the \( m_n = (n+1)^{-1}[CP^n] \) by the formula

\[
\nu(n)m_{n-1} = u_n + \sum_{d \mid n} \frac{\mu(n, d)\nu(n)}{\nu(d)} m_{\lfloor n/d \rfloor}u_d^{n/d}.
\]

These are the same generators as those written down by Kozma [14]. Note that the factor \( \nu(d)^{-1}\mu(n, d)\nu(n) \) is always an integer.

If one uses instead of the universal formal group \( F_U(X, Y) \) of [6], the universal formal group \( H_U(X, Y) \) over \( \mathbb{Z}[U] \) of [6] then, reasoning in exactly the same way, one finds generators \( \tilde{u}_i \) in \( MU_q(pt) \) which are related to the \( m_n \) by the formula

\[
\nu(n)m_{n-1} = \tilde{u}_n + \sum_{
rowd \geq 1} (-1)^{\sum d_i} \frac{\mu(n, d_i)\nu(n)}{\nu(d_i)} m_{n-1}^{d_{n-1}}\tilde{u}_n^{d_n} \cdots \tilde{u}_0^{d_0}.
\]

where \( \sum \) is the sum over all sequences \((d, d_i, d_{i-1}, \ldots, d_1) \) such that \( d, d_i, \ldots, d_1 \in \mathbb{N}, d_i \neq 1, n; d_i > 1 \) and not a power of a prime number for \( i = 2, \ldots, 1 \) and \( dd_i \ldots d_1 = n \). These are the generators given in [7] and [8, part II].
3.3. Remark. BP is a direct summand of MU_{\mathcal{P}}. If we identify \( u_{n} \) with \( v_{i} \), formula (3.2.6) (or formula (3.2.7) for that matter) reduces to formula (3.1.3) if \( n \) is a power of \( p \). It follows that the \( v_{i} \) are integral, i.e. they live in MU_{\mathcal{P}}(pt), not just in MU_{\mathcal{P}}(pt). This is also proved in [2].

4. Isomorphisms of \( p \)-typical formal groups and \( \eta_{\mathcal{P}} : BP_{\mathcal{P}}(pt) \rightarrow BP_{\mathcal{P}}(BP) \)

4.1. Universal strict isomorphisms of \( p \)-typical formal groups. In [5] we also constructed a universal strict isomorphism

\[
\alpha_{V,T}(X) : F_{V}(X,Y) \rightarrow F_{V,T}(X,Y)
\]

for \( p \)-typical formal groups over characteristic zero rings or \( \mathbb{Z}_{\mathcal{P}} \)-algebras. Here \( F_{V,T}(X,Y) \) is a \( p \)-typical formal group over \( \mathbb{Z}[V;T] = \mathbb{Z}[V_1, V_2, \ldots; T_1, T_2, \ldots] \) and the logarithm \( f_{V,T}(X) \) of \( F_{V,T}(X,Y) \) satisfies

\[
f_{V,T}(X) = \sum_{i=0} a_{i}(V,T)X^{p^{i}},
\]

\[
a_{i}(V,T) = a_{i}(V) + a_{i-1}(V)T_{1}^{p^{i-1}} + \ldots + a_{1}(V)T_{i-1}^{p} + T_{i}
\]

cf. formula (4.3.2) of [5].

Let \( I : \mathbb{Z}_{\mathcal{P}}\text{-Alg} \rightarrow \text{Sets} \) be the functor which associates to every \( \mathbb{Z}_{\mathcal{P}} \)-algebra \( A \) the set of all triples \((F(X,Y), a(X), G(X,Y))\) where \( F(X,Y) \) and \( G(X,Y) \) are \( p \)-typical formal groups over \( A \) and \( a(X) \) is a strict isomorphism from \( F(X,Y) \) to \( G(X,Y) \). If we restrict attention to \( \mathbb{Z}_{\mathcal{P}} \)-algebras theorem 2.12 of [5] says

4.2. Theorem. The \( \mathbb{Z}_{\mathcal{P}} \)-algebra \( \mathbb{Z}[V,T] \) represents the functor \( I \).

The isomorphism \( \mathbb{Z}_{\mathcal{P}}\text{-Alg} \rightarrow \text{Sets} \) looks as follows. Let \( \phi : \mathbb{Z}_{\mathcal{P}}[V,T] \rightarrow A \) be a \( \mathbb{Z}_{\mathcal{P}} \)-algebra homomorphism. Let \( v_{i} = \phi(V_{i}), t_{i} = \phi(T_{i}), i = 1, 2, \ldots \) then the triple associated to \( \phi \) is \((F_{\phi}(X,Y), \alpha_{\phi}(X), F_{\phi}(X,Y))\).

4.3. The homomorphism \( V_{i} \mapsto \bar{V}_{i} \) is a \( p \)-typical formal group over \( \mathbb{Z}[V;T] \). By the universality of \( F_{V,T}(X,Y) \) there are therefore unique polynomials \( \bar{V}_{i} \in \mathbb{Z}[V;T] \) such that \( F_{V,T}(X,Y) = F_{\bar{V}}(X,Y) \). Note that the \( \bar{V}_{i} \) have their coefficients in \( \mathbb{Z} \) not just in \( \mathbb{Z}_{\mathcal{P}} \).

We have just defined a homomorphism

\[
\nu_{\mathcal{P}} : \mathbb{Z}[V] \rightarrow \mathbb{Z}[V;T], \quad V_{i} \mapsto \bar{V}_{i}.
\]

A more functorial way of looking at this homomorphism is as follows. Let \( F : \mathbb{Z}_{\mathcal{P}}\text{-Alg} \rightarrow \text{Sets} \) be the functor which associates to a \( \mathbb{Z}_{\mathcal{P}} \)-algebra \( A \) the set of all \( p \)-typical formal groups over \( A \). Then \( F \) is represented by \( \mathbb{Z}_{\mathcal{P}}[V] \), (by the universality of \( F_{V}(X,Y) \)). There are two natural functor morphisms \( I \rightarrow F \), viz.
and because $\mathbb{Z}_p[\{V; T\}]$ represents $I$ and $\mathbb{Z}_p[\{V\}]$ represents $F$ we obtain two $\mathbb{Z}_p$-algebra homomorphisms $\mathbb{Z}_p[\{V\}] \to \mathbb{Z}_p[\{V; T\}]$. The homomorphism induced by (4.3.2) is the natural inclusion $\mathbb{Z}_p[\{V\}] \to \mathbb{Z}_p[\{V; T\}]$ and the homomorphism induced by (4.3.3) is the localization in $p$ of (4.3.1).

4.4. The Hopf-algebra $BP_*(BP)$. By Theorem 16.1 of [1, part II] we know that $BP_*(BP) = BP_*(pt)[t_1, t_2, \ldots ] = \mathbb{Z}_p[\{v_1, v_2, \ldots ; t_1, t_2, \ldots \}]$. It follows that $BP_*(BP)$ represents the functor $I$. This fact can be used to account for the Hopf-algebra structure of $BP_*(BP)$ by using various functor morphisms like (4.3.2) and (4.3.3) above. This was done in [15]. The structure of $BP_*(BP)$ as a left module over $BP_*(pt)$ is then given by the natural inclusion $BP_*(pt) \to BP_*(BP)$ and the structure of $BP_*(BP)$ as a right module over $BP_*(pt)$ is given by a homomorphism $\eta_R : BP_*(pt) \to BP_*(BP)$ which is the localization in $p$ of $v_R$ in (4.3.1) above if we identify $BP_*(pt)$ with $\mathbb{Z}_p[\{V\}]$ and $BP_*(BP)$ with $\mathbb{Z}_p[\{V; T\}]$ by means of $v_i \mapsto V_i$ and $t_i \mapsto T_i$ where the $v_i$ are the generators defined in 3.1 above. Alternatively we can appeal again to Theorem 16.1 of [1, part II] where it is shown that $\eta_R \otimes \mathbb{Q}$ is given by

\[(4.4.1) \quad l_n \mapsto \sum_{i=0}^{n} l_i v_i^{l_i-1},\]

where again $l_n = m_{p^{r-1}}$. Because $F_{v; T}(X, Y) = F_v(X, Y)$ and because of formula (4.1.3) this also shows that $\eta_R = v_R \otimes \mathbb{Z}_p$. (If $\phi : \mathbb{Z}_p[\{V\}] \to BP_*(pt)$ is the isomorphism $V_i \mapsto v_i$ then $\phi(a_i(V)) = l_i$ by (3.1.2) and (3.1.3), hence the right hand side of (4.1.3) becomes the right hand side of (4.4.1) under $\phi : \mathbb{Z}_p[\{V; T\}] \to BP_*(BP)$,

\[V_i \mapsto v_i, T_i \mapsto t_i, i = 1, 2, \ldots .\]

5. The isomorphism formula

The next thing we want to do is to give a recursion formula for the polynomials $\tilde{V}_n$ and hence also a recursive description of $\eta_R : BP_*(pt) \to BP_*(BP)$. To do so we first need a formula relating the $\tilde{V}_i$ and the $V_i$ which is also useful in its own right, especially when discussing reductions and liftings of formal groups and isomorphisms of formal groups. Cf [8, parts III and V].

5.1. Let $a_i = a_i(V)$ be defined by (3.1.2) and write $\bar{a}_i$ for $a_i(V, T)$, cf. (4.1.3). Then we have $f_v(X) = \sum a_i X^{p^i}$ and $f_{v; T}(X) = \sum \bar{a}_i X^{p^i}$ and because $f_{v; T}(X) = f_v(X)$, the $\bar{a}_i$ are given by the same formula (3.1.2) with bars over all the symbols occurring. I.e.

\[(5.1.1) \quad p\bar{a}_n = \bar{a}_{n-1} \tilde{V}_n^{p^{n-1}} + \ldots + \bar{a}_1 \tilde{V}_n^{p^{n-1}} + \tilde{V}_n.\]
In addition we define

\begin{equation}
Z_\nu^{(r)} = (V_i T_i^{\nu-i} - T_i V_i^{\nu-i}).
\end{equation}

5.2. Proposition.

\begin{equation}
p\bar{a}_n = \sum_{i=1}^n \bar{a}_n V_i^{\nu-i} + \sum_{i,j,n} a_{n-i,j}Z^{(n-i-j)}_\nu + pT_n.
\end{equation}

\textbf{Proof.} Using (4.1.3), (3.1.2) and (5.1.1) we have

\begin{align*}
p\bar{a}_n &= p\alpha_n + \sum_{i=1}^n p\alpha_{n-i} T_i^{\nu-i} \\
&= \sum_{i=1}^n a_{n-i} V_i^{\nu-i} + V_n + \sum_{i=1}^n \sum_{j=1}^i a_{n-i,j} V_j^{\nu-i-j} T_i^{\nu-i} + pT_n \\
&= \sum_{i=1}^n \bar{a}_n V_i^{\nu-i} - \sum_{i=1}^n \sum_{j=1}^i a_{n-i,j} V_j^{\nu-i-j} T_i^{\nu-i} + V_n \\
&\quad + \sum_{i=1}^n \sum_{j=1}^i a_{n-i,j} V_j^{\nu-i-j} T_i^{\nu-i} + pT_n \\
&= \sum_{i=1}^n \bar{a}_n V_i^{\nu-i} + \sum_{i,n} a_{n-i,j} (V_j^{\nu-i-j} - T_j V_j^{\nu-i-j}) + pT_n \\
&= \sum_{i=1}^n \bar{a}_n V_i^{\nu-i} + \sum_{i,n} a_{n-i,j} Z^{(n-i-j)}_\nu + pT_n.
\end{align*}

(Note that \(Z_0 + Z_1 = (V,T)^n - T_i V_i^{(n-1)}\) and similarly for \(Z^{(r)}_\nu\).)

5.3. Proposition.

\[\bar{V}_n = V_n + pT_n + \sum_{i=1}^n a_{n-i} \left\{ (V_i^{\nu-i} - \bar{V}_i^{\nu-i}) + \sum_{i,j,n} (V_i^{\nu-i-j} - T_i V_i^{\nu-i-j}) \right\},\]

\begin{equation}
(5.3.1)
\end{equation}

\[\quad + \sum_{i,j,n} (V_i T_i - T_i V_i).\]

\textbf{Proof.} This follows directly by substituting in (5.2.1) and (5.1.1) \(\bar{a}_{n-i} = \sum_{n-i} a_{n-i,j} T_j^{\nu-i,j},\) where \(T_0 = 1.\)

5.4. Remark. Formula (5.3.1) can be used to give an inductive proof that the \(\bar{V}_n\) are polynomials with integral coefficients in the \(V_1, \ldots, V_n; T_1, \ldots, T_n\). Indeed, we know that, cf. [5],

\begin{equation}
a_{n-k} = \sum_{i=1}^n p^{-1} V_i a_{n-k-i}^{(n)}
\end{equation}

and assuming that \(\bar{V}_n, i = 1, \ldots, n - 1\) is integral we also have that for all \(s \in \mathbb{N}\)
Finally \( p^i a \) is a polynomial with integral coefficients so that we have in \( \mathbb{Q}[V; T] \)

\[
\tilde{V}_n = V_n + p T_n + \sum_{i+j=n} (V T_i', T_j) - T_i \tilde{V}_i' \\
+ \sum_{i=1}^{n-1} \sum_{k=1}^{n-k} \frac{V_i}{p} \left\{ (V T_i^{x_{ik}} - \tilde{V}_i^{x_{ik}}) + \sum_{i+j=k} (V T_i^{x_{ij}} T_j^{x_{ij}} - T_j^{x_{ij}} \tilde{V}_i^{x_{ij}}) \right\} \\
= \sum_{i=1}^{n-1} \sum_{k=1}^{n-k} \frac{V_i}{p} \left\{ (V T_i^{x_{ik}} - \tilde{V}_i^{x_{ik}}) + \sum_{i+j=k} (V T_i^{x_{ij}} T_j^{x_{ij}} - T_j^{x_{ij}} \tilde{V}_i^{x_{ij}}) \right\} \\
+ \sum_{i=1}^{n} \frac{V_i}{p} \left\{ \tilde{V}_i^{x_{ii}} - \tilde{V}_i^{x_{ii}} + \sum_{i+j=n-i} (V T_i^{x_{ij}} T_j^{x_{ij}} - T_j^{x_{ij}} \tilde{V}_i^{x_{ij}}) \right\} \\
= \sum_{i=1}^{n} \frac{V_i}{p} (-p T_i') = 0
\]

where all congruences are modulo 1 in \( \mathbb{Q}[V; T] \). (Two polynomials in \( \mathbb{Q}[V; T] \) are \( \equiv \mod 1 \) if their difference is in \( \mathbb{Z}[V; T] \).) This proves the integrality of the \( \tilde{V}_n \), \( n = 1, 2, 3, \ldots \).

6. A generalization of the main lemma of Johnson and Wilson [13]

6.1. BP cohomology operations. The stable cohomology operations of BP cohomology can be described as \( BP_*(pt) \)-homomorphisms \( BP_*(BP) \rightarrow BP_*(pt) \), where \( BP_*(BP) \) is seen as a left module over \( BP_*(pt) \). Cf. [1] and also 4.3 and 4.4 above. To find out what a cohomology operation \( r \) does with elements of \( BP_*(pt) \), compose \( r \) with the right unit map \( \eta_k : BP_*(pt) \rightarrow BP_*(BP) \). Let \( E = (e_1, e_2, \ldots) \) be a sequence of \( \geq 0 \) integers of which only finitely many are nonzero. The cohomology operation \( r_E \) is defined as: coefficient of \( t^E \) in \( x \in BP_*(BP) = BP_*(pt) \). Thus \( r_E (v_n) = \) coefficient of \( t^E \) in \( \tilde{v}_n \), where \( \tilde{v}_n \) is obtained from \( \tilde{V}_n \) by replacing \( V_i \) with \( v_i \) and \( T_i \) with \( t_i \), \( i = 1, \ldots, n \).

Assign to an exponent sequence \( E = (e_1, e_2, \ldots) \) the weight \( \| E \| = e_1 (p-1) + e_2 (p^2-1) + \ldots \) and to \( v_i \) the weight \( p^i - 1 \). We then have

\[
(6.1.1) \quad \eta_e (v_n) = \tilde{v}_n = \sum_{\| E \| = p^i - 1} r_E (v_n) t^E
\]

where \( r_E (v_n) \) is homogeneous of weight \( p^i - 1 - \| E \| \).
In [13] Johnson and Wilson calculate \( r_E(v_a) \) modulo \((p, v_1, \ldots, v_{i-1})\) for \( \| E \| \gg p^\gamma - p^l \). ([13, Lemma 1.7] (sometimes known as the Budweiser lemma)).

As a first application of the recursion formula (5.3.1) we shall calculate in this section \( r_E(v_a) \) modulo \((p^\gamma, v_1, \ldots, v_{i-1})\) for all \( E \) with \( \| E \| \gg p^\gamma - p^l \).

6.2. Extension of the main lemma. Write \( \Delta_i \) for the exponent sequence \((0, 0, \ldots, 0, 1, 0, \ldots)\) with the 1 in the \( i \)-th place. We also write \( \Delta_0 = (0, 0, \ldots) \) and \( \| \Delta_0 \| = 0 \). Scalar multiplication and addition of exponent sequences are defined component-wise. The result now is

Lemma. (i) For \( n \geq 3 \) and \( 2 \leq l \leq n - 1 \) we have

(a) \( r_E(v_a) = 0 \mod (p^{\gamma + l}, v_1, \ldots, v_{i-1}) \) if \( p^\gamma - p^{l-1} > \| E \| \gg p^\gamma - p^l \) and \( E \) not equal to \( \Delta_i, \Delta_i + (p - 1)\Delta_{n-1} + p'\Delta_{n-1-i} \);

(b) \( r_E(v_a) = v_i \mod (p^{\gamma + l}, v_1, \ldots, v_{i-1}) \) if \( E = p'\Delta_{n-1} \);

(c) \( r_E(v_a) = -p^\gamma v_i \mod (p^{\gamma + l}, v_1, \ldots, v_{i-1}) \) if \( E = \Delta_i + (p - 1)\Delta_{n-1} + p'\Delta_{n-1-i} \).

(ii) For \( n \geq 3 \) (and \( l = 0 \)) we have

(a) \( r_E(v_a) = 0 \mod (p^{\gamma + 1}) \) if \( \| E \| \gg p^\gamma - 1 \) and \( E \) not equal to \( \Delta_n \) or \( \Delta_i + p\Delta_{n-1} \);

(b) \( r_E(v_a) = p \) if \( E = \Delta_n \);

(c) \( r_E(v_a) = -p^\gamma \mod (p^{\gamma + 1}) \) if \( E = p\Delta_{n-1} + \Delta_i \).

(iii) For \( n \geq 3 \) (and \( l = 1 \)) we have

(a) \( r_E(v_a) = 0 \mod (p^{\gamma + 1}) \) if \( p^\gamma - 1 > \| E \| \gg p^\gamma - p \) and \( E \) not equal to \( \Delta_{n-1} + \Delta_i \) or \( \Delta_i + (p - 1)\Delta_{n-1} + p\Delta_{n-2} \);

(b) \( r_E(v_a) = v_i (1 - p^{\gamma - 1}) \mod (p^{\gamma + 1}) \) if \( E = p\Delta_{n-1} \);

(c) \( r_E(v_a) = -p^\gamma v_i \mod (p^{\gamma + 1}) \) if \( E = \Delta_i + (p - 1)\Delta_{n-1} + p\Delta_{n-2} \).

(iv) For \( n = 1 \) we have

\( r_a(v_i) = p \).

(v) For \( n = 2 \) we have

(a) \( r_E(v_2) = 0 \) if \( \| E \| \gg p^2 - p \) and \( E \) not equal to \( \Delta_2, p\Delta_i, (p + 1)\Delta_i \);

(b) \( r_E(v_2) = p \) if \( E = \Delta_2 \);

(c) \( r_E(v_2) = -p^\gamma \) if \( E = (p + 1)\Delta_i \);

(d) \( r_E(v_2) = (1 - p^{\gamma - 1} - p^\gamma) v_i \) if \( E = p\Delta_i \).

The proof of this lemma goes in several steps.

6.3. Proof of Lemma 6.2. (iv) and (v). We have

\((6.3.1)\) \( \delta_i = v_i + pt_i \); \n
\((6.3.2)\) \( \delta_2 = -p^\gamma (v_i + pt_i) (v_i + pt_i)^{p^\gamma - v_i} + v_1 t_1^p + v_2 + v_1 t_2^p + pt_2 \).

Parts (iv) and (v) of Lemma 6.2 follow immediately from this.

6.4. Proof of Lemma 6.2. (ii). We prove by induction that for \( n \geq 2 \)

\((6.4.1)\) \( \delta_i = v_i + pt_i - p^\gamma t_{i-1}^p \mod (p^{\gamma + 2}, v_i, \ldots, v_{i-1}) \).

Formula (6.3.2) takes care of the case $n = 2$. Now suppose that $n \geq 3$. Because $a_{n-k} = 0 \mod(v_1, \ldots, v_{n-1})$ for $k = 1, \ldots, n-1$ we see from (5.3.1) that

$$\bar{v}_n = v_n + pt_n - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}.$$  

Now by induction we can assume that $\bar{v}_{n-j} = pt_{n-j} - p^{r_j} t_{n-j-1} \mod(p^{r_j}, v_1, \ldots, v_{n-1})$ for $j = 1, \ldots, n-2$ and $\bar{v}_1 = pt_1 \mod(p^{r_{n-1}}, v_1, \ldots, v_{n-1})$. Formula (6.4.1) now follows directly.

Part (ii) of Lemma 6.2 follows from (6.4.1) because of (6.1.1).

6.5. Proof of Lemma 6.2 (i) and (iii). Now let $n \geq 3$ and $1 \leq l \leq n-1$ and let $E$ be an exponent sequence such that $\| E \| \geq p^* - p^l$. If $Q$ is any polynomial in $v_1, v_2, \ldots$ we let $c_E(Q)$ denote the coefficient of $t^E$ in $Q$; $c_E(Q)$ is then a polynomial in $v_1, v_2, \ldots$. We have

(6.5.1) $r_E(v_n) = c_E(\bar{v}_n)$

and $c_E(\bar{v}_n)$ is homogeneous of weight $p^* - 1 - \| E \| \leq p^l - 1$, where $v_l$ has weight $p^l - 1$. In particular this means that $c_E(\bar{v}_n)$ cannot involve any $v_i$ with $i > l$ and that the only terms of $c_E(\bar{v}_n)$ involving $v_l$ are of the form $d v_i$ with $d \in \mathbb{Z}$. Now

(6.5.2) $a_{n-k} = \sum_{j=1}^{n-k} p^{-1} v_j a_{n-1-j}$

Substituting this in (5.3.1) and using the remarks just made we obtain, because $a_{n-k} = 0 \mod(v_1, v_2, \ldots)$ if $n > k + l$, that

$$c_E(\bar{v}_n) = c_E \left( pt_n + p^{-1} v_l \left\{ \left( v_{n-1}^{p^{l-1}} - \bar{v}_{n-1}^{p^{l-1}} \right) + \sum_{i+j=n-1} \left( v_i t_j - t_i v_j \right) \right\} \right)$$

(6.5.3) $+ c_E \left( \bar{v}_{n-1}^{p^{l-1}} - \sum_{j=1}^{l-1} t_j \bar{v}_{n-j}^{p^{l-j}} \right)$

where the congruence is $\mod(v_1, \ldots, v_{n-1})$. Now by (6.4.1)

(6.5.4) $\bar{v}_{n-i}^{p^j} = 0 \mod(v_1, \ldots, v_i, p^{r_i})$ if $l \geq 1, i \geq 1$,

$$\bar{v}_{n-l}^{p^j} = 0 \mod(v_1, \ldots, v_{n-l}, p^{r_{n-l}})$$ if $l \geq 2$,

$$\bar{v}_{n-1}^{p^1} = p^1 t_{n-1} \mod(v_1, \ldots, v_{n-1}, p^{r_{n-1}}).$$

It follows from (6.5.3), (6.5.4) and the fact that $c_E(\bar{v}_n)$ is homogeneous of weight $\leq p^l - 1$ that

(6.5.5) $c_E(\bar{v}_n) = c_E \left( pt_n - p^{r_n} v_1 t_{n-1} + v_l t_{n-1} - \sum_{j=1}^{l-1} t_j \bar{v}_{n-j}^{p^{l-j}} \right)$ if $l = 1$

where the congruence is $\mod(p^{r_{n-1}})$ (and $\| E \| \geq p^* - p$), and

(6.5.6) $c_E(\bar{v}_n) = c_E \left( pt_n + v_l t_{n-1} - \sum_{j=1}^{n-1-j} t_j \bar{v}_{n-j}^{p^{l-j}} \right)$ if $2 \leq l \leq n - 1$.
where the congruence is \( \text{mod}(p^{*r+1}, v_1, \ldots, v_{n-1}) \) (and \( \| E \| \geq p^n - p^i \)). It remains to calculate \( c_E(t \bar{v}_{n-j}^{*r}) \) for \( j = 1, \ldots, n - 1 \). We distinguish three cases: A) \( j > n - l \); B) \( j = n - l \); C) \( j > n - l \).

6.6. Case A. Calculation of \( c_E(t \bar{v}_{n-j}^{*r}) \) for \( j > n - l \). In this case we have \( n - j < l \) and hence by (6.4.1) that \( \bar{v}_{n-j} = pt_{n-j} - p^t t_{n-j} \text{ mod}(v_1, \ldots, v_{n-1}, p^{*r+1}) \) and as \( l \leq n - 1, j > n - l \), it follows that

\[
(6.6.1) \quad c_E(t \bar{v}_{n-j}^{*r}) = 0 \text{ mod}(p^{*r+1}, v_1, \ldots, v_{n-1}) \quad \text{if } j > n - l.
\]

6.7. Case B. Calculation of \( c_E(t \bar{v}_{n-1}^{*r}) \). In this case we have by (6.4.1) that \( \bar{v}_1 = v_1 + pt_1 - p^t t_1 \text{ mod}(v_1, \ldots, v_{n-1}, p^{*r+1}) \). Because \( \| E \| \geq p^n - p^i \) and \( v_i \) has weight \( p^i - 1 \) it follows that

\[
(6.7.1) \quad c_E(t \bar{v}_{n-1}^{*r}) = c_E(t \bar{v}_1^{*r}) = p^r \text{ mod}(p^{*r+1}, v_1, \ldots, v_{n-2}) \quad \text{if } E = \Delta_1 + p \Delta_{n-1}
\]

And we see that

\[
(6.7.2) \quad c_E(t \bar{v}_{n-1}^{*r}) = 0 \text{ mod}(p^{*r+1}, v_1, \ldots, v_{n-1}) \quad \text{if } n - l \geq 2.
\]

And for \( l = n - 1 \) we have

\[
(6.7.3) \quad c_E(t \bar{v}_{n-1}^{*r}) = 0 \text{ mod}(p^{*r+1}, v_1, \ldots, v_{n-2}) \quad \text{if } E \neq \Delta_1 + p \Delta_{n-1}.
\]

6.8. Case C. Calculation of \( c_E(t \bar{v}_{n-j}^{*r}) \) for \( 1 \leq j < n - l \). To deal with these terms we use induction. We have

\[
(6.8.1) \quad c_E(t \bar{v}_{n-j}^{*r}) = c_{E - A}(\bar{v}_{n-j}^{*r}).
\]

Write

\[
(6.8.2) \quad \bar{v}_{n-j} = \sum_{|F| \leq p^{*r+1}} r_F(v_{n-j})t^F.
\]

We then have

\[
(6.8.3) \quad \bar{v}_{n-j}^{*r} = \sum (\frac{p^i}{s_1, \ldots, s_m}) r_{F_1}(v_{n-j})^i \ldots r_{F_m}(v_{n-j})^{*r}t^{s_1F_1 + \ldots + s_mF_m}
\]

where \( F_1, \ldots, F_m \) is the set of all exponent sequences of weight \( \leq p^{*r+1} - 1 \) and the sum is over all \((s_1, \ldots, s_m)\) such that \( s_1 + \ldots + s_m = p^i, s_i \in \mathbb{N} \cup \{0\} \). The only terms of (6.8.3) which can contribute to \( c_{E - A}(\bar{v}_{n-j}^{*r}) \) are those with \( \| s_1F_1 + \ldots + s_mF_m \| = \| E - A \| \geq p^n - p^i - 1 \). This means that there must be at least one \( F_i \) with \( \| F_i \| > p^{*r+1} - p^i \), for which \( s_i \neq 0 \). Indeed if all \( F_i \) with \( s_i = 0 \) were of weight \( \leq p^{*r+1} - p^i \) then we would have \( \| s_1F_1 + \ldots + s_mF_m \| \leq p^i(p^{*r+1} - p^i) - p^i(p^{*r+1} - p^i + 1) \) because \( l \geq 1, j \geq 1 \). We can therefore assume that \( \| F_i \| \geq p^{*r+1} - p^i +
By induction (with respect to $n$) we have that $r_n(v_{n-j}) = 0 \mod (p^{p+1}, v_1, \ldots, v_{i-1})$ except in the following cases:

Case C1: $n - j \geq 3$, $F_i = \Delta_{n-j}$
Case C2: $n - j \geq 3$, $F_i = p \Delta_{n-j-1} + \Delta_1$
Case C3: $n - j = 2$, $F_i = \Delta_2$
Case C4: $n - j = 2$, $F_i = (p+1) \Delta_1$

In cases C2 and C4 we have $r_n(v_{n-j}) = 0 \mod (p^p)$.

In cases C1 and C3 where $F_i = \Delta_{n-j}$, suppose that there is an $i \geq 2$ with $\|F_i\| > p^{n-i} - p^i$, $s_i \neq 0$, $F_i \neq \Delta_{n-j}$, then by the previous reasoning we find a contribution $= 0 \mod (p^{p+1}, v_1, \ldots, v_{i-1})$. The only terms which can contribute something $\neq 0 \mod (p^{p+1}, v_1, \ldots, v_{i-1})$ are therefore of the form

\[(6.8.4) \quad \binom{p^i}{s_1, \ldots, s_m} r_n(v_{n-j})^{s_1} \cdots r_n(v_{n-j})^{s_m}\]

which can contribute something $\neq 0 \mod (p^{p+1}, v_1, \ldots, v_{i-1})$ are therefore of the form

\[(6.8.5) \quad F_i = \Delta_{n-j}, \quad \|F_i\| \leq p^{n-i} - p^i \quad \text{if } i \geq 2 \text{ and } s_i \neq 0.

We then have

\[(6.8.6) \quad \|s_i F_1 + \ldots + s_m F_m\| \leq s_1(p^{n-i} - 1) + (p^i - s_i)(p^{n-i} - p^i)\]

and we must have

\[(6.8.7) \quad \|s_i F_1 + \ldots + s_m F_m\| \geq p^n - p^i - p^i + 1.

If $j \geq 2$ then $p^i = p^{i+1} + p^i + p^i - p$ for all $l \geq 1$ and it follows that (6.8.6) and (6.8.7) can simultaneously hold only if $s_i \geq p + 1$. But then $r_n(v_{n-j})^{s_1} = 0 \mod (p^{p+1})$ so that we find no contributions $\neq 0 \mod (p^{p+1}, v_1, \ldots, v_{i-1})$ of the form (6.8.4) if $j \geq 2$.

Now suppose that $j = 1$, i.e. $F_i = \Delta_{n-i}$. Then we find from (6.8.6) and (6.8.7) that we must have $s_i \geq p - 1$. If $s_i \geq p + 1$ then we again find something $= 0 \mod (p^{p+1})$, so we are left with two subcases of C1 and C3 viz.

Case D: $j = 1$, $F_i = \Delta_{n-i}$, $s_i = p$
Case E: $j = 1$, $F_i = \Delta_{n-1}$, $s_i = p - 1$

In case D we have $s_1 + \ldots + s_m = p^i$, $s_1 = p$, hence $s_2 = \ldots = s_m = 0$ and (6.8.4) gives a contribution

\[(6.8.8) \quad r_{n-i}(v_{n-i})^p = p^p\]

to $c_{n-i}(v_{n-i})$.

Now suppose we are in case E. Then (6.8.4) reduces to

\[(6.8.9) \quad p^i r_{n-i}(v_{n-i})^{p-1} r_{n-i}(v_{n-i}) = p^p r_{n-i}(v_{n-i})\]

for a certain exponent sequence $F$ with $\|F\| \leq p^{n-1} - p^i$. On the other hand we must have $\|(p-1) \Delta_{n-i} + F\| \geq p^n - p^i - p + 1$. It follows that we must have
(6.8.10) \[ \| F \| = p^{n-1} - p^l. \]

But then by induction we know that \( r_F (v_{n-1}) = 0 \mod(p^{r+1}, v_1, \ldots, v_{l-1}) \) except in the following cases:

\[
 r_F (v_{n-1}) = v_1 \mod(p^{r+1}, v_1, \ldots, v_{l-1}) \quad \text{if} \quad F = p^l \Delta_{n-1-h} \quad n \geq 4,
\]

\[
 r_F (v_{n-1}) = -p^l v_i \mod(p^{r+1}, v_1, \ldots, v_{l-1}) \quad \text{if} \quad F = \Delta_1 + (p-1) \Delta_{n-2} + p^l \Delta_{n-2-h} \quad n \geq 4,
\]

\[
 r_F (v_1) = (1 - p^{r-1} - p^s) v_i \quad \text{if} \quad n = 3, \quad F = p \Delta_1 \quad \text{(and, necessarily,} \quad l = 1). \]

It follows that the only contribution \( \neq 0 \mod(p^{r+1}, v_1, \ldots, v_{l-1}) \) of the form (6.8.9) is congruent to \( p^l v_i \mod(p^{r+1}, v_1, \ldots, v_{l-1}). \) We have now proved that

6.9. Lemma. Let \( n \geq 3, \quad 1 \leq l \leq n-1, \quad \| E \| \equiv p^n - p^l, \) then \( c_E (t_i \delta_{n-l}^{x_i}) = 0 \mod(v_1, \ldots, v_{l-1}, p^{r+1}) \) except in the following cases:

(i) \( j = 1, \quad l = n-1, \quad E = \Delta_1 + p \Delta_{n-1}, \quad c_E (t_1 \delta_{n-1}^{x_1}) = p^r, \)

(ii) \( j = 1, \quad l < n-1, \quad E = \Delta_1 + p \Delta_{n-1}, \quad c_E (t_1 \delta_{n-1}^{x_1}) = p^r, \)

(iii) \( j = 1, \quad l < n-1, \quad E = \Delta_1 + (p-1) \Delta_{n-2} + p \Delta_{n-2-h}, \quad c_E (t_1 \delta_{n-2}^{x_1}) = p^s v_i \) where the congruences are all \( \mod(p^{r+1}, v_1, \ldots, v_{l-1}). \)

6.10. Proof of Lemma 6.2(i). Conclusion. According to (6.5.6) we have \( \mod(p^{r+1}, v_1, \ldots, v_{l-1}) \)

\[
c_E (\tilde{v}_n) = c_E \left( p t_n + v_i t_{n-1} - \sum_{j=1}^{l-1} t_j \delta_{n-1}^{x_j} \right).
\]

Now let \( p^n - p^{l-1} > \| E \| \equiv p^n - p^l. \) Then because \( l \geq 2 \) only case (iii) of Lemma 6.9 applies and we find that \( c_E (\tilde{v}_n) = 0 \mod(p^{r+1}, v_1, \ldots, v_{l-1}) \) except when \( E = p^l \Delta_{n-1} \) or \( E = \Delta_1 + (p-1) \Delta_{n-2} + p \Delta_{n-2-h} \) and in these two cases \( c_E (\tilde{v}_n) \) is respectively congruent to \( v_i \) and \( -p^s v_i. \)

6.11. Proof of Lemma 6.2 (iii). Conclusion. According to (6.5.5) we have \( \mod(p^{r+1}) \)

\[
c_E (\tilde{v}_n) = c_E \left( p t_n - p^{r-1} v_i t_{n-1} + v_i t_{n-2} - \sum_{j=1}^{l-1} t_j \delta_{n-1}^{x_j} \right).
\]

Now let \( p^n - 1 > \| E \| \equiv p^n - p. \) Then only case (iii) of Lemma 6.9 applies and we find that \( c_E (\tilde{v}_n) = 0 \mod(p^{r+1}) \) except when \( E = p \Delta_{n-1} \) or \( E = \Delta_1 + (p-1) \Delta_{n-2} + p \Delta_{n-2-h} \) and in these two cases \( c_E (\tilde{v}_n) \) is respectively congruent to \( (1 - p^{r-1}) v_i \) and \( -p^s v_i. \)

6.12. Lemma 6.2 is now completely proved. Note that cases (i) and (ii) of Lemma 6.9 deal with exponent sequences \( E \) with \( \| E \| = p^n - 1, \) which are therefore covered by part (ii) of Lemma 6.2.
7. The linear part of the Brown-Peterson cohomology operations map \( \eta_R \)

In this section we calculate \( \eta_R(v_n) \) modulo the ideal \((t_1, t_2, \ldots)^2\), or, equivalently, we calculate \( \bar{V}_n \) modulo \((T_1, T_2, \ldots)^2\).

7.1. We write \( B_i \) for the element \( p^i a_i(V) \in \mathbb{Z}[V_1, V_2, \ldots] \), where \( a_i(V) \) is defined by (3.1.2). Let \( J \) denote the ideal \((T_1, T_2, \ldots)^2\) in \( \mathbb{Z}[V; T] \).

**Theorem.** **Modulo** \( J \) we have

\[
\bar{V}_n = \sum (-1)^{y} (B_i V_n^{s_{i-1}}) (B_{i-1} V_n^{s_{i-2}}) \cdots (B_{i-k} V_n^{s_{i-k-1}}) \cdots \cdot (B_{i-k} V_n^{s_{i-k-1}}) \cdot (- T_i V_l^y)
\]

(7.1.1)

\[
+ \sum (-1)^{y} (B_i V_n^{s_{i-1}}) (B_{i-1} V_n^{s_{i-2}}) \cdots (B_{i-k} V_n^{s_{i-k-1}}) \cdot (p T_i) + V_n
\]

where the first sum is over all sequences \((s_1, \ldots, s_i, i)\) such that \( s_n, i, j \in \mathbb{N}, s_i + \ldots + s_i + i + j = n, t \in \mathbb{N} \cup \{0\}\) and the second sum is over all sequences \((s_1, \ldots, s_i, i)\) such that \( s_n, i \in \mathbb{N}, s_i + \ldots + s_i + i = n, t \in \mathbb{N} \cup \{0\}\).

7.2. **Example.**

\[
V_1 = B_1 V_n^{s_1} T_1 V_l^y - T_i V_n^{s_1} - T_2 T_1 V_l^y + B_1 V_n^{s_1} B_1 V_l^y (p T_i)
\]

\[
- B_2 V_n^{s_1} (p T_i) - B_1 V_n^{s_1} (p T_i) + p T_2 + V_n
\]

The proof of Theorem 7.1 uses the recursion formula (5.3.1). First two lemmas:

7.3. **Lemma.**

\[
\bar{V}_n = V_n + p T_n + \sum_{k=1}^{n} a_{n-k}(V) (V_k^{s_k-k} - \bar{V}_k^{s_k-k}) + \sum_{i=1}^{n} T_i \bar{V}^{s_i-i}_n
\]

where the congruence is modulo \( J = (T_1, T_2, \ldots)^2 \).

This follows immediately from formula (5.3.1)

7.4. **Lemma.** **Suppose that** \( \bar{V}_k = V_k + \sum T_i C_i \) **modulo** \( J \) **for certain** \( C_i \in \mathbb{Z}[V; T] \). Then

\[
\bar{V}_l = V_l^{s_l} + p^{s_l} V_l^{s_l-i} \left( \sum T_i C_i \right) \mod J.
\]

**Proof.** Obvious.

7.5. **Proof of Theorem 7.1.** Theorem 7.1 is proved by induction, the case \( n = 1 \) being trivial. Given formula (7.1.1) for all \( k < n \) we have that \( \bar{V}_k = V_k \mod (T_1, T_2, \ldots) \) so that we can apply Lemma 7.4. Substituting the result in (7.3.1) then proves (7.1.1).
7.6. Let \( b_n \in \text{BP}_n(pt) \) be the image of \( B_n \) under \( \mathbb{Z}[V_1, V_2, \ldots] \to \text{BP}_n(pt) \), where the \( V_i \) are the generators of \( \text{BP}_n(pt) \) determined by formula (3.1.3); i.e. \( b_n = p^n u_n = \text{CP}^{n-1} = p^nm_{p^{n-1}} \). In view of 4.4 we obtain

\[
7.7. \text{Corollary. For } 0 < i < n \text{ we have}
\]

\[
r_{n,i}(u_n) = \sum_{s_1 + \ldots + s_i = n - i} (-1)^s (b_{s_1} V^{p^{s_1}}_{n-s_1} \ldots V^{p^{s_i}}_{n-s_i}) (- u^{p^{s_i}}_{n-s_i} - \ldots - u^{p^{s_1}}_{n-s_1})
\]

(7.7.1)

where the first sum is over all sequences \( (s_1, \ldots, s_i) \) with \( s_s, t \in \mathbb{N} \) and \( s_i + \ldots + s_i < n - i \) and the second sum is over all sequences \( (s_1, \ldots, s_i) \), \( s_s \in \mathbb{N} \), with \( s_1 + \ldots + s_i = n - i \).

7.8. Let \( I \) denote the ideal of \( \mathbb{Z}[V; T] \) generated by the elements \( pT_i, i = 1, 2, \ldots \); \( T_i, T_j, i, j = 1, 2, \ldots \) Now

\[
7.8.1 \quad B_n = V_1V \ldots V^{p^{n-1}} \text{ mod}(p).
\]

It follows that

7.9. \text{Corollary. Modulo } I \text{ we have}

\[
7.9.1 \quad V_n = \sum (-1)^s T^s_{p^{s_1}} \ldots p^{s_i} V^{p^{s_i}}_{n-s_i} \ldots V^{p^{s_i}}_{n-s_i} (- T_i V^i_{p^i})
\]

where the sum is over all sequences \( (s_1, \ldots, s_i, i, j) \) such that \( s_s, i, j, t \in \mathbb{N} \) and \( s_i + \ldots + s_i + i + j = n \).

This corollary can be used to give a noncohomological proof of the Lubin-Tate formal moduli theorem. Cf. [8, part V]. Warning: the starting formula (2.2.1) in [8, part V] is not correct and should be replaced with (7.9.1) above; the proof of the Lubin-Tate Theorem remains mutatis mutandis the same.

7.10. \text{Corollary. For } 0 < i < n \text{ we have}

\[
r_{n,i}(u_n) = - u^{p^{n-i}}_{n-i}, \text{mod}(p, v_1).
\]

7.11. \text{Corollary. For } 0 < i < n - 1 \text{ we have}

\[
r_{n,i}(u_n) = - v^{p^{n-i}}_{n-i} + v_1 v^{p^{n-i-1}}_{n-i-1} \text{mod}(p, v_1^i).
\]

More generally let \( r = \min(n - i, -1, p) \), then we have

\[
r_{n,i}(u_n) = - v^{p^{n-i}}_{n-i} + v_1 v^{p^{n-i-1}}_{n-i-1} - v_2 v^{p^{n-i-2}}_{n-i-2} v^{p^{n-i-3}}_{n-i-3} + \ldots + (-1)^{i-1} v^{p^{i-1}}_{n-i} \ldots v^{p^{i-1}}_{n-i} \text{mod}(p, v_1^p).
\]
8. The functional equation lemma and multiplicative operations in $\text{BP}^*(\text{BP})$

As a final application of the universal isomorphism theorem 2.12 of [5] and the functional equation lemma 7.1 of [5] we reprove the main theorem of [20].

8.1. Choose a prime number $p$. Let $\sigma : \mathbb{Z}_p[V] \to \mathbb{Z}_p[V]$ be the ring homomorphism given by $V_i \mapsto V_i^p$, for $i = 1, 2, \ldots$. If $g(X)$ is a power series with coefficients in $\mathbb{Z}_p[V]$ or $Q[V]$ then $g^\sigma(X)$ denotes the power series obtained by applying $\sigma$ to the coefficients of $g(X)$. We also write $a^\sigma$ for $\sigma(a)$ if $a \in Q[V]$. Part of the functional equation lemma 7.1 of [5] now says

8.2. Functional equation lemma. If $d(X) = X + d_2X^2 + \ldots$ is a power series with $d_i \in \mathbb{Z}_p[V]$ and $f_v(X)$ is the logarithm of the $p$-typically universal formal group $F_v(X, Y)$ of [5] and [7], then there are unique elements $e_1, e_2, \ldots \in \mathbb{Z}_p[V]$ such that

\[(\text{8.2.1}) \quad g(X) - \sum_{i=1}^p p^i V g^{\sigma^i}(X^{p^i}) = X + \sum_{i=1}^p e_i X^i\]

where $g(X) = f_v(d(X))$. Inversely given a power series $g(X) = X + \sum_{i=1}^c c_i X^i$, $c_i \in Q[V]$ such that (8.2.1) holds for certain $e_i \in \mathbb{Z}_p[V]$, then there exists a unique power series $d(X) = X + d_2X^2 + \ldots$ with $d_i \in \mathbb{Z}_p[V]$ such that $g(X) = f_v(d(X))$.

8.3. Corollary. If $d(X)$ is such that $g(X) = X + \sum_{i=1}^c c_i X^i$, i.e. $c_i = 0$ if $i$ is not a power of $p$, then $e_i = 0$ if $i$ is not a power of $p$ and writing $s_\alpha$ for $e_\alpha$, we have

\[(\text{8.3.1}) \quad c_\alpha = \sum_{k=0}^\alpha a_{\alpha-k} s_k^{\alpha-k}\]

where $a_\alpha$ is the coefficient of $X^\alpha$ in $f_v(X)$.

This follows immediately from (8.2.1) above because $a_\alpha$ satisfies

\[(\text{8.3.2}) \quad a_\alpha = \sum_{i=1}^\alpha p^{-1} V a^{\sigma^i} \quad \text{and} \quad a_\alpha = \sum_{i=1}^\alpha p^{-1} a^{\sigma^i} V^{\sigma^{i}}\]

Let $\text{BP}_*(pt) = \mathbb{Z}_p[v_1, v_2, \ldots]$, where the $v_i$ are the free polynomial generators defined by formula (3.1.3) above. Define the homomorphism $\sigma : \text{BP}_*(pt) \to \text{BP}_*(pt)$ by $v_i \mapsto v_i^p$. Let $l_i = m_{i-1} \in \text{BP}_*(pt) \otimes Q$, cf. 3.1.

8.4. Theorem (Ravenel [20]). For every sequence of elements $(r_1, r_2, \ldots)$ in $\text{BP}_*(pt)$ there is a unique sequence of elements $(s_1, s_2, \ldots)$ in $\text{BP}_*(pt)$ such that

\[(\text{8.4.1}) \quad \sum_{i=0}^\alpha l_i r_i^{p^i} = \sum_{i=0}^\alpha l_i s_i^{p^i}\]

for every $n > 0$. Inversely for every sequence $(s_1, s_2, \ldots)$ in $\text{BP}_*(pt)$ there is a unique sequence $(r_1, r_2, \ldots)$ such that (8.3.1) holds for all $n > 0$. 

Proof. Identify $\mathbb{Z}_p[V]$ with $\text{BP}_*(pt)$ via $V_i \mapsto v_i$. The element $a_i$ in $\mathbb{Q}[V]$ then corresponds with $l_i \in \text{BP}_*(pt) \otimes \mathbb{Q}$. Take a sequence of elements $(r_1, r_2, \ldots)$ in $\text{BP}_*(pt)$. Let $\phi : \mathbb{Z}_p[V; T] \to \text{BP}_*(pt)$ be the ring homomorphism defined by $V_i \mapsto v_i, T_i \mapsto r_i$. Write $G(X, Y) = F^* \tau(X, Y)$. Let $g(X)$ be the logarithm of $G(X, Y)$, then, cf. (4.1.3)

$$g(X) = X + \sum_{n=1}^{\infty} c_n X^n, \quad c_n = \sum_{i=0}^{n} l_i \tau_i^{*n}.$$  

The formal group $G(X, Y)$ is strictly isomorphic to $F^*(X, Y) = \mu_{\text{BP}}(X, Y)$ over $\text{BP}_*(pt)$, and the isomorphism is equal to the inverse of $\alpha^* \tau(X)$. Cf. [5, Theorem 2.12] and 4.1 above. It follows that there is a power series $d(X) = X + d_2 X^2 + \ldots$ with $d_i \in \text{BP}_*(pt)$ such that $g(X) = f_*(d(X))$. (In fact $d^{-1}(X) = \alpha^* \tau(X)$.) Now apply Corollary 8.3, to find $s_i$ such that (8.4.1) holds.

Inversely given elements $(s_1, s_2, \ldots)$ in $\text{BP}_*(pt)$, let $g(X)$ be the power series

$$g(X) = X + \sum_{n=1}^{\infty} \sum_{i=0}^{n} l_i s_i^{*n}$$

then $g(X)$ satisfies a functional equation (8.2.1) and hence again by the functional equation lemma, there exists a power series $d(X) = X + d_2 X^2 + \ldots, d_i \in \text{BP}_*(pt)$ such that $g(X) = f_*(d(X))$. It follows that $g(X)$ is the logarithm of a $p$-typical formal group $G(X, Y)$ which is strictly isomorphic over $\text{BP}_*(pt)$ to $F^*(X, Y) = \mu_{\text{BP}}(X, Y)$. By the universality of the triple $(F^*(X, Y), \alpha^* \tau(X), F^* \tau(X, Y))$ there is therefore a unique homomorphism $\psi : \mathbb{Z}_p[V; T] \to \text{BP}_*(pt)$, such that $\psi(V_i) = v_i$ and $f_*(\psi(T)) = g(X)$. Let $r_i = \psi(T_i) \in \text{BP}_*(pt)$. Then because of (4.1.3)

$$g(X) = X + \sum_{n=1}^{\infty} \sum_{i=0}^{n} l_i \tau_i^{*n}.$$ 

This concludes the proof of the theorem.

References