ON THE KALMAN FILTER AND THE ECONOMETRIC GENERAL LINEAR MODEL

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July 11, 1975
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In this note we show how a particular econometric estimator for the general linear model (the one best adapted to the noise present in the observations) arises as the limit of Kalman state estimators of a discrete dynamical system with trivial dynamics.

1. First consider a discrete dynamical system given by the equations

\[ x_{k+1} = A_k x_k + B_k u_k + f_k + C_k w_k \]
\[ y_k = H_k x_k \]
\[ z_k = y_k + v_k \]

where \( x_k \) is an n-dimensional state vector; \( u_k \) are deterministic controls; \( f_k \) a deterministic forcing vector; \( w_k \) is random noise in the system; \( y_k \) is the p-dimensional output vector; \( z_k \) are the observations of the \( y_k \) corrupted by random noise \( v_k \); \( A_k, B_k, C_k, H_k \) are matrices of appropriate sizes, which are assumed to be known. One further makes the assumptions

\[ E(w_k) = 0 = E(v_k), E(w_k w_k^T) = \delta_{kl} Q_k, E(v_k v_k^T) = \delta_{kl} R_k, E(w_k v_k^T) = 0 \]

where \( E \) denotes expectation and \( T \) denotes transposition. The covariance matrices \( Q_k \) and \( R_k \) are also assumed to be known.

Given all this, the Kalman filter with starting values \( \hat{x}_0 = 0, P_0 \) is given by the equations (cf. e.g. [2]).

\[ P_k' = A_{k-1} P_{k-1} A_{k-1}^T + C_{k-1} Q_{k-1} C_{k-1}^T \]
\[ K_k = P_k' H_k (H_k P_k' H_k^T + R_k)^{-1} \]
\[ \hat{x}_k' = A_{k-1} \hat{x}_{k-1} + f_{k-1} + B_{k-1} u_{k-1} \]
\[ \hat{x}_k = \hat{x}_k' + K_k (z_k - H_k \hat{x}_k') \]
2. Now consider the general linear model as it is often used in econometrics

\[ z = Hx + v \]

(H is usually written \( X \), and \( x \) is usually written \( \beta \); cf. [1] for a discussion of the linear model), where \( x \) is to be estimated from the observations \( z \) and \( v \) is random noise with \( \mathbb{E}(v) = 0 \), \( \mathbb{E}(vv^T) = R \), where \( R \) is nonnegative definite.

We write the model as (a rather trivial) discrete dynamical system

\[ x_{k+1} = x_k, \quad y_k = Hx_k, \quad z_k = y_k + v \]

Now we are going to apply the discrete Kalman filter (3) to it starting with \( \hat{x}_0 = 0 \) and an arbitrary initial covariance matrix \( P_0 \). Using a double induction one finds

\[ K_1 = P_0H^T(HP_0H^T+R)^{-1}, \quad K_n = P_0H^T(nHP_0H^T+R)^{-1} \]

It follows from this that

\[ K_n = K_{n-1} - K_nH_n^{-1} \]

and hence, using induction again, that the estimate for \( \hat{x}_n \) is equal to

\[ \hat{x}_n = nP_0H^T(nHP_0H^T+R)^{-1}z \]

3. We are interested in what happens as \( n \) goes to infinity. There are (at least) two interesting cases.

Case A: \( HP_0H^T \) is nonsingular. (I.e. in any case less outputs than the dimension of the system). In this case the \( R \) in equation (8) can be neglected as \( n \) goes to infinity and we find the estimator

\[ \hat{x} = P_0H^T(HP_0H^T)^{-1}z \]
Case B. R is nonsingular, \( p > n \), \( \text{rank}(H) = n \). This is what is usually assumed in the econometric general linear model.

Premultiplication with \( H^T R^{-1} H \) of the matrix in (8) gives

\[
(10) \quad (H^T R^{-1} H)(nH_0^T H^T(nH_0^T H^T + R))^{-1} = H^T R^{-1} - H^T (nH_0^T H^T + R)^{-1}
\]

Now

\[
(11) \quad \lim_{n \to \infty} H^T (nH_0^T H^T + R)^{-1} = 0
\]

which is seen as follows. Because \( \text{rank}(H^T) = \text{rank}(H_0^T H^T) \) it suffices to prove that \( H_0^T H^T (nH_0^T H^T + R)^{-1} \) goes to zero. But

\[
(12) \quad H_0^T H^T (nH_0^T H^T + R)^{-1} = n^{-1} I - n^{-1} R (nH_0^T H^T + R)^{-1},
\]

and one easily sees that the terms in \( (nH_0^T H^T + R)^{-1} \) are bounded independently of \( n \) (e.g. by diagonalizing \( H_0^T H^T \)). This proves (11).

Using (10) and (11) in (8) it follows that in case B one obtains the limit estimator

\[
(13) \quad \hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} z
\]

which is the econometric generalized least squares estimator with weighting matrix equal to \( R = \text{E}(vv^T) \).

REFERENCES