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CONSTRUCTING FORMAL GROUPS III: APPLICATIONS TO COMPLEX COBORDISM AND BROWN–PETERSON COHOMOLOGY

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1. Introduction

In parts I and II, cf. [5], [6] and also [7], of this series of papers we constructed a universal p-typical one dimensional commutative formal group and a universal one dimensional commutative formal group. The extraordinary cohomology theories BP (Brown–Peterson cohomology) and MU (complex cobordism cohomology) are complex oriented and hence define one dimensional formal groups over $\mathrm{BP}_*(pt)$ and $\mathrm{MU}_*(pt)$ respectively. Cf. [1]. These formal groups are respectively p-typically universal and universal. Cf. [1], [3], [4] and [18]. Let μ_{BP} and μ_{MU} denote these formal groups. The logarithms of the formal groups μ_{BP} and μ_{MU} are known, cf. [17], and have a very simple expression in terms of the cobordism classes of the complex projective spaces. Using the formulas for the logarithms of the universal formal groups of [5] and [6] one then obtains a free polynomial basis for $\mathrm{BP}_*(pt)$ and $\mathrm{MU}_*(pt)$ in terms of the classes of the complex projective spaces. This is the subject matter of Sections 2, 3 below.

In [5] we also constructed a universal isomorphism between p-typical formal groups. The associated map $\mathbf{Z}[V_1, V_2, \dots] \to \mathbf{Z}[V_1, V_2, \dots; T_1, T_2, \dots]$ (when localized at p) can be identified with the right unit map $\mu_{\mathrm{R}} : \mathrm{BP}_*(pt) \to \mathrm{BP}_*(\mathrm{BP})$ of the Hopf algebra $\mathrm{BP}_*(\mathrm{BP})$.

In Sections 4, 5 below, we use the universal isomorphism of [5] to obtain a recursive description of the homomorphism η_{R}. This description is useful in the calculation of various BP cohomology operations, cf. Sections 6, 7 below. To obtain this recursive description of η_{R} we need an isomorphism formula (Section 5 below) which is also useful in the theory of formal groups itself, cf. part III of [8].

Finally in Section 8, we use the universal isomorphism and the functional equation lemma of [5] to derive the main theorem of [20]. All formal groups in this paper will be commutative and one dimensional. Some of the results of this paper were announced in [7], [11].

Acknowledgements. Luilevicius [16] was the first to write down a formula similar to (3.1.3) and to prove that it gives generators for $\mathrm{BP}_*(pt)$ in the case $p = 2$.

Once one has the various universal p-typical formal groups (which are more or less canonical) they can be fitted together in various ways (all noncanonical). One way to do this is described in part II of [8] and gives the generators for $\mathrm{MU}_*(pt)$ described in [8, part II] and [7]. Subsequently Kozma [14] wrote down a different set of polynomial generators for $\mathrm{MU}_*(pt)$, which satisfy more elegant recursion formulas. These generators correspond to a different way of fitting the various universal p-typical formal groups together, which, however, does not generalize to more dimensional formal groups, but does generalize if one restricts attention to more dimensional curvilinear formal groups. Cf. the introduction of [6] and [12] for more details.

2. The formal groups of complex cobordism and Brown–Peterson cohomology

2.1. Complex oriented cohomology theories. Let h^* be a complex oriented cohomology theory (defined on finite CW complexes); and let $e^h(L)$ denote the Euler class in $h^*(X)$ of a complex line bundle L over X. Cf. [3, part I, §5], [1, part II, §2], or [19] for a definition of "complex oriented".

For complex line bundles L_1, L_2 one has

$$(2.1.1) \qquad e^h(L_1 \otimes L_2) = \sum_{i,j} a_{ij} e^h(L_1)^i e^h(L_2)^j$$

with $a_{ij} \in h_*(pt)$, and by naturality the coefficients a_{ij} do not depend on L_1 and L_2. So we have a well-defined formal power series

$$(2.1.2) \qquad F_h(X, Y) = \sum a_{ij} X^i Y^j$$

which in fact defines a (one dimensional commutative) formal group over $h_*(pt)$ by commutativity and associativity of tensor products and naturality of Euler classes.

2.2. The formal groups of MU and BP. Choose a prime number p. Let MU stand for the complex cobordism spectrum and BP for the Brown–Peterson spectrum associated to the prime number p. These theories are complex oriented. Let μ_{MU} and μ_{BP} be the associated formal groups. Cf. [1], [3], [19]. Let $\log_{\mathrm{MU}}$ and $\log_{\mathrm{BP}}$ be their logarithmic series, i.e.

$$(2.2.1) \qquad \mu_{\mathrm{MU}}(X, Y) = \log_{\mathrm{MU}}^{-1}(\log_{\mathrm{MU}}(X) + \log_{\mathrm{MU}}(Y)),$$

(2.2.2) $\mu_{BP}(X, Y) = \log_{BP}^{-1}(\log_{BP}(X) + \log_{BP}(Y))$.

One then has (Miščenko's theorem, cf. [17])

$$\log_{MU}(X) = \sum_{n\geq 0} m_n X^{n+1}, \tag{2.2.3}$$

$$\log_{BP}(X) = \sum_{n\geq 0} m_{p^n-1} X^{p^n} \tag{2.2.4}$$

with $m_0 = 1$ and $m_n = (n+1)^{-1}[\mathbf{CP}^n]$, where $[\mathbf{CP}^n]$ is the cobordism class of complex projective space of (complex) dimension n. Cf. [1], [3], [17], [18].

The formal group μ_{MU} is universal by a theorem of Quillen [18] and it follows immediately that μ_{BP} is p-typically universal. Cf. also [3].

3.1. Generators for $BP_*(pt)$. Choose a prime number p. Let $f_V(X)$ be the power series defined by formula (2.2.1) in [5] (cf. also [7]) and let $F_V(X, Y) = f_V^{-1}(f_V(X) + f_V(Y))$. According to Theorems 2.3 and 2.8 of [5] $F_V(X, Y)$ is a p-typically universal formal group over $\mathbf{Z}[V] = \mathbf{Z}[V_1, V_2, V_3, \dots]$. Write

$$f_V(X) = \sum_{i=0}^{\infty} a_i(V) X^{p^i}, \qquad a_0(V) = 1 \tag{3.1.1}$$

then we have according to formula (4.3.1) of [5]:

$$p a_n(V) = a_{n-1}(V) V_1^{p^{n-1}} + \dots + a_1(V) V_{n-1}^p + V_n. \tag{3.1.2}$$

Because $F_V(X, Y)$ over $\mathbf{Z}_{(p)}[V]$ and $\mu_{BP}(X, Y)$ over $BP_*(pt)$ are both p-typically universal formal groups (for p-typical formal groups over $\mathbf{Z}_{(p)}$-algebras) there exist (cf. [5, Definition 2.4]) mutually inverse isomorphims $\phi : \mathbf{Z}_{(p)}[V] \to BP_*(pt)$, $\psi : BP_*(pt) \to \mathbf{Z}_{(p)}[V]$ such that ϕ applied to the coefficients of $f_V(X)$ gives the coefficients of $\mu_{BP}(X)$. Applying ϕ to (3.1.2) and writing v_i for $\phi(V_i)$ we therefore find elements $v_1, v_2, v_3, \dots$ of $BP_*(pt)$ which constitute a free polynomial basis for $BP_*(pt)$ and which are related to the $m_n = (n+1)^{-1}[\mathbf{CP}^n]$ of (2.2.4) above by the relations

$$p l_n = l_{n-1} v_1^{p^{n-1}} + l_{n-2} v_2^{p^{n-2}} + \dots + l_1 v_{n-1}^p + v_n \tag{3.1.3}$$

where we have written l_n for m_{p^n-1}.

3.2. Generators for $MU_*(pt)$. Let $f_U(X)$ be the power series defined by formulas (2.2.1) and (2.2.4) of [6], and let $F_U(X, Y) = f_U^{-1}(f_U(X) + f_U(Y))$. According to Theorems 2.3 and 2.4 of [6] $F_U(X, Y)$ is a universal formal group over $\mathbf{Z}[U] = \mathbf{Z}[U_2, U_3, U_4, \dots]$. Write

$$f_U(X) = \sum_{n=1}^{\infty} b_n(U) X^n, \qquad b_1 = 1. \tag{3.2.1}$$

Then if we specify the coefficients $n(i_1, \dots, i_s)$ occurring in the definition of $f_U(X)$ according to [6, Section 7] we have the following recursion formula:

$$(3.2.2)\qquad \nu(n)b_n(U) = U_n + \sum_{\substack{d \mid n \\ d \neq 1, n}} \frac{\mu(n,d)\nu(n)}{\nu(d)} b_{n/d}(U)U_d^{n/d}$$

where the integers $\nu(n)$ and $\mu(n, d)$ are defined as follows:

$$(3.2.3)\qquad \begin{aligned} &\nu(n) = 1 \text{ if } n \text{ is not a power of a prime number} \\ &\nu(p^r) = p \text{ for all prime numbers } p \text{ and } r \in \mathbf{N} = \{1, 2, 3, \dots\}, \end{aligned}$$

$$(3.2.4)\qquad \mu(n, d) = \prod_{p \mid n} c(p, d)$$

where the product is defined over all prime numbers p dividing n and the $c(p, d)$ are integers which can be chosen arbitrarily subject to

$$(3.2.5)\qquad \begin{aligned} &c(p, d) = 1 \quad \text{if } \nu(d) = 1, p, \\ &c(p, d) \equiv \begin{cases} 1 & \bmod p \\ 0 & \bmod q \end{cases} \quad \text{if } \nu(d) = q \neq p. \end{aligned}$$

More precisely: first one chooses $c(p, d) \in \mathbf{Z}$ for all prime numbers p and $d \in \mathbf{N}$ such that (3.2.5) holds: then one constructs $f_U(X)$ and $F_U(X, Y)$ according to the formulas (7.1.2), (7.1.3), (2.2.1), (2.2.4) and (2.2.7) of [6]; the result is then a universal formal group $F_U(X, Y)$ over $\mathbf{Z}[U]$ with logarithm $f_U(X)$ satisfying (3.2.2) with $\nu(n)$ and $\mu(n, d)$ given by (3.2.3) and (3.2.4). Different choices for the $c(p, d)$ result in different universal formal groups $F_U(X, Y)$. Because $F_U(X, Y)$ over $\mathbf{Z}[U]$ and $\mu_{\mathrm{MU}}(X, Y)$ over $\mathrm{MU}_*(pt)$ are both universal formal groups there are mutually inverse isomorphisms $\phi : \mathbf{Z}[U] \to \mathrm{MU}_*(pt)$, $\psi : \mathrm{MU}_*(pt) \to \mathbf{Z}[U]$ such that ϕ applied to the coefficients of $f_U(X)$ gives the coefficients of $\mu_{\mathrm{MU}}(X)$. Applying ϕ to (3.2.2) and writing u_i, $i = 1, 2, \dots$ for $\phi(U_i)$ we find elements $u_2, u_3, \dots$ in $\mathrm{MU}_*(pt)$ which constitute a free polynomial basis for $\mathrm{MU}_*(pt)$ and which are related to the $m_n = (n+1)^{-1}[\mathbf{CP}^n]$ by the formula

$$(3.2.6)\qquad \nu(n)m_{n-1} = u_n + \sum_{\substack{d \mid n \\ d \neq 1, n}} \frac{\mu(n,d)\nu(n)}{\nu(d)} m_{(n/d)-1}u_d^{n/d}.$$

These are the same generators as those written down by Kozma [14]. Note that the factor $\nu(d)^{-1}\mu(n, d)\nu(n)$ is always an integer.

If one uses instead of the universal formal group $F_U(X, Y)$ of [6], the universal formal group $H_U(X, Y)$ over $\mathbf{Z}[U]$ of [6] then, reasoning in exactly the same way, one finds generators $\bar{u}_i$ in $\mathrm{MU}_*(pt)$ which are related to the m_n by the formula

$$(3.2.7)\qquad \nu(n)m_{n-1} = \bar{u}_n + \sum_{i=1}^{\infty} (-1)^i \sum^{(i)} \frac{\mu(n,d_1)\nu(n)}{\nu(d_1)} m_{d-1}\bar{u}_{d_i}^{d}\bar{u}_{d_{i-1}}^{dd_i} \cdots \bar{u}_{d_1}^{dd_i \cdots d_2}$$

where $\Sigma^{(i)}$ is the sum over all sequences $(d, d_i, d_{i-1}, \dots, d_1)$ such that $d, d_i, \dots, d_1 \in \mathbf{N}$, $d_1 \neq 1, n$; $d_j > 1$ and not a power of a prime number for $j = 2, \dots, i$ and $dd_i \dots d_1 = n$. These are the generators given in [7] and [8, part II].

3.3. Remark. BP is a direct summand of $\mathrm{MU}\mathbf{Z}_{(p)}$. If we identify u_{p^i} with v_i formula (3.2.6) (or formula (3.2.7) for that matter) reduces to formula (3.1.3) if n is a power of p. It follows that the v_i are integral, i.e. they live in $\mathrm{MU}_*(pt)$, not just in $\mathrm{MU}\mathbf{Z}_{(p)*}(pt)$. This is also proved in [2].

4. Isomorphisms of p-typical formal groups and $\eta_{\mathrm{R}}: \mathrm{BP}_*(pt) \to \mathrm{BP}_*(\mathrm{BP})$

4.1. Universal strict isomorphisms of p-typical formal groups. In [5] we also constructed a universal strict isomorphism

(4.1.1) $\qquad \alpha_{V,T}(X): F_V(X,Y) \to F_{V,T}(X,Y)$

for p-typical formal groups over characteristic zero rings or $\mathbf{Z}_{(p)}$-algebras. Here $F_{V,T}(X,Y)$ is a p-typical formal group over $\mathbf{Z}[V;T] = \mathbf{Z}[V_1, V_2, \ldots; T_1, T_2, \ldots]$ and the logarithm $f_{V,T}(X)$ of $F_{V,T}(X,Y)$ satisfies

(4.1.2) $\qquad f_{V,T}(X) = \sum_{i=0}^{\infty} a_i(V,T)X^{p^i},$

(4.1.3) $\qquad a_i(V,T) = a_i(V) + a_{i-1}(V)T_1^{p^{i-1}} + \ldots + a_1(V)T_{n-1}^p + T_n$

cf. formula (4.3.2) of [5].

Let $I: \mathbf{Z}_{(p)}\text{-Alg} \to \text{Sets}$ be the functor which associates to every $\mathbf{Z}_{(p)}$-algebra A the set of all triples $(F(X,Y), \alpha(X), G(X,Y))$ where $F(X,Y)$ and $G(X,Y)$ are p-typical formal groups over A and $\alpha(X)$ is a strict isomorphism form $F(X,Y)$ to $G(X,Y)$. If we restrict attention to $\mathbf{Z}_{(p)}$-algebras theorem 2.12 of [5] says

4.2. Theorem. *The $\mathbf{Z}_{(p)}$-algebra $\mathbf{Z}_{(p)}[V,T]$ represents the functor I.*

The isomorphism $\mathbf{Z}_{(p)}\text{-Alg}\ (\mathbf{Z}_{(p)}[V,T], A) \xrightarrow{\sim} I\ (A)$ looks as follows. Let $\phi: \mathbf{Z}_{(p)}[V,T] \to A$ be a $\mathbf{Z}_{(p)}$-algebra homomorphism. Let $v_i = \phi(V_i)$, $t_i = \phi(T_i)$, $i = 1, 2, \ldots$ then the triple associated to ϕ is $(F_v(X,Y), \alpha_{v,t}(X), F_{v,t}(X,Y))$.

4.3. The homomorphism $V_i \mapsto \bar{V}_i$. $F_{V,T}(X,Y)$ is a p-typical formal group over $\mathbf{Z}[V;T]$. By the universality of $F_V(X,Y)$ there are therefore unique polynomials $\bar{V}_i \in \mathbf{Z}[V;T]$ such that $F_{V,T}(X,Y) = F_{\bar{V}}(X,Y)$. Note that the $\bar{V}_i$ have their coefficients in $\mathbf{Z}$ not just in $\mathbf{Z}_{(p)}$.

We have just defined a homomorphism

(4.3.1) $\qquad \nu_{\mathrm{R}}: \mathbf{Z}[V] \to \mathbf{Z}[V;T], \qquad V_i \mapsto \bar{V}_i.$

A more functorial way of looking at this homomorphism is as follows. Let $F: \mathbf{Z}_{(p)}\text{-Alg} \to \text{Sets}$ be the functor which associates to a $\mathbf{Z}_{(p)}$-algebra A the set of all p-typical formal groups over A. Then F is represented by $\mathbf{Z}_{(p)}[V]$, (by the universality of $F_V(X,Y)$). There are two natural functor morphims $I \to F$, viz.

$$(4.3.2) \qquad I(A) \to F(A), \qquad (F(X,Y), \alpha(X), G(X,Y)) \mapsto F(X,Y),$$

$$(4.3.3) \qquad I(A) \to F(A), \qquad (F(X,Y), \alpha(X), G(X,Y)) \mapsto G(X,Y)$$

and because $\mathbf{Z}_{(p)}[V;T]$ represents I and $\mathbf{Z}_{(p)}[V]$ represents F we obtain two $\mathbf{Z}_{(p)}$-algebra homomorphisms $\mathbf{Z}_{(p)}[V] \to \mathbf{Z}_{(p)}[V,T]$. The homomorphism induced by (4.3.2) is the natural inclusion $\mathbf{Z}_{(p)}[V] \to \mathbf{Z}_{(p)}[V,T]$ and the homomorphism induced by (4.3.3) is the localization in p of (4.3.1).

4.4. The Hopf-algebra $\mathrm{BP}_*(\mathrm{BP})$. By Theorem 16.1 of [1, part II] we know that $\mathrm{BP}_*(\mathrm{BP}) = \mathrm{BP}_*(pt)[t_1, t_2, \ldots] = \mathbf{Z}_{(p)}[v_1, v_2, \ldots; t_1, t_2, \ldots]$. It follows that $\mathrm{BP}_*(\mathrm{BP})$ represents the functor I. This fact can be used to account for the Hopf-algebra structure of $\mathrm{BP}_*(\mathrm{BP})$ by using various functor morphisms like (4.3.2) and (4.3.3) above. This was done in [15]. The structure of $\mathrm{BP}_*(\mathrm{BP})$ as a left module over $\mathrm{BP}_*(pt)$ is then given by the natural inclusion $\mathrm{BP}_*(pt) \hookrightarrow \mathrm{BP}_*(pt)[t_1, t_2, \ldots]$ and the structure of $\mathrm{BP}_*(\mathrm{BP})$ as a right module over $\mathrm{BP}_*(pt)$ is given by a homomorphism $\eta_{\mathrm{R}}: \mathrm{BP}_*(pt) \to \mathrm{BP}_*(\mathrm{BP})$ which is the localization in p of ν_{R} in (4.3.1) above if we identify $\mathrm{BP}_*(pt)$ with $\mathbf{Z}_{(p)}[V]$ and $\mathrm{BP}_*(\mathrm{BP})$ with $\mathbf{Z}_{(p)}[V;T]$ by means of $v_i \leftarrow V_i$, $t_i \leftarrow T_i$, where the v_i are the generators defined in 3.1 above. Alternatively we can appeal again to Theorem 16.1 of [1, part II] where it is shown that $\eta_{\mathrm{R}} \otimes \mathbf{Q}$ is given by

$$(4.4.1) \qquad l_n \mapsto \sum_{i=0}^{n} l_i t_{n-i}^{p^i}$$

where again $l_n = m_{p^n-1}$. Because $F_{V,T}(X,Y) = F_{\bar{V}}(X,Y)$ and because of formula (4.1.3) this also shows that $\eta_{\mathrm{R}} = \nu_{\mathrm{R}} \otimes \mathbf{Z}_{(p)}$. (If $\phi: \mathbf{Z}_{(p)}[V] \to \mathrm{BP}_*(pt)$ is the isomorphism $V_i \mapsto v_i$, then $\phi(a_i(V)) = l_i$ by (3.1.2) and (3.1.3), hence the right hand side of (4.1.3) becomes the right hand side of (4.4.1) under $\phi: \mathbf{Z}_{(p)}[V;T] \xrightarrow{\sim} \mathrm{BP}_*(\mathrm{BP})$, $V_i \mapsto v_i$, $T_i \mapsto t_i$, $i = 1, 2, \ldots$.)

5. The isomorphism formula

The next thing we want to do is to give a recursion formula for the polynomials $\bar{V}_i$, and hence also a recursive description of $\eta_{\mathrm{R}}: \mathrm{BP}_*(pt) \to \mathrm{BP}_*(\mathrm{BP})$. To do so we first need a formula relating the $\bar{V}_i$ and the V_i which is also useful in its own right, especially when discussing reductions and liftings of formal groups and isomorphisms of formal groups. Cf [8, parts III and V].

5.1. Let $a_i = a_i(V)$ be defined by (3.1.2) and write $\bar{a}_i$ for $a_i(V,T)$, cf. (4.1.3). Then we have $f_V(X) = \sum a_i X^{p^i}$ and $f_{V,T}(X) = \sum \bar{a}_i X^{p^i}$ and because $f_{V,T}(X) = f_{\bar{V}}(X)$, the $\bar{a}_i$ are given by the same formula (3.1.2) with bars over all the symbols occurring. I.e.

$$(5.1.1) \qquad p\bar{a}_n = \bar{a}_{n-1}\bar{V}_1^{p^{n-1}} + \ldots + \bar{a}_1 \bar{V}_{n-1}^{p} + \bar{V}_n.$$

In addition we define

(5.1.2) $$Z_{ij}^{(r)} = (V_i^{p^r} T_j^{p^{r+i}} - T_j^{p^r} V_i^{p^{r+j}}).$$

5.2. Proposition.

(5.2.1) $$p\bar{a}_n = \sum_{i=1}^{n} \bar{a}_{n-i} V_i^{p^{n-i}} + \sum_{i,j\geq 1, i+j\leq n} a_{n-i-j} Z_{i,j}^{(n-i-j)} + pT_n.$$

Proof. Using (4.1.3), (3.1.2) and (5.1.1) we have

$$\begin{aligned}
p\bar{a}_n &= pa_n + \sum_{i=1}^{n} pa_{n-i} T_i^{p^{n-i}} \\
&= \sum_{i=1}^{n-1} a_{n-i} V_i^{p^{n-i}} + V_n + \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} a_{n-i-j} V_j^{p^{n-i-j}} T_i^{p^{n-i}} + pT_n \\
&= \sum_{i=1}^{n-1} \bar{a}_{n-i} V_i^{p^{n-i}} - \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} a_{n-i-j} T_j^{p^{n-i-j}} V_i^{p^{n-i}} + V_n \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} a_{n-i-j} V_j^{p^{n-i-j}} T_i^{p^{n-i}} + pT_n \\
&= \sum_{i=1}^{n} \bar{a}_{n-i} V_i^{p^{n-i}} + \sum_{i,j\geq 1, i+j\leq n} a_{n-i-j} (V_j^{p^{n-i-j}} T_i^{p^{n-i}} - T_j^{p^{n-i-j}} V_i^{p^{n-i}}) + pT_n \\
&= \sum_{i=1}^{n} \bar{a}_{n-i} V_i^{p^{n-i}} + \sum_{i,j\geq 1, i+j\leq n} a_{n-i-j} Z_{ij}^{(n-i-j)} + pT_n.
\end{aligned}$$

(Note that $Z_{ij} + Z_{ji} = (V_i T_j^{p^i} - T_i V_j^{p^i}) + (V_j T_i^{p^j} - T_j V_i^{p^j})$ and similarly for $Z_{ij}^{(r)}$.)

5.3. Proposition.

(5.3.1) $$\begin{aligned}
\bar{V}_n = V_n + pT_n + \sum_{k=1}^{n-1} a_{n-k} &\left\{ (V_k^{p^{n-k}} - \bar{V}_k^{p^{n-k}}) + \sum_{\substack{i+j=k \\ i,j\geq 1}} (V_i^{p^{n-k}} T_j^{p^{n-j}} - T_j^{p^{n-k}} \bar{V}_i^{p^{n-i}}) \right\} \\
&+ \sum_{\substack{i+j=n \\ i,j\geq 1}} (V_i T_j^{p^i} - T_j \bar{V}_i^{p^j}).
\end{aligned}$$

Proof. This follows directly by substituting in (5.2.1) and (5.1.1) $\bar{a}_{n-i} = \sum_{j=0}^{n-i} a_{n-i-j} T_j^{p^{n-i-j}}$, where $T_0 = 1$.

5.4. Remark. Formula (5.3.1) can be used to give an inductive proof that the $\bar{V}_n$ are polynomials with integral coefficients in the $V_1, \ldots, V_n$; $T_1, \ldots, T_n$. Indeed, we know that, cf. [5],

(5.4.1) $$a_{n-k} = \sum_{l=1}^{n-k} p^{-1} V_l a_{n-k-l}^{(p^l)}$$

and assuming that $\bar{V}_i$, $i = 1, \ldots, n-1$ is integral we also have that for all $s \in \mathbf{N}$

(5.4.2) $\bar{V}_i^{p^{s+t}} \equiv (\bar{V}_i^{(p^s)})^{p^t} \bmod p^{t+1}$.

Finally $p^i a_i$ is a polynomial with integral coefficients so that we have in $\mathbf{Q}[V;T]$

$$\begin{aligned}
\bar{V}_n &= V_n + pT_n + \sum_{\substack{i+j=n\\ i,j\geq 1}} (V_i T_j^{p^i} - T_j \bar{V}_i^{p^j}) \\
&\quad + \sum_{k=1}^{n-1} a_{n-k} \left\{ (V_k^{p^{n-k}} - \bar{V}_k^{p^{n-k}}) + \sum_{\substack{i+j=k\\ i,j\geq 1}} (V_i^{p^{n-k}} T_j^{p^{n-k}} - T_j^{p^{n-k}} \bar{V}_i^{p^{n-i}}) \right\} \\
&\equiv \sum_{k=1}^{n-1} \sum_{l=1}^{n-k} \frac{V_l}{p} a_{n-k-l}^{(p^l)} \left\{ (V_k^{p^{n-k}} - \bar{V}_k^{p^{n-k}}) + \sum_{\substack{i+j=k\\ i,j\geq 1}} (V_i^{p^{n-k}} T_j^{p^{n-j}} - T_j^{p^{n-k}} \bar{V}_i^{p^{n-i}}) \right\} \\
&\equiv \sum_{l=1}^{n-1} \frac{V_l}{p} \sum_{k=1}^{n-l-1} a_{n-l-k}^{(p^l)} \left\{ (V_k^{p^l})^{p^{n-l-k}} - (\bar{V}_k^{(p^l)})^{p^{n-l-k}} + \right. \\
&\qquad\qquad \left. + \sum_{\substack{i+j=k\\ i,j\geq 1}} ((V_i^{p^l})^{p^{n-l-k}} (T_j^{p^l})^{p^{n-l-j}} - (T_j^{p^l})^{p^{n-l-k}} (\bar{V}_i^{(p^l)})^{p^{n-l-i}}) \right\} \\
&\quad + \sum_{l=1}^{n-1} \frac{V_l}{p} \left\{ \bar{V}_{n-l}^{(p^l)} - \bar{V}_{n-l}^{(p^l)} + \sum_{i+j=n-l} (V_i^{p^l} (T_j^{p^l})^{p^i} - T_j^{p^l} (\bar{V}_i^{(p^l)})^{p^j}) \right\} \\
&= \sum_{l=1}^{n-1} \frac{V_l}{p} (-pT_l^{p^l}) \equiv 0
\end{aligned}$$

where all congruences are modulo 1 in $\mathbf{Q}[V;T]$. (Two polynomials in $\mathbf{Q}[V;T]$ are $\equiv$ mod 1 if their difference is in $\mathbf{Z}[V;T]$.) This proves the integrality of the $\bar{V}_n$, $n = 1, 2, 3, \ldots$.

6. A generalization of the main lemma of Johnson and Wilson [13]

6.1. BP cohomology operations. The stable cohomology operations of BP cohomology can be described as $\mathrm{BP}_*(pt)$-homomorphisms $\mathrm{BP}_*(\mathrm{BP}) \to \mathrm{BP}_*(pt)$, where $\mathrm{BP}_*(\mathrm{BP})$ is seen as a left module over $\mathrm{BP}_*(pt)$. Cf. [1] and also 4.3 and 4.4 above. To find out what a cohomology operation r does with elements of $\mathrm{BP}_*(pt)$, compose r with the right unit map $\eta_R : \mathrm{BP}_*(pt) \to \mathrm{BP}_*(\mathrm{BP})$. Let $E = (e_1, e_2, \ldots)$ be a sequence of ≥ 0 integers of which only finitely many are nonzero. The cohomology operation r_E is defined as: coefficient of t^E in $x \in \mathrm{BP}_*(\mathrm{BP}) = \mathrm{BP}_*(pt)[t_1, t_2, \ldots]$. Thus $r_E(v_n) =$ coefficient of t^E in $\bar{v}_n$, where $\bar{v}_n$ is obtained from $\bar{V}_n$ by replacing V_i with v_i and T_i with t_i, $i = 1, \ldots, n$.

Assign to an exponent sequence $E = (e_1, e_2, \ldots)$ the weight $\|E\| = e_1(p-1) + e_2(p^2-1) + \ldots$ and to v_i the weight $p^i - 1$. We then have

(6.1.1) $$\eta_R(v_n) = \bar{v}_n = \sum_{\|E\| \leq p^n - 1} r_E(v_n) t^E$$

where $r_E(v_n)$ is homogeneous of weight $p^n - 1 - \|E\|$.

In [13] Johnson and Wilson calculate $r_E(v_n)$ modulo $(p, v_1, \ldots, v_{l-1})$ for $\|E\| \geq p^n - p^l$, ([13, Lemma 1.7] (sometimes known as the Budweiser lemma)).

As a first application of the recursion formula (5.3.1) we shall calculate in this section $r_E(v_n)$ modulo $(p^{p+1}, v_1, \ldots, v_{l-1})$ for all E with $\|E\| \geq p^n - p^l$.

6.2. Extension of the main lemma. Write Δ_i for the exponent sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ with the 1 in the i-th place. We also write $\Delta_0 = (0, 0, \ldots)$ and $\|\Delta_0\| = 0$. Scalar multiplication and addition of exponent sequences are defined component-wise. The result now is

Lemma. (i) *For $n \geq 3$ and $2 \leq l \leq n-1$ we have*

(a) $r_E(v_n) \equiv 0 \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$ if $p^n - p^{l-1} > \|E\| \geq p^n - p^l$ *and E not equal to $p^l\Delta_{n-l}$ or $\Delta_1 + (p-1)\Delta_{n-1} + p^l\Delta_{n-l-1}$,*

(b) $r_E(v_n) \equiv v_l \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$ *if $E = p^l\Delta_{n-l}$,*

(c) $r_E(v_n) \equiv -p^p v_l \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$ *if $E = \Delta_1 + (p-1)\Delta_{n-1} + p^l\Delta_{n-l-1}$.*

(ii) *For $n \geq 3$ (and $l = 0$) we have*

(a) $r_E(v_n) \equiv 0 \bmod (p^{p+2})$ *if $\|E\| \geq p^n - 1$ and E not equal to Δ_n or $\Delta_1 + p\Delta_{n-1}$,*

(b) $r_E(v_n) = p$ *if $E = \Delta_n$,*

(c) $r_E(v_n) \equiv -p^p \bmod (p^{p+2})$ *if $E = p\Delta_{n-1} + \Delta_1$.*

(iii) *For $n \geq 3$ (and $l = 1$) we have*

(a) $r_E(v_n) \equiv 0 \bmod (p^{p+1})$ *if $p^n - 1 > \|E\| \geq p^n - p$ and E not equal to $p\Delta_{n-1}$ or $\Delta_1 + (p-1)\Delta_{n-1} + p\Delta_{n-2}$,*

(b) $r_E(v_n) \equiv v_1(1 - p^{p-1}) \bmod (p^{p+1})$ *if $E = p\Delta_{n-1}$,*

(c) $r_E(v_n) \equiv -p^p v_1 \bmod (p^{p+1})$ *if $E = \Delta_1 + (p-1)\Delta_{n-1} + p\Delta_{n-2}$.*

(iv) *For $n = 1$ we have*

$r_{\Delta_1}(v_1) = p.$

(v) *For $n = 2$ we have*

(a) $r_E(v_2) = 0$ *if $\|E\| \geq p^2 - p$ and E not equal to Δ_2, $p\Delta_1$, $(p+1)\Delta_1$,*

(b) $r_E(v_2) = p$ *if $E = \Delta_2$,*

(c) $r_E(v_2) = -p^p$ *if $E = (p+1)\Delta_1$,*

(d) $r_E(v_2) = (1 - p^{p-1} - p^p)v_1$ *if $E = p\Delta_1$.*

The proof of this lemma goes in several steps.

6.3. Proof of Lemma 6.2. (iv) and (v). We have

(6.3.1) $\bar{v}_1 = v_1 + pt_1,$

(6.3.2) $\bar{v}_2 = -p^{-1}(v_1 + pt_1)(v_1 + pt_1)^p + p^{-1}v_1v_1^p + v_2 + v_1t_1^p + pt_2.$

Parts (iv) and (v) of Lemma 6.2 follow immediately from this.

6.4. Proof of Lemma 6.2. (ii). We prove by induction that for $n \geq 2$

(6.4.1) $\bar{v}_n \equiv v_n + pt_n - p^p t_1 t_{n-1}^p \bmod (p^{p+2}, v_1, \ldots, v_{n-1}).$

Formula (6.3.2) takes care of the case $n=2$. Now suppose that $n \geqslant 3$. Because $a_{n-k} \equiv 0 \bmod (v_1, \ldots, v_{n-1})$ for $k=1, \ldots, n-1$ we see from (5.3.1) that

$$\bar{v}_n \equiv v_n + pt_n - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}^{p^j}.$$

Now by induction we can assume that $\bar{v}_{n-j} \equiv pt_{n-j} - p^p t_1 t_{n-j-1}^p \bmod (p^{p+2}, v_1, \ldots, v_{n-1})$ for $j=1, \ldots, n-2$ and $\bar{v}_1 \equiv pt_1 \bmod (p^{p+2}, v_1, \ldots, v_{n-1})$. Formula (6.4.1) now follows directly.

Part (ii) of Lemma 6.2 follows from (6.4.1) because of (6.1.1).

6.5. Proof of Lemma 6.2 (i) and (iii). Now let $n \geqslant 3$ and $1 \leqslant l \leqslant n-1$ and let E be an exponent sequence such that $\|E\| \geqslant p^n - p^l$. If Q is any polynomial in $v_1, v_2, \ldots; t_1, t_2, \ldots$ we let $c_E(Q)$ denote the coefficient of t^E in Q; $c_E(Q)$ is then a polynomial in $v_1, v_2, \ldots$. We have

$$r_E(v_n) = c_E(\bar{v}_n) \tag{6.5.1}$$

and $c_E(\bar{v}_n)$ is homogeneous of weight $p^n - 1 - \|E\| \leqslant p^l - 1$, where v_i has weight $p^i - 1$. In particular this means that $c_E(\bar{v}_n)$ cannot involve any v_i with $i > l$ and that the only terms of $c_E(\bar{v}_n)$ involving v_l are of the form dv_l with $d \in \mathbf{Z}$. Now

$$a_{n-k} = \sum_{s=1}^{n-k} p^{-1} v_s a_{n-k-s}^{(p^s)}. \tag{6.5.2}$$

Substituting this in (5.3.1) and using the remarks just made we obtain, because $a_{n-k-l} \equiv 0 \bmod (v_1, v_2, \ldots)$ if $n > k+l$, that

$$\begin{aligned} c_E(\bar{v}_n) \equiv\ & c_E\left(pt_n + p^{-1} v_l \left\{ (v_{n-1}^{p^l} - \bar{v}_{n-l}^{p^l}) + \sum_{\substack{i+j=n-l \\ i,j \geqslant 1}} (v_i^{p^l} t_j^{p^{l+i}} - t_j^{p^l} \bar{v}_i^{p^{l+i}}) \right\}\right) \\ & + c_E\left(v_l t_{n-l}^{p^l} - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}^{p^j}\right) \end{aligned} \tag{6.5.3}$$

where the congruence is $\bmod (v_1, \ldots, v_{l-1})$. Now by (6.4.1)

$$\begin{aligned} & \bar{v}_i^{p^{l+i}} \equiv 0 \bmod (v_1, \ldots, v_i, p^{p+2}) \quad \text{if } l \geqslant 1,\ i \geqslant 1, \\ & \bar{v}_{n-l}^{p^l} \equiv 0 \bmod (v_1, \ldots, v_{n-l}, p^{p+2}) \quad \text{if } l \geqslant 2, \\ & \bar{v}_{n-l}^{p} \equiv p^p t_{n-l}^p \bmod (v_1, \ldots, v_{n-l}, p^{p+2}). \end{aligned} \tag{6.5.4}$$

It follows from (6.5.3), (6.5.4) and the fact that $c_E(\bar{v}_n)$ is homogeneous of weight $\leqslant p^l - 1$ that

$$c_E(\bar{v}_n) \equiv c_E\left(pt_n - p^{p-1} v_1 t_{n-1}^p + v_1 t_{n-1}^p - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}^{p^j}\right) \quad \text{if } l = 1 \tag{6.5.5}$$

where the congruence is $\bmod (p^{p+1})$ (and $\|E\| \geqslant p^n - p$), and

$$c_E(\bar{v}_n) \equiv c_E\left(pt_n + v_l t_{n-l}^{p^l} - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}^{p^j}\right) \quad \text{if } 2 \leqslant l \leqslant n-1 \tag{6.5.6}$$

where the congruence is $\bmod(p^{p+1}, v_1, \ldots, v_{l-1})$ (and $\|E\| \geq p^n - p^l$). It remains to calculate $c_E(t_j\bar{v}_{n-j}^{p^j})$ for $j = 1, \ldots, n-1$. We distinguish three cases: A) $j > n - l$; B) $j = n - l$; C) $j > n - l$.

6.6. Case A. Calculation of $c_E(t_j\bar{v}_{n-j}^{p^j})$ **for** $j > n - l$. In this case we have $n - j < l$ and hence by (6.4.1) that $\bar{v}_{n-j} \equiv pt_{n-j} - p^p t_1 t_{n-j} \bmod(v_1, \ldots, v_{l-1}, p^{p+2})$ and as $l \leq n-1$, $j > n - l$, it follows that

$$c_E(t_j\bar{v}_{n-j}^{p^j}) \equiv 0 \bmod(p^{p+1}, v_1, \ldots, v_{l-1}) \quad \text{if } j > n - l. \tag{6.6.1}$$

6.7. Case B. Calculation of $c_E(t_{n-l}\bar{v}_l^{p^{n-l}})$. In this case we have by (6.4.1) that $\bar{v}_l \equiv v_l + pt_l - p^p t_1 t_{l-1}^p \bmod(v_1, \ldots, v_{l-1}, p^{p+2})$. Because $\|E\| \geq p^n - p^l$ and v_l has weight $p^l - 1$ it follows that

$$\begin{aligned} c_E(t_{n-l}\bar{v}_l^{p^{n-l}}) &\equiv c_E(t_{n-l}(pt_l - p^p t_1 t_{l-1}^p)^{p^{n-l}}) \\ &\quad + c_E(t_{n-l}p^{n-l}v_l(pt_l - p^p t_1 t_{l-1}^p)^{p^{n-l}-1}). \end{aligned} \tag{6.7.1}$$

And we see that

$$c_E(t_{n-l}\bar{v}_l^{p^{n-l}}) \equiv 0 \bmod(p^{p+1}, v_1, \ldots, v_{l-1}) \quad \text{if } n - l \geq 2. \tag{6.7.2}$$

And for $l = n - 1$ we have

$$\begin{aligned} c_E(t_1\bar{v}_{n-1}^p) &\equiv p^p \bmod(p^{p+1}, v_1, \ldots, v_{n-2}) \quad \text{if } E = \Delta_1 + p\Delta_{n-1} \\ c_E(t_1\bar{v}_{n-1}^p) &\equiv 0 \bmod(p^{p+1}, v_1, \ldots, v_{n-2}) \quad \text{if } E \neq \Delta_1 + p\Delta_{n-1}. \end{aligned} \tag{6.7.3}$$

6.8. Case C. Calculation of $c_E(t_j\bar{v}_{n-j}^{p^j})$ **for** $1 \leq j < n - l$. To deal with these terms we use induction. We have

$$c_E(t_j\bar{v}_{n-j}^{p^j}) = c_{E-\Delta_j}(\bar{v}_{n-j}^{p^j}). \tag{6.8.1}$$

Write

$$\bar{v}_{n-j} = \sum_{\|F\| \leq p^{n-j}-1} r_F(v_{n-j})t^F. \tag{6.8.2}$$

We then have

$$\bar{v}_{n-j}^{p^j} = \sum \binom{p^j}{s_1 \ldots s_m} r_{F_1}(v_{n-j})^{s_1} \ldots r_{F_m}(v_{n-j})^{s_m} t^{s_1F_1 + \ldots + s_mF_m} \tag{6.8.3}$$

where $F_1, \ldots, F_m$ is the set of all exponent sequences of weight $\leq p^{n-j} - 1$ and the sum is over all $(s_1, \ldots, s_m)$ such that $s_1 + \ldots + s_m = p^j$, $s_r \in \mathbf{N} \cup \{0\}$. The only terms of (6.8.3) which can contribute to $c_{E-\Delta_j}(\bar{v}_{n-j}^{p^j})$ are those with $\|s_1F_1 + \ldots + s_mF_m\| = \|E - \Delta_j\| \geq p^n - p^l - p^j + 1$. This means that there must be at least one F_i with $\|F_i\| > p^{n-j} - p^l$, for which $s_i \neq 0$. Indeed if all F_i with $s_i \neq 0$ were of weight $\leq p^{n-j} - p^l$ then we would have $\|s_1F_1 + \ldots + s_mF_m\| \leq p^j(p^{n-j} - p^l) = p^n - p^{l+j} < p^n - p^l - p^j + 1$ because $l \geq 1$, $j \geq 1$. We can therefore assume that $\|F_1\| \geq$

$p^{n-j}-p^l+1$. By induction (with respect to n) we have that $r_{F_1}(v_{n-j})\equiv 0 \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$ except in the following cases:

Case C_1: $n-j\geq 3$, $F_1=\Delta_{n-j}$,

Case C_2: $n-j\geq 3$, $F_1=p\Delta_{n-j-1}+\Delta_1$,

Case C_3: $n-j=2$, $F_1=\Delta_2$,

Case C_4: $n-j=2$, $F_1=(p+1)\Delta_1$.

In cases C_2 and C_4 we have $r_{F_1}(v_{n-j})\equiv 0 \bmod p^p$. And it follows that $\bar{v}_{n-j}^{p^j}\equiv 0 \bmod (p^{p+1})$ in these cases because either $s_1>1$ or $s_1=1$ and then the binomial coefficient is divisible by p.

So we are left with the cases C_1 and C_3 where $F_1=\Delta_{n-j}$. Suppose that there is an $i\geq 2$ with $\|F_i\|>p^{n-j}-p^l$, $s_i\neq 0$, $F_i\neq \Delta_{n-j}$, then by the previous reasoning we find a contribution $\equiv 0 \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$. The only terms

$$(6.8.4)\qquad \binom{p^j}{s_1\ldots s_m} r_{F_1}(v_{n-j})^{s_1}\ldots r_{F_m}(v_{n-j})^{s_m}$$

which can contribute something $\neq 0 \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$ are therefore of the form

$$(6.8.5)\qquad F_1=\Delta_{n-j},\qquad \|F_i\|\leq p^{n-j}-p^l \quad \text{if } i\geq 2 \text{ and } s_i\neq 0.$$

We then have

$$(6.8.6)\qquad \|s_1F_1+\ldots+s_mF_m\|\leq s_1(p^{n-j}-1)+(p^j-s_1)(p^{n-j}-p^l)$$

and we must have

$$(6.8.7)\qquad \|s_1F_1+\ldots+s_mF_m\|\geq p^n-p^l-p^j+1.$$

If $j\geq 2$ then $p^{l+j}\geq p^{l+1}+p^l+p^j-p$ for all $l\geq 1$ and it follows that (6.8.6) and (6.8.7) can simultaneously hold only if $s_1\geq p+1$. But then $r_{F_1}(v_{n-j})^{s_1}\equiv 0 \bmod p^{p+1}$ so that we find no contributions $\not\equiv 0 \bmod (p^{p+1}, v_1, \ldots, v_{l-1})$ of the form (6.8.4) if $j\geq 2$.

Now suppose that $j=1$, i.e. $F_1=\Delta_{n-1}$. Then we find from (6.8.6) and (6.8.7) that we must have $s_1\geq p-1$. If $s_1\geq p+1$ then we again find something $\equiv 0 \bmod (p^{p+1})$, so we are left with two subcases of C_1 and C_3 viz.

Case D: $j=1$, $F_1=\Delta_{n-1}$, $s_1=p$,

Case E: $j=1$, $F_1=\Delta_{n-1}$, $s_1=p-1$.

In case D we have $s_1+\ldots+s_m=p^j$, $s_1=p$, hence $s_2=\ldots=s_m=0$ and (6.8.4) gives a contribution

$$(6.8.8)\qquad r_{\Delta_{n-1}}(v_{n-1})^p=p^p$$

to $c_{E-\Delta_1}(\bar{v}_{n-1}^p)$.

Now suppose we are in case E. Then (6.8.4) reduces to

$$(6.8.9)\qquad pr_{\Delta_{n-1}}(v_{n-1})^{p-1}r_F(v_{n-1})=p^pr_F(v_{n-1})$$

for a certain exponent sequence F with $\|F\|\leq p^{n-1}-p^l$. On the other hand we must have $\|(p-1)\Delta_{n-1}+F\|\geq p^n-p^l-p+1$. It follows that we must have

(6.8.10) $\quad \|F\| = p^{n-1} - p^l.$

But then by induction we know that $r_F(v_{n-1}) \equiv 0 \bmod(p^{p+1}, v_1, \ldots, v_{l-1})$ except in the following cases:

$$r_F(v_{n-1}) \equiv v_1 \bmod(p^{p+1}, v_1, \ldots, v_{l-1}) \quad \text{if } F = p^l\Delta_{n-1-l},\ n \geq 4,$$

$$r_F(v_{n-1}) \equiv -p^p v_l \bmod(p^{p+1}, v_1, \ldots, v_{l-1}) \quad \text{if } F = \Delta_1 + (p-1)\Delta_{n-2} + p^l\Delta_{n-2-l},\ n \geq 4, \tag{6.8.11}$$

$$r_F(v_2) = (1 - p^{p-1} - p^p)v_1 \quad \text{if } n = 3,\ F = p\Delta_1 \text{ (and, necessarily, } l = 1).$$

It follows that the only contribution $\not\equiv 0 \bmod(p^{p+1}, v_1, \ldots, v_{l-1})$ of the form (6.8.9) is congruent to $p^p v_l \bmod(p^{p+1}, v_1, \ldots, v_{l-1})$. We have now proved that

6.9. Lemma. *Let* $n \geq 3$, $1 \leq l \leq n-1$, $\|E\| \geq p^n - p^l$, *then* $c_E(t_j\bar{v}_{n-j}^{p^j}) \equiv 0 \bmod(v_1, \ldots, v_{l-1}, p^{p+1})$ *except in the following eases*:

(i) $j = 1$, $l = n-1$, $E = \Delta_1 + p\Delta_{n-1}$, $c_E(t_1\bar{v}_{n-1}^p) \equiv p^p$,

(ii) $j = 1$, $l < n-1$, $E = \Delta_1 + p\Delta_{n-1}$, $c_E(t_1\bar{v}_{n-1}^p) \equiv p^p$,

(iii) $j = 1$, $l < n-1$, $E = \Delta_1 + (p-1)\Delta_{n-1} + p^l\Delta_{n-1-l}$, $c_E(t_1\bar{v}_{n-1}^p) \equiv p^p v_l$ *where the congruences are all* $\bmod(p^{p+1}, v_1, \ldots, v_{l-1})$.

6.10. Proof of Lemma 6.2(i). Conclusion. According to (6.5.6) we have $\bmod(p^{p+1}, v_1, \ldots, v_{l-1})$

$$c_E(\bar{v}_n) \equiv c_E\left(pt_n + v_l t_{n-l}^{p^l} - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}^{p^j}\right).$$

Now let $p^n - p^{l-1} > \|E\| \geq p^n - p^l$. Then because $l \geq 2$ only case (iii) of Lemma 6.9 applies and we find that $c_E(\bar{v}_n) \equiv 0 \bmod(p^{p+1}, v_1, \ldots, v_{l-1})$ except when $E = p^l\Delta_{n-l}$ or $E = \Delta_1 + (p-1)\Delta_{n-1} + p^l\Delta_{n-1-l}$ and in these two cases $c_E(\bar{v}_n)$ is respectively congruent to v_l and $-p^p v_l$.

6.11. Proof of Lemma 6.2 (iii). Conclusion. According to (6.5.5) we have $\bmod(p^{p+1})$

$$c_E(\bar{v}_n) \equiv c_E\left(pt_n - p^{p-1}v_1 t_{n-1}^p + v_1 t_{n-1}^p - \sum_{j=1}^{n-1} t_j \bar{v}_{n-j}^{p^j}\right).$$

Now let $p^n - 1 > \|E\| \geq p^n - p$. Then only case (iii) of Lemma 6.9 applies and we find that $c_E(\bar{v}_n) \equiv 0 \bmod(p^{p+1})$ except when $E = p\Delta_{n-1}$ or $E = \Delta_1 + (p-1)\Delta_{n-1} + p\Delta_{n-2}$ and in these two cases $c_E(\bar{v}_n)$ is respectively congruent to $(1 - p^{p-1})v_1$ and $-p^p v_1$.

6.12. Lemma 6.2 is now completely proved. Note that cases (i) and (ii) of Lemma 6.9 deal with exponent sequences E with $\|E\| = p^n - 1$, which are therefore covered by part (ii) of Lemma 6.2.

7. The linear part of the Brown–Peterson cohomology operations map η_R

In this section we calculate $\eta_R(v_n)$ modulo the ideal $(t_1, t_2, \dots)^2$, or, equivalently, we calculate $\bar{V}_n$ modulo $(T_1, T_2, \dots)^2$.

7.1. We write B_i for the element $p^i a_i(V) \in \mathbf{Z}[V_1, V_2, \dots]$, where $a_i(V)$ is defined by (3.1.2). Let J denote the ideal $(T_1, T_2, \dots)^2$ in $\mathbf{Z}[V; T]$.

Theorem. *Modulo J we have*

$$\begin{aligned}\bar{V}_n \equiv {} & \sum (-1)^t (B_{s_1} V_{n-s_1}^{p^{s_1}-1})(B_{s_2} V_{n-s_1-s_2}^{p^{s_2}-1}) \cdot \ldots \cdot (B_{s_t} V_{n-s_1-\dots-s_t}^{p^{s_t}-1})(-T_i V_j^{p^i}) \\ & + \sum (-1)^t (B_{s_1} V_{n-s_1}^{p^{s_1}-1})(B_{s_2} V_{n-s_1-s_2}^{p^{s_2}-1}) \cdot \ldots \cdot (B_{s_t} V_{n-s_1-\dots-s_t}^{p^{s_t}-1})(pT_i) + V_n\end{aligned} \tag{7.1.1}$$

where the first sum is over all sequences $(s_1, \dots, s_t, i, j)$ such that $s_k, i, j \in \mathbf{N}$, $s_1 + \dots + s_t + i + j = n$, $t \in \mathbf{N} \cup \{0\}$ and the second sum is over all sequences $(s_1, \dots, s_t, i)$ such that s_k, $i \in \mathbf{N}$, $s_1 + \dots + s_t + i = n$, $t \in \mathbf{N} \cup \{0\}$.

7.2. Example.

$$\begin{aligned}\bar{V}_3 \equiv {} & B_1 V_2^{p-1} T_1 V_1^p - T_1 V_2^p - T_2 V_1^{p^2} + B_1 V_2^{p-1} B_1 V_1^{p-1}(pT_1) \\ & - B_2 V_1^{p^2}(pT_1) - B_1 V_2^{p-1}(pT_2) + pT_3 + V_3.\end{aligned}$$

The proof of Theorem 7.1 uses the recursion formula (5.3.1). First two lemmas:

7.3. Lemma.

$$\bar{V}_n \equiv V_n + pT_n + \sum_{k=1}^{n-1} a_{n-k}(V)(V_k^{p^{n-k}} - \bar{V}_k^{p^{n-k}}) + \sum_{j=1}^{n-1} - T_j \bar{V}_{n-j}^{p^j} \tag{7.3.1}$$

where the congruence is modulo $J = (T_1, T_2, \dots)^2$.

This follows immediately from formula (5.3.1)

7.4. Lemma. *Suppose that $\bar{V}_k \equiv V_k + \sum T_i C_i$ modulo J for certain $C_i \in \mathbf{Z}[V; T]$. Then*

$$\bar{V}_k^{p^l} \equiv V_k^{p^l} + p^l V_k^{p^l-1}\left(\sum T_i C_i\right) \bmod J. \tag{7.4.1}$$

Proof. Obvious.

7.5. Proof of Theorem 7.1. Theorem 7.1 is proved by induction, the case $n = 1$ being trivial. Given formula (7.1.1) for all $k < n$, we have that $\bar{V}_k \equiv V_k \bmod(T_1, T_2, \dots)$ so that we can apply Lemma 7.4. Substituting the result in (7.3.1) then proves (7.1.1).

7.6. Let $b_n \in \mathrm{BP}_*(pt)$ be the image of B_n under $\mathbf{Z}[V_1, V_2, \ldots] \to \mathrm{BP}_*(pt)$, $V_i \mapsto v_i$ where the v_i are the generators of $\mathrm{BP}_*(pt)$ determined by formula (3.1.3); i.e. $b_n = p^n l_n = [\mathbf{CP}^{p^n-1}] = p^n m_{p^n-1}$. In view of 4.4 we obtain

7.7. Corollary. *For* $0 < i < n$ *we have*

$$r_{\Delta_i}(v_n) = \sum_{s_1+\ldots+s_t<n-i} (-1)^t (b_{s_1} v_{n-s_1}^{p^{s_1}-1}) \ldots (b_{s_t} v_{n-s_1-\ldots-s_t}^{p^{s_t}-1})(-v_{n-s_1-\ldots-s_t-i}^{p^i})$$

(7.7.1)

$$-v_{n-i}^{p^i} + p \sum_{s_1+\ldots+s_t=n-i} (-1)^t (b_{s_1} v_{n-s_1}^{p^{s_1}-1}) \ldots (b_{s_t} v_{n-s_1-\ldots-s_t}^{p^{s_t}-1})$$

where the first sum is over all sequences $(s_1, \ldots, s_t)$ with $s_k, t \in \mathbf{N}$ and $s_1 + \ldots + s_t < n - i$ and the second sum is over all sequences $(s_1, \ldots, s_t)$, $s_k \in \mathbf{N}$, with $s_1 + \ldots + s_t = n - i$.

7.8. Let I denote the ideal of $\mathbf{Z}[V; T]$ generated by the elements pT_i, $i = 1, 2, \ldots$; $T_i T_j$, $i, j = 1, 2, \ldots$. Now

(7.8.1) $\quad B_n \equiv V_1 V_1^p \ldots V_1^{p^{n-1}} \bmod (p).$

It follows that

7.9. Corollary. *Modulo I we have*

$$V_n \equiv \sum (-1)^t V_1^{(p-1)^{-1}(p^{s_1}+\ldots+p^{s_t}-t)} V_{n-s_1}^{p^{s_1}-1} \cdot \ldots \cdot V_{n-s_1-\ldots-s_t}^{p^{s_t}-1} (-T_i V_j^{p^i})$$

(7.9.1)

$$+ V_n - T_1 V_{n-1}^p - T_2 V_{n-2}^{p^2} - \ldots - T_{n-1} V_1^{p^{n-1}}$$

where the sum is over all sequences $(s_1, \ldots, s_t, i, j)$ *such that* s_k, i, j, $t \in \mathbf{N}$ *and* $s_1 + \ldots + s_t + i + j = n$.

This corollary can be used to give a noncohomological proof of the Lubin–Tate formal moduli theorem. Cf. [8, part V]. Warning: the starting formula (2.2.1) in [8, part V] is not correct and should be replaced with (7.9.1) above; the proof of the Lubin–Tate Theorem remains mutatis mutandis the same.

7.10. Corollary. *For* $0 < i < n$ *we have*

$$r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} \bmod (p, v_1).$$

7.11. Corollary. *For* $0 < i < n - 1$ *we have*

$$r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} + v_1 v_{n-i-1}^{p^i} v_{n-1}^{p-1} \bmod (p, v_1^2).$$

More generally let $r = \min(n - i - 1, p)$, then we have

$$r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} + v_1 v_{n-1}^{p-1} v_{n-i-1}^{p^i} - v_1^2 v_{n-1}^{p-1} v_{n-2}^{p-1} v_{n-i-2}^{p^i} + \ldots$$

$$+ (-1)^{r+1} v_1^r v_{n-1}^{p-1} \ldots v_{n-r}^{p-1} v_{n-i-r}^{p^i} \bmod (p, v_1^{p+1}).$$

8. The functional equation lemma and multiplicative operations in BP*(BP)

As a final application of the universal isomorphism theorem 2.12 of [5] and the functional equation lemma 7.1 of [5] we reprove the main theorem of [20].

8.1. Choose a prime number p. Let $\sigma : \mathbf{Z}_{(p)}[V] \to \mathbf{Z}_{(p)}[V]$ be the ring homomorphism given by $V_i \mapsto V_i^p$, for $i = 1, 2, \ldots$. If $g(X)$ is a power series with coefficients in $\mathbf{Z}_{(p)}[V]$ or $\mathbf{Q}[V]$ then $g^\sigma(X)$ denotes the power series obtained by applying σ to the coefficients of $g(X)$. We also write a^σ for $\sigma(a)$ if $a \in \mathbf{Q}[V]$. Part of the functional equation lemma 7.1 of [5] now says

8.2. Functional equation lemma. *If $d(X) = X + d_2X^2 + \ldots$ is a power series with $d_i \in \mathbf{Z}_{(p)}[V]$ and $f_V(X)$ is the logarithm of the p-typically universal formal group $F_V(X, Y)$ of* [5] *and* [7], *then there are unique elements $e_2, e_3, \ldots \in \mathbf{Z}_{(p)}[V]$ such that*

$$g(X) - \sum_{i=1}^{\infty} p^{-1} V_i g^{\sigma^i}(X^{p^i}) = X + \sum_{i=2}^{\infty} e_i X^i \tag{8.2.1}$$

where $g(X) = f_V(d(X))$. Inversely given a power series $g(X) = X + \sum_{i=2}^{\infty} c_i X^i$, $c_i \in \mathbf{Q}[V]$ such that (8.2.1) *holds for certain $e_i \in \mathbf{Z}_{(p)}[V]$, then there exists a unique power series $d(X) = X + d_2X^2 + \ldots$ with $d_i \in \mathbf{Z}_{(p)}[V]$ such that $g(X) = f_V(d(X))$.*

8.3. Corollary. *If $d(X)$ is such that $g(X) = X + \sum_{n=1}^{\infty} c_{p^n} X^{p^n}$, i.e. $c_i = 0$ if i is not a power of p, then $e_i = 0$ if i is not a power of p and writing s_n for e_{p^n} we have*

$$c_{p^n} = \sum_{k=0}^{n} a_{n-k} s_k^{\sigma^{n-k}} \tag{8.3.1}$$

where a_n is the coefficient of X^{p^n} in $f_V(X)$.

This follows immediately from (8.2.1) above because a_n satisfies

$$a_n = \sum_{k=1}^{n} p^{-1} V_k a_{n-k}^{\sigma^k} \quad \text{and} \quad a_n = \sum_{k=1}^{n-1} p^{-1} a_{n-k} V_k^{p^{n-k}}. \tag{8.3.2}$$

Let $\mathrm{BP}_*(pt) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$, where the v_i are the free polynomial generators defined by formula (3.1.3) above. Define the homomorphism $\sigma : \mathrm{BP}_*(pt) \to \mathrm{BP}_*(pt)$ by $v_i \mapsto v_i^p$. Let $l_i = m_{p^i-1} \in \mathrm{BP}_*(pt) \otimes \mathbf{Q}$, cf. 3.1.

8.4. Theorem (Ravenel [20]). *For every sequence of elements $(r_1, r_2, \ldots)$ in $\mathrm{BP}_*(pt)$ there is a unique sequence of elements $(s_1, s_2, \ldots)$ in $\mathrm{BP}_*(pt)$ such that*

$$\sum_{i=0}^{n} l_{n-i} r_i^{p^{n-i}} = \sum_{i=0}^{n} l_{n-i} s_i^{\sigma^{n-i}} \tag{8.4.1}$$

for every $n > 0$. Inversely for every sequence $(s_1, s_2, \ldots)$ in $\mathrm{BP}_*(pt)$ there is a unique sequence $(r_1, r_2, \ldots)$ such that (8.3.1) *holds for all $n > 0$.*

Proof. Identify $\mathbf{Z}_{(p)}[V]$ with $\mathrm{BP}_*(pt)$ via $V_i \mapsto v_i$. The element a_i in $\mathbf{Q}[V]$ then corresponds with $l_i \in \mathrm{BP}_*(pt) \otimes \mathbf{Q}$. Take a sequence of elements $(r_1, r_2, \dots)$ in $\mathrm{BP}_*(pt)$. Let $\phi : \mathbf{Z}_{(p)}[V; T] \to \mathrm{BP}_*(pt)$ be the ring homomorphism defined by $V_i \mapsto v_i$, $T_i \mapsto r_i$. Write $G(X, Y) = F^{\phi}_{V,T}(X, Y)$. Let $g(X)$ be the logarithm of $G(X, Y)$, then, cf. (4.1.3)

$$(8.4.2) \qquad g(X) = X + \sum_{n=1}^{\infty} c_n X^{p^n}, \qquad c_n = \sum_{i=0}^{n} l_{n-i} r_i^{p^{n-i}}.$$

The formal group $G(X, Y)$ is strictly isomorphic to $F_v(X, Y) = \mu_{\mathrm{BP}}(X, Y)$ over $\mathrm{BP}_*(pt)$, and the isomorphism is equal to the inverse of $\alpha^{\phi}_{V,T}(X)$. Cf. [5, Theorem 2.12] and 4.1 above. It follows that there is a power series $d(X) = X + d_2X^2 + \dots$ with $d_i \in \mathrm{BP}_*(pt)$ such that $g(X) = f_v(d(X))$. (In fact $d^{-1}(X) = \alpha^{\phi}_{V,T}(X)$.) Now apply Corollary 8.3, to find s_i such that (8.4.1) holds.

Inversely given elements $(s_1, s_2, \dots)$ in $\mathrm{BP}_*(pt)$, let $g(X)$ be the power series

$$(8.4.3) \qquad g(X) = X + \sum_{n=1}^{\infty} \sum_{i=0}^{n} l_{n-i} s_i^{\sigma^{n-i}}$$

then $g(X)$ satisfies a functional equation (8.2.1) and hence again by the functional equation lemma, there exists a power series $d(X) = X + d_2X^2 + \dots$, $d_i \in \mathrm{BP}_*(pt)$ such that $g(X) = f_V(d(X))$. It follows that $g(X)$ is the logarithm of a p-typical formal group $G(X, Y)$ which is strictly isomorphic over $\mathrm{BP}_*(pt)$ to $F_v(X, Y) = \mu_{\mathrm{BP}}(X, Y)$. By the universality of the triple $(F_V(X, Y), \alpha_{V,T}(X), F_{V,T}(X, Y))$ there is therefore a unique homomorphism $\psi : \mathbf{Z}_{(p)}[V; T] \to \mathrm{BP}_*(pt)$, such that $\psi(V_i) = v_i$ and $f^{\psi}_{V,T}(X) = g(X)$. Let $r_i = \psi(T_i) \in \mathrm{BP}_*(pt)$. Then because of (4.1.3)

$$(8.4.4) \qquad g(X) = X + \sum_{n=1}^{\infty} \sum_{i=0}^{n} l_{n-i} r_i^{p^{n-i}}.$$

This concludes the proof of the theorem.

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