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A UNIVERSAL ISOMORPHISM FOR P-TYPICAL FORMAL
GROUPS AND OPERATIONS IN BROWN-PETERSON
COHOMOLOGY

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MATHEMATICS

A UNIVERSAL ISOMORPHISM FOR P -TYPICAL FORMAL
 GROUPS AND OPERATIONS IN BROWN-PETERSON
 COHOMOLOGY

BY

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ABSTRACT

We construct an abstract isomorphism of p -typical formal groups which is universal for isomorphisms of p -typical formal groups over $\mathbf{Z}_{(p)}$ -algebras or characteristic zero rings. Associated to this universal isomorphism is a homomorphism of rings $\mathbf{Z}[V_1, V_2, \dots] \rightarrow \mathbf{Z}[V_1, V_2, \dots; T_1, T_2, \dots]$ which (after localization at p) can be identified with the right unit homomorphism $\eta_R: BP_*(pt) \rightarrow BP_*(BP)$ of the Hopf-algebra $BP_*(BP)$ of Brown-Peterson (co)homology. We calculate η_R modulo the ideal $(T_1, T_2, \dots)^2$. These results are then used to obtain information on some of the operations of Brown-Peterson cohomology.

1. INTRODUCTION

Choose a prime number p and let \mathbf{Q} denote the rational numbers. Let $a_i(V)$, $a_i(V, T)$ in

$$\mathbf{Q}[V] = \mathbf{Q}[V_1, V_2, \dots] \text{ and } \mathbf{Q}[V; T] = \mathbf{Q}[V_1, V_2, \dots; T_1, T_2, \dots]$$

be the polynomials defined by the equations

$$(1.1) \quad pa_i(V) = \sum_{k=1}^i a_{i-k}(V) V_k^{p^i - k}, \quad a_0(V) = 1$$

$$(1.2) \quad a_i(V, T) = \sum_{k=0}^i a_k(V) T_k^{p^i - k}, \quad a_0(V, T) = 1.$$

Now define the power series

$$(1.3) \quad f_V(X) = \sum_{n=0}^{\infty} a_n(V) X^{p^n}, \quad f_{V,T}(X) = \sum_{n=0}^{\infty} a_n(V, T) X^{p^n}$$

$$(1.4) \quad F_V(X, Y) = f_V^{-1}(f_V(X) + f_V(Y)), \quad F_{V,T}(X, Y) = f_{V,T}^{-1}(f_{V,T}(X) + f_{V,T}(Y))$$

$$(1.5) \quad \alpha_{V,T}(X) = f_{V,T}^{-1}(f_V(X))$$

where $f_{\bar{V}}^{-1}(X)$ and $f_{\bar{V},T}^{-1}(X)$ are the inverse power series to $f_V(X)$ and $f_{V,T}(X)$ respectively, i.e. $f_{\bar{V}}^{-1}(f_V(X))=X$ and $f_{\bar{V},T}^{-1}(f_{V,T}(X))=X$. One then has (cf. [3], [4] and [5] part I).

1.6. THEOREM.

The power series $F_V(X, Y)$, $F_{V,T}(X, Y)$, $\alpha_{V,T}(X)$ have their coefficients in $\mathbf{Z}[V]$, $\mathbf{Z}[V; T]$, $\mathbf{Z}[V; T]$.

The power series $F_V(X, Y)$ and $F_{V,T}(X, Y)$ therefore define p -typical (one dimensional commutative) formal groups over $\mathbf{Z}[V]$ and $\mathbf{Z}[V; T]$ respectively, which are strictly isomorphic via $\alpha_{V,T}(X)$. In addition one has (cf. [4] and [5] part I).

1.7. THEOREM.

The triple $(F_V(X, Y), \alpha_{V,T}(X), F_{V,T}(X, Y))$ over $\mathbf{Z}[V; T]$ is universal for triples $(F(X, Y), \alpha(X), G(X, Y))$ consisting of two p -typical formal groups and a strict isomorphism between them defined over a ring A which is a $\mathbf{Z}_{(p)}$ -algebra or a characteristic zero ring.

I.e. for every such triple $(F(X, Y), \alpha(X), G(X, Y))$ there is a unique homomorphism $\phi: \mathbf{Z}[V; T] \rightarrow A$ such that $F(X, Y) = F_{\bar{V}}^{\phi}(X, Y)$, $\alpha(X) = \alpha_{\bar{V},T}^{\phi}(X)$, $G(X, Y) = F_{\bar{V},T}^{\phi}(X, Y)$.

If we restrict attention to $\mathbf{Z}_{(p)}$ -algebras A theorem 1.7 implies that $\mathbf{Z}_{(p)}[V; T]$ represents the functor $\mathcal{S}: A \mapsto$ set of all triples $(F(X, Y), \alpha(X), G(X, Y))$. Now $BP_*(BP) = \mathbf{Z}_{(p)}[V; T]$, cf. [1] part II, theorem 16.1, or [2], so that \mathcal{S} is also represented by $BP_*(BP)$ where BP is the Brown-Peterson spectrum. This fact has been used to derive all the structure maps of the Hopf algebra $BP_*(BP)$, cf. [7]. $F_V(X, Y)$ is a p -typically universal p -typical formal group and $F_{V,T}(X, Y)$ is a p -typical formal group. It follows that there are unique polynomials $\bar{V}_n \in \mathbf{Z}[V; T]$ such that $F_{\bar{V}}(X, Y) = F_{V,T}(X, Y)$. It follows that we have for the polynomials \bar{V}_n

$$(1.8) \quad pa_n(V, T) = \sum_{k=1}^n a_{n-k}(V, T) \bar{V}_k^{p^{n-k}}.$$

The assignment $V_n \mapsto \bar{V}_n$ defines a homomorphism $\mathbf{Z}[V] \rightarrow \mathbf{Z}[V, T]$. Now $BP_*(pt) = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ where the v_i are defined by

$$(1.9) \quad pl_n = l_{n-1}v_1^{p^{n-1}} + \dots + l_1v_{n-1}^p + v_n$$

where $l_n = p^{-n}[\mathbf{C}P^{p^n-1}] \in BP_*(pt) \otimes \mathbf{Q} \subset MU_*(pt) \otimes \mathbf{Q}$. Now identify $\mathbf{Z}_{(p)}[V]$ with $BP_*(pt)$ by means of $V_i \mapsto v_i$ and $\mathbf{Z}_{(p)}[V; T]$ with $BP_*(BP)$ by means of $V_i \mapsto v_i$, $T_i \mapsto t_i$ where the t_i are the elements of $BP_*(BP)$ described in theorem 16.1 of part II of [1]. The homomorphism $\mathbf{Z}[V] \rightarrow \mathbf{Z}[V; T]$ (when localized at p) then becomes the right unit map $\eta_R: BP_*(pt) \rightarrow BP_*(BP)$.

correct and should be replaced with (2.1.3) above. The proofs in [5] part V remain mutatis mutandi the same.

3. APPLICATIONS TO BROWN-PETERSON COHOMOLOGY OPERATIONS

A stable BP cohomology operation can be described as a $BP_*(pt)$ -linear homomorphism $BP_*(BP) \rightarrow BP_*(pt)$, where $BP_*(BP)$ is seen as a left $BP_*(pt)$ module. To find out what such an operation does to elements of $BP_*(pt)$ one composes with $\eta_R: BP_*(pt) \rightarrow BP_*(BP)$. Cf. [1] part II, section 16 for all this. Let $E=(e_1, e_2, \dots)$ be a sequence of integers ≥ 0 which are almost all zero.

Write $BP_*(BP)=BP_*(pt) [t_1, t_2, \dots]$ where the t_i are as in [1] part II, section 16. The cohomology operation r_E is defined as: = coefficient of t^E . One assigns to the exponent sequence E the weight

$$\|E\|=(p-1)e_1+(p^2-1)e_2+\dots$$

Let Δ_i denote the exponent sequence $\Delta_i=(0, \dots, 0, 1, 0, \dots)$ with the 1 in the i -th place, let $\Delta_0=(0, 0, \dots)$. Scalar multiplication with an element of \mathbb{N} and addition of exponent sequences are defined component wise.

A first application of (2.1.1) is then the following slight generalization of lemma 1.9 of [6] (sometimes known as the Budweiser lemma).

3.1. LEMMA.

- (i) For $n \geq 3$ and $2 < l < n-1$ we have that

$$r_E(v_n) \equiv 0 \pmod{(p^{p+1}, v_1, \dots, v_{l-1})}$$

if $p^n - p^{l-1} > \|E\| \geq p^n - p^l$ except in the cases

$$E = p^l \Delta_{n-l}, \quad E = \Delta_1 + (p-1)\Delta_{n-1} + p^l \Delta_{n-l-1}.$$

In these two cases $r_E(v_n)$ is respectively congruent to v_l and $-p^p v_l \pmod{(p^{p+1}, v_1, \dots, v_{l-1})}$.

- (ii) For $n \geq 3$ (and $l=1$) we have that $r_E(v_n) \equiv 0 \pmod{(p^{p+1})}$ if $p^n - 1 > \|E\| \geq p^n - p$ except in the cases

$$E = p \Delta_{n-1}, \quad E = \Delta_1 + (p-1)\Delta_{n-1} + p \Delta_{n-2}.$$

In these two cases $r_E(v_n)$ is respectively congruent to $v_1(1-p^{p-1})$ and $-p^p v_1 \pmod{(p^{p+1})}$.

- (iii) For $n \geq 3$ (and $l=0$) we have that $r_E(v_n) \equiv 0 \pmod{(p^{p+2})}$ if $\|E\| \geq p^n - 1$ except in the cases $E = \Delta_n, E = \Delta_1 + p \Delta_{n-1}$. In these two cases $r_E(v_n)$ is respectively congruent to p and $-p^p \pmod{(p^{p+2})}$.

(There are slightly different formulae for the cases $n=1, 2$).

A second application is the calculation of the $r_{\Delta_i}(v_n)$. Let $b_n \in BP_*(pt)$ stand for the element $p^n l_n$. Then we have immediately from (2.1.2).

3.2. THEOREM.

For $0 < i < n$ we have

$$(3.2.1) \quad \left\{ \begin{aligned} r_{\mathcal{A}_i}(v_n) &= \sum (-1)^t (b_{s_1} v_{n-s_1}^{s_1-1}) \cdot \dots \cdot (b_{s_t} v_{n-s_1-\dots-s_t}^{s_t-1}) (-v_{n-s_1-\dots-s_t-i}^{s_t}) \\ &+ p \sum (-1)^t (b_{s_1} v_{n-s_1}^{s_1-1}) \cdot \dots \cdot (b_{s_t} v_{n-s_1-\dots-s_t}^{s_t-1}) - v_{n-i}^{s_t} \end{aligned} \right.$$

where the first sum is over all sequences (s_1, \dots, s_t) with $s_1 + \dots + s_t < n - i$, $s_k, t \in \mathbb{N}$ and the second sum is over all sequences (s_1, \dots, s_t) with $s_1 + \dots + s_t = n - i$, $s_k, t \in \mathbb{N}$. Modulo p we have for $0 < i < n$.

$$(3.2.2) \quad \left\{ \begin{aligned} r_{\mathcal{A}_i}(v_n) &\equiv -v_{n-i}^{s_t} + \sum (-1)^t v_1^{(p-1)^{-1}(s_1+\dots+s_t-i)} \\ &v_{n-s_1}^{s_1-1} \cdot \dots \cdot v_{n-s_1-\dots-s_t}^{s_t-1} (-v_1^{s_t}) \end{aligned} \right.$$

where the sum is over all sequences (s_1, \dots, s_t, j) such that $s_k, t, j \in \mathbb{N}$, $s_1 + \dots + s_t + j = n - i$.

3.3. COROLLARY.

For $0 < i < n$ we have $r_{\mathcal{A}_i}(v_n) \equiv -v_{n-i}^{s_t} \pmod{(p, v_1)}$. More generally: let $r = \min(n - i - 1, p)$, then we have $\pmod{(p, v_1^{r+1})}$

$$(3.3.1) \quad r_{\mathcal{A}_i}(v_n) \equiv -v_{n-i}^{s_t} + \sum_{t=1}^r (-1)^{t+1} v_1^t (v_{n-1} \dots v_{n-t})^{p-1} v_{n-i-t}^{s_t}.$$

REFERENCES

1. Adams, J. F. - Stable Homotopy and Generalized Homology. Univ. of Chicago Press, 1974.
2. Araki, S. - Typical Formal Groups in Complex Cobordism and K -theory. Kinokuniya Book-Store Cy, 1973.
3. Hazewinkel, M. - A Universal Formal Group and Complex Cobordism. Bull. Amer. Math. Soc. 81, 930-933 (1975).
4. Hazewinkel, M. - Constructing Formal Groups I. The Local one-dimensional Case. (To appear, a Preliminary Version of this is part I of [5]).
5. Hazewinkel, M. - Constructing Formal Groups I, II, III, IV, V. Reports 7119, 7201, 7207, 7322, 7514. Econometric Inst., Erasmus Univ. Rotterdam, 1971, 1972, 1973, 1975.
6. Johnson, D. C. and W. S. Wilson - BP -operations and Morava's extraordinary K -theories. Math. Z. 144, 55-75 (1975).
7. Landweber, P. S. - $BP^*(BP)$ and Typical Formal Groups. Osaka J. Math. 12, 357-363 (1975).
8. Lazard, M. - Sur les groupes de Lie formels à un paramètre. Bull. Soc. Math. France 83, 251-274 (1955).
9. Lubin, J. and J. Tate - Formal Moduli for One Parameter Formal Lie Groups. Bull. Soc. Math. France 94, 49-60 (1966).