ERASMUS UNIVERSITY ROTTERDAM

ECONOMETRIC INSTITUTE

Report 7512/M,S,E

ON THE KALMAN FILTER AND THE ECONOMETRIC GENERAL LINEAR MODEL

by Michiel Hazewinkel

July 11, 1975

SA

ON THE KALMAN FILTER AND THE ECONOMETRIC GENERAL LINEAR MODEL

by Michiel Hazewinkel

In this note we show how a particular econometric estimator for the general linear model (the one best adapted to the noise present in the observations) arises as the limit of Kalman state estimators of a discrete dynamical system with trivial dynamics.

1. First consider a discrete dynamical system given by the equations

$$
(1)\qquad
\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k + f_k + C_k w_k \\
y_k &= H_k x_k \\
z_k &= y_k + v_k
\end{aligned}
$$

where x_k is an n-dimensional state vector; u_k are deterministic controls; f_k a deterministic forcing vector; w_k is random noise in the system; y_k is the p-dimensional output vector; z_k are the observations of the y_k corrupted by random noise v_k; A_k, B_k, C_k, H_k are matrices of appropriate sizes, which are assumed to be known. One further makes the assumptions

$$
(2)\qquad E(w_k) = 0 = E(v_k),\ E(w_k w_l^T) = \delta_{kl} Q_k,\ E(v_k v_l^T) = \delta_{kl} R_k,\ E(w_k v_l^T) = 0
$$

where E denotes expectation and T denotes transposition. The covariance matrices Q_k and R_k are also assumed to be known.
Given all this, the Kalman filter with starting values $\hat{x}_o = 0$, P_o is given by the equations (cf. e.g. [2]).

$$
(3)\qquad
\begin{aligned}
P'_k &= A_{k-1} P_{k-1} A_{k-1}^T + C_{k-1} Q_{k-1} C_{k-1}^T \\
P_k &= P'_k - K_k H_k P'_k \\
K_k &= P'_k H_k^T (H_k P'_k H_k^T + R_k)^{-1} \\
\hat{x}'_k &= A_{k-1} \hat{x}_{k-1} + f_{k-1} + B_{k-1} u_{k-1} \\
\hat{x}_k &= \hat{x}'_k + K_k (z_k - H_k \hat{x}'_k)
\end{aligned}
$$

2. Now consider the general linear model as it is often used in econometrics

$$(4) \qquad z = Hx + v$$

(H is usually written X, and x is usually written β; cf. [1] for a discussion of the linear model), where x is to be estimated from the observations z and v is random noise with $E(v) = 0$, $E(vv^T) = R$, where R is nonnegative definite.
We write the model as (a rather trivial) discrete dynamical system

$$(5) \qquad x_{k+1} = x_k,\ y_k = Hx_k,\ z_k = y_k + v$$

Now we are going to apply the discrete Kalman filter (3) to it starting with $\hat{x}_o = 0$ and an arbitrary initial covariance matrix P_o. Using a double induction one finds

$$(6) \qquad K_1 = P_oH^T(HP_oH^T+R)^{-1},\ K_n = P_oH^T(nHP_oH^T+R)^{-1}$$

It follows from this that

$$(7) \qquad K_n = K_{n-1} - K_nHK_{n-1}$$

and hence, using induction again, that the estimate for $\hat{x}_n$ is equal to

$$(8) \qquad \hat{x}_n = nP_oH^T(nHP_oH^T+R)^{-1}z$$

3. We are interested in what happens as n goes to infinity. There are (at least) two interesting cases.
Case A: HP_oH^T is nonsingular. (I.e. in any case less outputs than the dimension of the system). In this case the R in equation (8) can be neglected as n goes to infinity and we find the estimator

$$(9) \qquad \hat{x} = P_oH^T(HP_oH^T)^{-1}z$$

Case B. R is nonsingular, $p \geq n$, rank(H) = n. This is what is usually assumed in the econometric general linear model.
Premultiplication with $H^TR^{-1}H$ of the matrix in (8) gives

$$(10) \qquad (H^TR^{-1}H)(nP_oH^T)(nHP_oH^T+R)^{-1} = H^TR^{-1} - H^T(nHP_oH^T+R)^{-1}$$

Now

$$(11) \qquad \lim_{n\to\infty} H^T(nHP_oH^T+R)^{-1} = 0$$

which is seen as follows. Because $\mathrm{rank}(H^T) = \mathrm{rank}(HP_oH^T)$ it suffices to prove that $HP_oH^T(nHP_oH^T+R)^{-1}$ goes to zero. But

$$(12) \qquad HP_oH^T(nHP_oH^T+R)^{-1} = n^{-1}I - n^{-1}R(nHP_oH^T+R)^{-1},$$

and one easily sees that the terms in $(nHP_oH^T+R)^{-1}$ are bounded independantly of n (e.g. by diagonalizing HP_oH^T). This proves (11).
Using (10) and (11) in (8) it follows that in case B one obtains the limit estimator

$$(13) \qquad \hat{x} = (H^TR^{-1}H)^{-1}H^TR^{-1}z$$

which is the econometric generalized least squares estimator with weighting matrix equal to $R = E(vv^T)$.

REFERENCES

1. J. Koerts, A.P.J. Abrahamse. On the Theory and Applications of the General Linear Model, Rotterdam Univ. Pr., 1969.
2. H.W. Sorenson. Kalman Filtering Techniques. In: C.T. Leondes (ed). Advances in Control Systems. Vol. 3, Acad. Pr., 1966, 219-292.