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CONSTRUCTING FORMAL GROUPS. V: THE LUBIN-TATE FORMAL MODULI THEOREM AND THE LAZARD CLASSIFICATION THEOREM FOR ONE DIMENSIONAL FORMAL GROUPS OVER ALGEBRAICALLY CLOSED FIELDS

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Preliminary

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CONSTRUCTING FORMAL GROUPS V

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1. INTRODUCTION

Let $F_V(X,Y)$ be the p-typically universal formal group over $\mathbb{Z}[V] = \mathbb{Z}[V_1,V_2,\ldots]$ constructed in [2] cf. also [3] . In [2] we also constructed a universal strict isomorphism $\alpha_{V,T}(X)$ between formal groups:
$\alpha_{V,T}\colon F_V \to F_{V,T}$. Now $F_{V,T}(X,Y)$ over $\mathbb{Z}[V;T] = \mathbb{Z}[V_1,V_2,\ldots;T_1,T_2,\ldots]$ is a p-typical formal group. Hence there exist polynomials $\bar{V}_i$(in $T_1,\ldots,T_i;V_1,\ldots V_i$) such that $F_{V,T}(X,Y) = F_{\bar{V}}(X,Y)$. The map $V_i \mapsto \bar{V}_i$ can also be interpreted as the map $\eta_R\colon BP_*(pt) \to BP_*(BP)$ of Brown-Peterson cohomology. Cf. [1], [6], [7]. In [4] we gave a recursive procedure for calculating the polynomials $\bar{V}_i$ and in [5] we used this result to give a formula for $\bar{V}_i$ $\bmod (T_1,T_2,\ldots)^2$.
In this note we use these two results to give (i) a noncohomological proof for the Lubin-Tate formal moduli theorem for one dimensional formal groups over complete, separated, local rings an (ii) a proof of Lazard's classification theorem for one dimensional formal groups over an algebraically closed field.
The phrase "formal group" will be used as an abbreviation of "one dimensional commutative formal group (law)".

2. THE RECURSION PROCEDURE AND THE FORMULA FOR $\bar{V}_i$.

2.1. <u>The Recursion Procedure</u>.

U_r, Y_r, $W_{r,\ell}$ are the polynomials with integer coefficients in the variables $V_1,V_2,\ldots;\ T_1,T_2,\ldots;\ S_1,S_2,\ldots$ defined as follows

$$U_1 = V_1,\quad U_r = \sum_{k=1}^{r-1} V_k W_{r-k,k} + Y_r + V_r$$

(2.1.1)
$$W_{r,} = p^{-1}(U_r^{(p\ )} - (U_r + pT_r)^{p\ })$$

$$Y_r = (V_1 T_{r-1}^p - T_{r-1} S_1^{p^{r-1}}) + \ldots + (V_{r-1} T_1^{p^{r-1}} - T_1 S_{r-1}^p)$$

where $U_s^{(p^\ell)}$ is the polynomial obtained from U_s by replacing each T_i, V_i, S_i, $i = 1,2,\ldots$ with their p^ℓ-th powers.

Let $\bar{U}_r$, $\bar{Y}_r$, $\bar{W}_{s,\ell}$ be the polynomials obtained from U_r, Y_r, $W_{s,\ell}$ by substituting $\bar{V}_i$ for S_i, $i = 1,2,\ldots$. Then one has $\bar{V}_1 = V_1 + pT_1$ and given $\bar{V}_1, \ldots, \bar{V}_{n-1}$ the polynomial $\bar{V}_n$ is given by

$$\bar{V}_n = \bar{U}_n + pT_n = \sum_{k=1}^{n-1} V_k \bar{W}_{n-k,k} + \bar{Y}_n + V_n + pT_n \tag{2.1.2}$$

For a proof of these assertions cf. [4].

2.2. A Formula for $\bar{V}_n$.

Using (2.1.1) and (2.1.2) we deduced in [5] the following congruence for $\bar{V}_n$

$$\begin{aligned} \bar{V}_n \equiv V_n &- (T_1 V_{n-1}^p + \ldots + T_{n-1} V_1^{p^{n-1}}) \\ &+ V_1 V_{n-1}^{p-1} (T_1 V_{n-2}^p + \ldots + T_{n-2} V_1^{p^{n-2}}) + \ldots \\ &+ (-1)^{n-2} V_1 V_{n-1}^{p-1} V_1 V_{n-2}^{p-1} \ldots V_1 V_2^{p-1} (T_1 V_1^p) \end{aligned} \tag{2.2.1}$$

where the congruence is modulo the ideal in $\mathbb{Z}[V,T]$ generated by the elements pT_i, $i = 1,2,\ldots$ and the elements T_iT_j, $i,j = 1,2,\ldots$.

3. THE LUBIN-TATE FORMAL MODULI THEOREM

Let A be a local ring with maximal ideal $\mathfrak{m}$ complete and separated in the $\mathfrak{m}$-adic topology. Let $k = A/\mathfrak{m}$ be the residue field and let $\Phi(X,Y)$ be a formal group over k of height h. We assume that k has characteristic $p > 0$. A formal group $F(X,Y)$ over A is a lift of $\Phi(X,Y)$ if $F^\pi(X,Y) = \Phi(X,Y)$ where $\pi : A \to k$ is the natural projection. We are interested in the lifts of Φ modulo (strict) isomorphisms which reduce to the identity isomorphism mod $\mathfrak{m}$.

3.1. Proposition.

Let $F(X,Y)$, $G(X,Y)$ be two p-typical formal groups over A such that $F^\pi(X,Y) = G^\pi(X,Y)$. Let $v_1, v_2, \ldots \in A$ be such that $F(X,Y) = F_v(X,Y)$, let h be the height of $F(X,Y)$ and $G(X,Y)$. Then there are $w_1, \ldots, w_h \in A$, and $\phi(X) \in A[[X]]$ such that

(i) $\phi(X) \equiv X \bmod (X^2)$

(ii) $\phi(X) \equiv X \bmod \mathfrak{m}$

(iii) $\phi G(X,Y) = F_w(\phi X, \phi Y)$ where $w = (w_1, \ldots, w_h, v_{h+1}, v_{h+2}, \ldots)$

(iv) $w_i \equiv v_i \bmod \mathfrak{m}$, $i = 1,2,\ldots$

Proof. Let $v_1', v_2', \ldots \in A$ be such that $G(X,Y) = F_{v'}(X,Y)$. (Such v_i' exist because $F_V(X,Y)$ is a p-typically universal formal group, cf [2]). Let $\alpha_{V,T}(X)\colon F_V(X,Y) \to F_{V,T}(X,Y) = F_{\bar{V}}(X,Y)$ be the universal strict isomorphism between p-typical formal groups. Because $F^{\pi}(X,Y) = G^{\pi}(X,Y)$ we have that

$$v_i \equiv v_i' \bmod \mathfrak{m} \tag{3.1.1}$$

We are now going to construct inductively sequences of elements $v(n) = (v_1(n), v_2(n), \ldots)$, $n = 1,2,\ldots$ and power series $\psi_n(X) \in A[[X]]$ such that

$$\psi_n(X)\colon F_{v(n)}(X,Y) \to F_{v(n+1)}(X) \text{ is a strict isomorphism} \tag{3.1.2}$$

$$\psi_n(X) \equiv X \bmod (X^2),\ \psi_n(X) \equiv X \bmod \mathfrak{m}^n \tag{3.1.3}$$

$$v_i(n) \equiv v_i \bmod \mathfrak{m}^n,\quad i = h+1, h+2, \ldots \tag{3.1.4}$$

First assume that $h > 1$, where h is the height of $F(X,Y)$ and $G(X,Y)$. Take $v_i(1) = v_i'$. Assume we have already found $v_i(n)$. Define $t_i(n)$ inductively by the formula.

$$t_i(n) = v_h(n)^{-p^i}(v_{i+h}(n) - v_{i+h} - t_1(n)v_{i+h-1}(n)^p - \ldots - t_{i-1}(n)v_{h+1}(n)^{p^{i-1}}) \tag{3.1.5}$$

This is well defined because the height of $F_{v(n)}(X,Y)$ is h, hence $v_h(n) \in A^*$, the units of A. It follows easily by induction that $t_i(n) \in \mathfrak{m}^n$. Let

$$\psi_n(X) = \alpha_{v(n),t(n)}(X),\quad v_i(n+1) = \bar{V}_i(v(n), t(n)) \tag{3.1.6}$$

where $t(n) = (t_1(n), t_2(n), \ldots)$. Because $v_1, \ldots, v_{h-1} \in \mathfrak{m}$ it follows from 3.1.5 and (2.2.1) that $v_i(n+1) \equiv v_i' \bmod \mathfrak{m}^{n+1}$, and because $\alpha_{V,T}(X) \equiv X \bmod (T_1, T_2, \ldots)$ we have $\psi_n(X) \equiv X \bmod \mathfrak{m}^n$. The rest of (3.1.2) -

– (3.1.4) is automatic. Because of (3.1.3) and (3.1.4) one has that the composed strict isomorphisms

$$F_{v'}(X,Y) \to F_{v(2)}(X,Y) \to \dots \to F_{v(n)}(X,Y)$$

converge to an isomorphism

$$\phi(X): F_{v'}(X,Y) \to F_w(X,Y)$$

where because of (3.1.4) $w_i = v_i$ for $i = h+1, h+2, \dots$. This proves the proposition for the case $h > 1$. The case $h = 1$ is handled in the same way except that the formula for $t_i(n)$ now becomes

$$t_i(n) = v_1(n)^{-p^i}(v_{i+1}(n)-v_{i+1}-t_1(n)v_i(n)^p-\dots-t_{i-1}(n)v_2(n)^{p^{i-1}})$$

$$(3.1.7) \qquad + v_1(n)^{-p^i}(v_1(n)v_i(n)^{p-1})(t_1(n)v_{i-1}(n)^p+\dots+t_{i-1}(n)v_1(n)^{p^{i-1}}) +$$

$$\dots$$

$$+ v_1(n)^{-p^i}(-1)^{i-1}(v_1(n)v_i(n)^{p-1}) \dots (v_1(n)v_2(n)^{p-1})(t_1(n)v_1(n)^p)$$

This proves the proposition.

Two formal groups $F(X,Y), G(X,Y)$ over A will be said to be *-isomorphic if there is a power series $\phi(X)$ such that $\phi(X) \equiv X \bmod \mathfrak{m}$ and $\phi F(X,Y) = G(\phi X, \phi Y)$. If in addition $\phi(X) \equiv X \bmod (X^2)$ then $F(X,Y)$ and $G(X,Y)$ are said to be strict *-isomorphic.

3.2. Lemma.

Let $v = (v_1, v_2, \dots)$, $v' = (v'_1, v'_2, \dots)$ be two sequences of elements of A such that $v_i, v'_i \in \mathfrak{m}$, $i = 1, \dots, h-1$, $v_h \in A^*$, the units of A, and such that $v_i = v'_i$ for $i = h+1, h+2, \dots$. Then $F_v(X,Y)$ and $F_{v'}(X,Y)$ are strict *-isomorphic iff $v_i = v'_i$, $i = 1, \dots, h$.

Proof. If also $v_i = v'_i$, $i = 1, \dots, h$, then $F_v(X,Y) = F_{v'}(X,Y)$. Inversely suppose that there is a strict *-isomorphism from $F_v(X,Y)$ to $F_{v'}(X,Y)$.

Because of the universality of $\alpha_{V,T}(X)$, cf [2], this isomorphism is of the form $\alpha_{v,t}(X)$ for certain $t_1, t_2, \ldots \in A$. It follows, cf. [2], that $t_i \in \mathfrak{m}$, because $\alpha_{v,t}(X)$ is a *-isomorphism. Suppose that $\alpha_{v,t}(X) \neq X$. Let $n, r \in \mathbb{N}$ be such that $t_n \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$, $t_i \in \mathfrak{m}^{r+1}$ for $i < n$, and $t_i \in \mathfrak{m}^r$, $i \geq n$. It then follows from (2.2.1) that $v'_{n+h} = \bar{V}_{n+h}(v,t) \neq v_{n+h}$ contradicting the assumptions.

q.e.d.

3.3. <u>The Lubin-Tate Formal Moduli Theorem (Cf. [9])</u>.

Let $\Phi(X,Y)$ be a formal group over k of height $h \in \mathbb{N}$. We are interested in the lifts of $\Phi(X,Y)$ over A modulo (strict) *-isomorphism. Because every formal group over k is (strictly) isomorphic to a p-typical one, we can as well assume that $\Phi(X,Y)$ is p-typical; i.e. that $\Phi(X,Y) = F_{v^*}(X,Y)$ for certain $v_1^*, v_2^*, \ldots \in k$. Because $\Phi(X,Y)$ has height h we have $v_1^* = \ldots = v_{h-1}^* = 0$. Choose $v_i \in A$, $i = 1,2,\ldots$ such that $\pi(v_i) = v_i^*$.

<u>Theorem</u>.

(i) Every strict *-equivalence class of lifts of $\Phi(X,Y)$ contains precisely one element of the form $F_w(X,Y)$, with $w_i = v_i + s_i$, $s_i \in \mathfrak{m}$, $i = 1, \ldots, h$; $w_j = v_j$, $j = h+1, h+2, \ldots$.

(ii) Every *-equivalence class of lifts of $\Phi(X,Y)$ contains precisely one element of the form $F_w(X,Y)$, with $w_i = v_i + s_i$, $s_i \in \mathfrak{m}$, $i = 1, \ldots, h-1$; $w_j = v_j$, $j = h, h+1, h+2, \ldots$.

I.e. there are h formal moduli for strict *-isomorphism classes and (h-1) formal moduli for *-isomorphism classes.

Proof. Let $F(X,Y)$ be a lift of $\Phi(X,Y)$. There is a universal way of making formal groups p-typical which is the identity on p-typical groups. Cf. [2]. Applying this to $F(X,Y)$ we find a strict *-isomorphic p-typical formal group $F'(X,Y)$ which also is a lift of $\Phi(X,Y)$ (Because $\Phi(X,Y)$ is already p-typical). Now apply proposition 3.1 and lemma 3.2 to obtain the first part of the theorem. To prove the second part we need the analogues of 3.1 and 3.2 above for the case of *-isomorphisms instead of strict *-isomorphisms. Now every *-isomorphism is composed of a strict *-isomorphism followed by an isomorphism of the form $\phi(X) = (1+t_o)X$ for a certain $t_o \in \mathfrak{m}$. Now an isomorphism $\phi(X) = (1+t_o)X$ applied to $F_v(X,Y)$ changes this formal group into $F_{v'}(X,Y)$, with $v'_i = (1+t_o)^{p^i-1} v_i$. So that the 'universal' formula replacing (2.2.1) for the case of *-isomorphisms becomes

$$\bar{\bar{v}}_n \equiv \{1+t_o(p^n-1)\}v_n - (t_1v_{n-1}^p+\ldots+t_{n-1}v_1^{p^{n-1}})$$

$$+ v_1v_{n-1}^{p-1}(t_1v_{n-2}+\ldots+t_{n-2}v_1^{p^{n-1}}) + \tag{3.3.1}$$

$$\cdots$$

$$+ (-1)^{n-2}(v_1v_{n-1}^{p-1}) \ldots (v_1v_2^{p-1})(t_1v_1^p)$$

where now the congruence is mod $(pt_i, i = 1,2,\ldots;\ t_it_j,\ i,j = 0,1,2,\ldots)$. Using this instead of (2.2.1) it is an easy exercise to prove the necessary analogues of proposition 3.1 and lemma 3.2. (In proving the analogon of 3.1 the first $t_i(n)$ to be defined is (of course) $t_o(n)$, $t_o(n) = (p^h-1)^{-1}v_h(n)^{-1}(v_h-v_h(n))$; the other $t_i(n)$ are defined as before except that a term $-\ t_o(p^i-1)v_i(n)$ is added; the proof of the analogon of lemma 3.2 is as before except that now $n \in \mathbb{N} \cup \{0\}$ instead of $n \in \mathbb{N}$). This concludes the proof of the theorem.

4. LAZARD'S CLASSIFICATION THEOREM FOR ONE DIMENSIONAL FORMAL GROUPS OVER AN ALGEBRAICALLY CLOSED FIELD.

Let k be an algebraically closed field. The case char(k) = 0 is not interesting because every formal group over k is then isomorphic to the additive one. We assume therefore that char(k) = p > 0. It is the aim of this section to give a new proof of Lazard's theorem that the one dimensional formal groups over k are classified by their heights. To do this we need some lemmas. Let U_n, $W_{n,\ell}$, Y_n be the polynomials defined in (2.1) above.

4.1. Lemma.

$$U_n \equiv V_n \mod (T_1,\ldots,T_{n-1})$$

Proof. By induction $U_1 = V_1$. Assuming the lemma for $s < n$, we see from (2.1.1) that $W_{n-\ell,\ell}$, $\ell = 1, \ldots, n-1$, is $\equiv 0 \mod(T_1,\ldots,T_{n-1})$ and that $Y_n \equiv 0 \mod(T_1,\ldots,T_{n-1})$, which proves the lemma also for n.

q.e.d.

4.2. Corollary.

$W_{n,\ell} \equiv 0 \bmod(T_1,\ldots,T_n)$, $\bar{W}_{n,\ell} \equiv 0 \bmod(T_1,\ldots,T_{n-1},p)$

4.3. Lemma.

Fix $h \in \mathbb{N}$. Then for all $n \in \mathbb{N}$

$$(4.3.1) \qquad \bar{V}_n \equiv V_n \mod(V_1,\ldots,V_{h-1},V_{h+1},\ldots,V_{n-1},T_1,\ldots,T_{n-h},p)$$

Proof. If $n \leq h$, this follows (by induction) directly from (2.1.1) and (2.1.2). Now let $n > h$, and suppose the statement has been proved for $s < n$. We have modulo $(V_1,\ldots,V_{h-1},V_{h+1},\ldots,V_{n-1})$

$$\bar{V}_n \equiv V_h\bar{W}_{n-h,h} + V_h T_{n-h}^{p^h} - T_1\bar{V}_{n-1}^p - \ldots - T_{n-1}\bar{V}_1^{p^{n-1}} + V_n + pT_n$$

Now use the induction hypothesis and corollary 4.2 above to conclude that (4.3.1) holds also for n.

q.e.d.

4.4. Lemma.

Fix $h \in \mathbb{N}$, then for all $n \in \mathbb{N}$

$$\bar{V}_{n+h} \equiv V_{n+h} - T_n V_h^{p^n} + V_h T_n^{p^h} \mod(V_1,\ldots,V_{h-1},V_{h+1},\ldots,V_{n+h-1},T_1,\ldots,T_{n-1},p)$$

Proof. We have

$$\bar{V}_{n+h} = \Sigma\, V_\ell \bar{W}_{n+h-\ell,\ell} + \bar{Y}_{n+h} + V_{n+h} + pT_{n+h}$$

$$\equiv V_h\bar{W}_{n,h} + V_h T_n^{p^h} - T_1\bar{V}_{n+h-1}^p - \ldots - T_{n+h-1}\bar{V}_1^{p^{n+h-1}} + V_{n+h}$$

Now apply lemma 4.2 and lemma 4.3.

4.5. Lazard's theorem (Cf. [8]).

Two formal groups over an algebraically closed field k of characteristic $p > 0$ are isomorphic if and only if they are of the same height.

Proof. Two isomorphic formal groups are certainly of the same height. Let F(X,Y) be a formal group over k of height h. We are going to prove that F(X,Y) is isomorphic to the formal group $F_{e(h)}(X,Y)$, where e(h) is

the sequence of elements (0,0,...,0,1,0,...) with the 1 in the h-th spot. First, because every formal group is strictly isomorphic to a p-typical one, we can assume that $F(X,Y) = F_v(X,Y)$ for certain $v_1, v_2, \ldots \in k$. Then $v_1 = \ldots = v_{h-1} = 0$.
We are going to construct sequences $v(n) = (v_1(n), v_2(n), \ldots)$ and strict power series $\psi_n(X)$ such that

(4.4.1) $$v_i(n) = 0 \text{ for } i = h+1, \ldots, h+n-1$$

(4.4.2) $$\psi_n(X) \equiv X \mod(X^{p^n})$$

(4.4.3) $$\psi_n(X) \colon F_{v(n)}(X,Y) \to F_{v(n+1)}(X,Y) \text{ is a strict isomorphism}$$

Take $v_i(1) = v_i$. Suppose we have already found $v_i(n)$, $i = 1,2,\ldots$. We define $t_i(n) = 0$, $i = 1, \ldots, n-1, n+1, n+2, \ldots$. Choose $t_n(n)$ such that

(4.4.4) $$v_{n+h}(n) - t_n(n)v_h(n)^{p^n} + v_h(n)t_n(n)^{p^h} = 0$$

This can be done because $v_h(n) \neq 0$. (The height of $F_{v(n)}(X,Y)$ being h).
Now define

(4.4.5) $$v_i(n+1) = \bar{V}_i(v(n), t(n)), \ \psi_n(X) = \alpha_{v(n),t(n)}(X).$$

The $v_i(n+1)$ satisfy (4.4.1) because of lemma's 4.3 and 4.4; (4.4.3) follows from (4.4.5) and (4.4.2) holds because $t_i(n) = 0$ for $i = 1, \ldots, n-1$.
The composed strict isomorphisms

$$F_v(X,Y) \to F_{v(2)}(X,Y) \to \ \ldots \to F_{v(n)}(X,Y)$$

converge to a strict isomorphism

$$\psi(X) \colon F_v(X,Y) \to F_w(X,Y)$$

because of (4.4.2), and because of (4.4.1) we have that $w_{h+1} = w_{h+2} = \ldots = 0$.
Also $w_1 = \ldots = w_{h-1} = 0$ because the height of $F_w(X,Y)$ is h.
Let $\phi(X) = a^{-1}X$, where a is a (p^h-1)-th root of w_h. Then $\phi(X)$ is an isomorphism

$$\phi(X): F_w(X,Y) \to F_{e(h)}(X,Y)$$

This concludes the proof of the theorem.

4.5. Remark.

We have also shown that over an algebraically closed field k every one dimensional formal group is strictly isomorphic to one of the form $F_w(X,Y)$ with $w_1 = \dots = w_{h-1} = 0 = w_{h+1} = w_{h+2} = \dots$.

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