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# THE LINEAR PART OF THE BROWN-PETERSON COHOMOLOGY OPERATIONS MAP $\eta_R$: $BP_*(pt) \to BP_*(BP)$

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## 1. INTRODUCTION.

This paper could equally well have been called: 'Calculation of the stable cohomology operations $r_{\Delta_i}$ of Brown-Peterson cohomology', where $\Delta_i$ stands for the exponent sequence $(0,\dots,0,1,0,\dots)$ with the 1 in the i-th place.

Let BP stand for the Brown-Peterson spectrum. The stable cohomology operations of BP cohomology can be described as continuous homomorphisms over $BP_*(pt)$ from $BP_*(BP)$ to $BP_*(pt)$, where $BP_*(BP)$ is seen as a left $BP_*(pt)$ module. Cf. [1]. To find out what such an operation does to the elements of $BP_*(pt)$ compose such a homomorphism with the right unit map $\eta_R$: $BP_*(pt) \to BP_*(BP)$; cf [1].

Now $BP_*(pt) = \mathbb{Z}_{(p)}[v_1,v_2,\dots]$ cf. [2], and $BP_*(BP) = BP_*(pt)[t_1,t_2,\dots]$.

In this paper we calculate $\eta_R(v_i) \in BP_*(BP) \bmod (t_1,t_2,\dots)^2$.

As a corollary we find $r_{\Delta_n}(v_i)$ for all i, n = 1, 2,...

## 2. THE RECURSION FORMULA

We know that $BP_*(pt) \otimes \mathbb{Q} = \mathbb{Q}[m_1,m_2,\dots]$, where $m_i = p^{-i}[\mathbb{C}P^{p^i-1}]$ where the square brackets denote cobordism classes. The $v_i$ and $m_i$ are related as follows

$$(2.1) \qquad m_k = \sum_{i_1+\dots+i_r=k,\, i_j\geq 1,\, r\geq 1} \frac{v_{i_1} v_{i_2}^{p^{i_1}} \cdots v_{i_r}^{p^{i_1+\dots+i_{r-1}}}}{p^r}$$

The map $\eta_R$: $BP_*(pt) \to BP_*(BP)$ is given over $\mathbb{Q}$ by

$$(2.2) \qquad m_k \mapsto \bar{m}_k = \sum_{i=o}^{k} m_i t_{k-i}^{p^i}$$

where $m_o = \bar{m}_o = t_o = 1$. Write

(2.3) $$\bar{m}_k = \sum_{i_1+\dots+i_r=k,\, i_j\geq 1} \frac{\bar{v}_{i_1}\bar{v}_{i_2}^{p^{i_1}}\cdots\bar{v}_{i_r}^{p^{i_1+\dots+i_{r-1}}}}{p^r}$$

(This determines the $\bar{v}_i \in BP_*(BP) \otimes \mathbb{Q}$ uniquely). It turns out that the $\bar{v}_i$ are polynomials with integer coefficients in the $v_1,\dots,v_i$; $t_1,\dots,t_i$. Cf. [3]. The map $\eta_R$: $BP_*(pt) \to BP_*(BP)$ is now given by $v_i \mapsto \bar{v}_i$.
In [3] we gave the following recursion procedure for calculating the $\bar{v}_i$. Define the polynomials $U_r$, $W_{s,\ell}$, $Y_r$ with integer coefficients in the symbols $v_1, v_2, \dots$; $t_1, t_2, \dots$; $V_1, V_2, \dots$ as follows

(2.4)
$$U_1 = v_1,\ U_r = \sum_{k=1}^{r-1} v_k W_{r-k,k} + Y_r + v_r$$
$$W_{s,\ell} = p^{-1}(U_s^{(p^\ell)} - (U_s + pt_s)^{p^\ell})$$
$$Y_r = \sum_{k=1}^{r-1} (v_k t_{r-k}^{p^k} - t_{r-k} V_k^{p^{r-k}})$$

where $U_s^{(p^\ell)}$ is the polynomial obtained from $U_s$ by replacing each of the $v_i, t_i, V_i$, $i = 1,2,\dots$ with their $p^\ell$-th powers.
Once we know $\bar{v}_1,\dots,\bar{v}_{r-1}$, the polynomial $\bar{v}_r$ is then given by

(2.5) $$v_r = \bar{U}_r + pt_r = \sum_{k=1}^{r-1} v_k \bar{W}_{r-k,k} + \bar{Y}_r + v_r + pt_r$$

where $\bar{U}_r$, $\bar{W}_{s,\ell}$, $\bar{Y}_r$ are obtained from $U_r$, $W_{s,\ell}$, $Y_r$, respectively, by substituting $\bar{v}_i$ for $V_i$, $i = 1,2,\dots$ .

## 3. CALCULATION of $\bar{v}_i$ mod J.

Let J be the ideal generated by the $t_i t_j$, $i,j = 1,2,\dots$; i.e. $J = (t_1, t_2, \dots)^2$. We now use the recursion scheme given above in section 2 to calculate $\bar{v}_i$ mod J. Let R be short for $BP_*(BP) = BP_*(pt)[t_1, t_2, \dots]$.

3.1. <u>Lemma</u>.

Let $U_s$, $W_{s,\ell}$ be as in (2.4) above. Suppose that $U_s \equiv v_s + \Sigma t_i z_i$ mod J for certain $z_i \in R$. Then

(3.1.1) $$W_{s,\ell} \equiv -p^\ell t_s v_s^{p^\ell - 1} - (\Sigma t_i z_i) p^{\ell-1} v_s^{p^\ell - 1} \quad \text{mod } J$$

Proof. It follows from the hypothesis concerning $U_s$ that

$$U_s^{(p^\ell)} \equiv v_s^{p^\ell} \quad \bmod J$$

$$(U_s+pt_s)^{p^\ell} \equiv v_s^{p^\ell} + \binom{p^\ell}{1}(\Sigma t_i z_i)v_s^{p^\ell-1} + \binom{p^\ell}{1}v_s^{p^\ell-1}(pt_s) \quad \bmod J$$

The lemma follows from this because $W_{s,\ell} = p^{-1}(U_s^{(p^\ell)}-(U_s+pt_s)^{p^\ell})$ and because if x, y ∈ R and $x \equiv y \bmod (J \otimes \mathbb{Q})$ in $R \otimes \mathbb{Q}$, then $x \equiv y \bmod J$.

3.2. Proposition.

Let $U_n$ be as in (2.4). Then we have mod J

(3.2.1)

$$U_n + pt_n \equiv \sum_{(s_1,\ldots,s_t,i,j)} (-1)^t (v_{s_1} v_{n-s_1}^{p^{s_1}-1} p^{s_1-1})(v_{s_2} v_{n-s_1-s_2}^{p^{s_2}-1} p^{s_2-1}) \ldots$$

$$\cdot (v_{s_t} v_{n-s_1-\ldots-s_{t-1}}^{p^{s_t}-1} p^{s_t-1}).(-t_i v_j^{p^i})$$

$$+ \sum_{(s_1,\ldots,s_t,i)} (-1)^t (v_{s_1} v_{n-s_1}^{p^{s_1}-1} p^{s_1-1})(v_2 v_{n-s_1-s_2}^{p^{s_2}-1} p^{s_2-1}). \ldots .$$

$$\cdot (v_{s_t} v_{n-s_1-\ldots-s_{t-1}}^{p^{s_t}-1} p^{s_t-1})(pt_i)$$

$$+ v_n$$

where the first sum is over all sequences $(s_1,\ldots,s_t,i,j)$ such that $s_k,i,j \in \mathbb{N}$, $s_1+ \ldots + s_t + i + j = n$, $t \in \mathbb{N} \cup \{0\}$ and the second sum is over all sequences $(s_1,\ldots,s_t,i)$ such that $s_k$, $i \in \mathbb{N}$, $s_1+ \ldots + s_t + i = n$, $t \in \mathbb{N} \cup \{0\}$.

Proof. According to (2.4) we have

(3.2.2) $$U_n = v_n + Y_n + \sum_{k=1}^{n-1} v_k W_{n-k,k}\ , \quad U_1 = v_1$$

Use induction with respect to n, noting that formula (3.2.1) holds trivially for n = 1. Now remark that the hypothesis of lemma 3.1 holds for $U_s$, s < n. Using formula (3.1.1) and (3.2.1) for s < n, formula (3.2.1) itself readily follows.

3.3. Corollary.

The formula for $\bar{v}_n$ is obtained from formula (3.2.1) by substituting $v_i$ for $V_i$, $i = 1, \ldots, n-1$ in the righthand side of 3.2.1.

Proof. We claim that

$$\bar{v}_n \equiv v_n \quad (\mathrm{mod}(t_1, t_2, \ldots)) \tag{3.3.1}$$

This claim is proved by induction simultaneously with the corollary itself. The congruence (3.3.1) obvious holds for $n = 1$, as $\bar{v}_1 = v_1 + pt_1$. Suppose that the corollary and (3.3.1) both have been proved for $s < n$. Then the corollary follows for $s = n$ because of (3.2.1) and (2.5), which in turn proves (3.3.1) for $s = n$.

3.4. Corollary.

Let I denote the ideal $(\ldots, pt_i, \ldots; \ldots, t_i t_j, \ldots)$. Then we have modulo I

$$\begin{aligned} \bar{v}_n \equiv\ & v_n - t_1 v_{n-1}^p - \ldots - t_{n-1} v_1^{p^{n-1}} \\ & + v_1 v_{n-1}^{p-1} (t_1 v_{n-2}^p + \ldots + t_{n-2} v_1^{p^{n-2}}) \\ & - v_1 v_{n-1}^{p-1} v_1 v_{n-2}^{p-1} (t_1 v_{n-3}^p + \ldots + t_{n-3} v_1^{p^{n-3}}) \\ & + (-1)^{n-2} v_1 v_{n-1}^{p-1} \ldots v_1 v_2^{p-1} (t_1 v_1^p) \end{aligned} \tag{3.4.1}$$

## 4. APPLICATION TO BROWN-PETERSON COHOMOLOGY OPERATIONS.

Let $\Delta_i$ be the exponent sequence $(0,0,\ldots,0,1,0,\ldots)$ with the 1 in the i-th place, and let $r_{\Delta_i}$ denote the corresponding BP-cohomology operation. I.e. $r_{\Delta_i}(v_n) =$ coefficient of $t_i$ in $\bar{v}_n$.

4.1. Theorem.

For all $0 < i < n$ we have

$$r_{\Delta_i}(v_n) = \sum_{(s_1,\dots,s_t,j)} (-1)^{t+1}(v_{s_1} v_{n-s_1}^{p^{s_1}-1}) \dots (v_{s_t} v_{n-s_1-\dots-s_{t-1}}^{p^{s_t}-1}) v_j^{p^i} p^{n-t-i-j}$$

(4.1.1)

$$+ \sum_{(s_1,\dots,s_t)} (-1)^{t}(v_{s_1} v_{n-s_1}^{p^{s_1}-1}) \dots (v_{s_t} v_{n-s_1-\dots-s_{t-1}}^{p^{s_t}-1}) p^{n-i-t+1}$$

where the first sum is over all sequences $(s_1,\dots,s_t,j)$ such that $s_k, j \in \mathbb{N} = \{1,2,3,\dots\}$, $t \in \mathbb{N} \cup \{0\}$, and $s_1 + \dots + s_t + j = n - i$, and the second sum is over all sequences $(s_1,\dots,s_t)$ such that $s_k \in \mathbb{N}$, $t \in \mathbb{N} \cup \{0\}$, $s_1 + \dots + s_t = n - i$.

4.2. Addendum.

$$r_{\Delta_n}(v_n) = p, \quad r_{\Delta_i}(v_n) = 0 \quad \text{if } i > n$$

4.3. Proof of (4.1) and (4.2): write down the coefficient of $t_i$ in $\bar{v}_n$ using (3.2.1) and corollary 3.3.

4.4. Theorem.

For all $0 < i \leq n$ we have the congruence

$$(4.4.1) \quad r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} + v_1 v_{n-1}^{p-1} v_{n-i-1}^{p^i} + \dots + (-1)^{n-i} v_1 v_{n-1}^{p-1} \dots v_1 v_{i+1}^{p-1} v_1^{p^i} \mod p$$

Proof. This follows directly from either (4.1.1) or (3.4.1)

4.5. Remark.

Some of the terms in the two sums of (3.2.1) give rise to the same monomials in $v_1, v_2, \dots; t_1, t_2, \dots$ E.g. the terms with $s_t = 1$, $i = 1$ in the second sum gives rise to the same monomials as the terms $(s_1,\dots,s_{t-1},1,1)$ in the first sum.

4.6. Some Examples.

Modulo J we have

$$\bar{v}_4 \equiv -v_1^{p^3}t_3 - p^3v_1^{p^3-1}v_3t_1 + v_1^{p^2+1}v_3^{p-1}t_2 + p^2v_1^{p^2}v_2v_3^{p-1}t_1$$

$$-(p+1)v_1^{p+2}v_2^{p-1}v_3^{p-1}t_1 + (p^2+p)v_1^pv_2^{p^2}t_1 + pv_1^2v_2^{p-1}v_3^{p-1}t_2$$

$$+ v_1v_2^pv_3^{p-1}t_1 - pv_1v_3^{p-1}t_3 - (p^2+1)v_2^{p^2}t_2 - v_3^pt_1 + pt_4$$

and from this the $r_{\Delta_i}(v_4)$, $i = 1,2,3,4$ can be read off immediately. The results modulo p are

$$r_{\Delta_4}(v_4) \equiv 0 \pmod p$$

$$r_{\Delta_3}(v_4) \equiv -v_1^{p^3} \pmod p$$

$$r_{\Delta_2}(v_4) \equiv +v_1^{p^2+1}v_3^{p-1} - v_2^{p^2} \pmod p$$

$$r_{\Delta_1}(v_4) \equiv -v_1^{p+2}v_2^{p-1}v_3^{p-1} + v_1v_2^pv_3^{p-1} - v_3^p \pmod p$$

which (of course) agrees with (4.4.1).

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