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ON INVARIANTS, CANONICAL FORMS AND MODULI FOR LINEAR, CONSTANT, FINITE DIMENSIONAL, DYNAMICAL SYSTEMS

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ON INVARIANTS, CANONICAL FORMS AND MODULI FOR LINEAR, CONSTANT, FINITE DIMENSIONAL, DYNAMICAL SYSTEMS

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1. INTRODUCTION AND SURVEY OF RESULTS

A linear, constant, finite dimensional dynamical system is thought of as being represented by a triple of matrices (F,G,H), where F is an n x n matrix, G an n x m matrix, and H an p x n matrix; i.e. there are m inputs, p outputs and the state space dimension is n. The dynamical system itself is

$$\dot{x} = Fx + Gu, \quad y = Hx \tag{1.1}$$

or, if one prefers discrete time systems

$$x_{t+1} = Fx_t + Gu_t, \quad y_t = Hx_t \tag{1.2}$$

A change of coordinates in state space changes the triple of matrices (F,G,H) into the triple (SFS^{-1}, SG, HS^{-1}). Let $\underline{DS}$ denote the space of all triples (F,G,H); i.e. $\underline{DS}$ is affine space of dimension $np + n^2 + nm$.
Then we have just defined an action of GL_n on $\underline{DS}$. This paper is concerned with the following type problems. To what extent does the quotient $\underline{DS}/GL_n$ exist ? Does the quotient have a nice geometric structure ? Do there exist globally defined algebraic continuous canonical forms for triples (F,G,H)?
Most of the paper is concerned with the input aspect only, i.e. instead of studying triples (F,G,H) under the action $(F,G,H)^S = (SFS^{-1}, SG, HS^{-1})$ we study pairs (F,G) under the action $(F,G)^S = (SFS^{-1}, SG)$. Let $\underline{IS}$ be the affine space of all pairs (F,G) and $\underline{IS}_{cr}$ the open subvariety of all completely reachable pairs. It turns out that the orbit space $\underline{IS}_{cr}/GL_n$ has a nice geometric structure. In fact, it is a quasi-projective algebraic variety. Moreover this variety $\underline{M}_{m,n} = \underline{IS}_{cr}/GL_n$ turns out to be a fine moduli space for algebraic families of completely reachable pairs (suitably defined). I.e. the points of $\underline{M}_{m,n}$ correspond bijectively to equivalence classes of completely reachable pairs and there exists over $\underline{M}_{m,n}$ a universal family from which every family can be obtained (uniquely) by pullback.

However, if there are two or more inputs the underlying bundle of this universal family is non trivial and this ruins all chances of finding continuous algebraic canonical forms for $\underline{IS}_{cr}$. This in turn also implies the nonexistence of continuous algebraic canonical forms for $\underline{IS}$, $\underline{DS}_{cr}$ and $\underline{DS}$. There exist of course (many) discontinuous canonical forms. (To keep the non existence result in proper perspective: the Jordan canonical form for square matrices is also not continuous).

In this paper we shall work over an arbitrary field k, which, for convenience, can be taken to be algebraically closed. However, all the constructions performed yield varieties defined over k itself. The category of varieties over k is denoted $\underline{Sch}_k$. Much of the material which follows is also contained in [2] in one way or another. The emphasis and presentation are different, however; here we stress the underlying ideas rather than the algebraic geometric techniques. Also this paper contains additional new material, notably subsections 3.9, 6.1, 6.3, 7.1, 7.2, 7.3, 7.4, 7.5.

2. GRASSMANN VARIETIES

Let $\underline{A}_{n,s}$ denote the affine space of all n x s matrices, where s > n; i.e. $\underline{A}_{n,s}$ is affine space of dimension ns. Let $\underline{A}^{reg}_{n,s}$ denote the (Zariski) open dense subvariety of $\underline{A}_{n,s}$ consisting of matrices of maximal rank. The group GL_n acts on $\underline{A}_{n,s}$ (and $\underline{A}^{reg}_{n,s}$) by multiplication on the left: $(S,A) \mapsto SA$.

The orbit space $\underline{A}^{reg}_{n,s}/GL_n$ has a nice geometric structure; it is a smooth projective algebraic variety of dimension n x (s-n), known as the Grassmann variety of n-planes in s-space, and denoted $\underline{G}_{n,s}$. This interpretation arises as follows. Let A be an n x s matrix of rank n. The n rows of A span an n-dimensional subspace of affine space of dimension s, and, clearly, the rows of SA span the same subspace.

The (canonical) projective embedding of $\underline{G}_{n,s}$ is obtained as follows.

A selection α of $\{1,\ldots,s\}$ is a subset of size n. For each selection α and n x s matrix A, let A_α be the submatrix of A consisting of those columns of A which are indexed by an element of α. Let $N = \binom{n}{s} - 1$, the number of selections minus 1. We now define a morphism from $\underline{G}_{n,s}$ to projective N-space

$$(2.1) \qquad G_{n,s} \to \underline{P}^N,\ A \to (\det(A_\alpha))_\alpha$$

where det denotes determinant. This is an embedding and exhibits $\underline{G}_{n,s}$ as a closed subvariety of $\underline{P}^N$.

Choose a selection α. The open subvariety of $\underline{G}_{n,s}$ where $\det(A_\alpha) \neq 0$ is isomorphic to affine n x (s-n) space: a point $x \in \underline{G}_{n,s}$ corresponds to the unique n x s matrix A_x for which (i) $(A_x)_\alpha = I_n$, the n x n unit matrix and (ii) the rows of

A_x span the linear subspace x.
For further details concerning $\underline{G}_{n,s}$, e.g. for a description of the equations defining $\underline{G}_{n,s}$ as a closed subvariety of $\underline{P}^N$ cf. e.g. [4]. For more details concerning $\underline{G}_{n,s}$ from the differential topological point of view cf. e.g. [3].

3. THE COARSE MODULI SPACE $\underline{M}_{m,n}$

Let $\underline{IS}$ denote the affine space of all pairs of matrices (F,G). The group GL_n acts on $\underline{IS}$ by $(F,G) \to (SFS^{-1},SG)$.

3.1. The Morphism R and Completely Reachable Pairs.

We define the morphism R from $\underline{IS}$ to $\underline{A}_{n,(n+1)m}$ by

$$R(F,G) = (G\ FG\ \dots\ F^nG) \tag{3.1.1}$$

The pair (F,G) is said to be <u>completely reachable</u> if R(F,G) has rank n. Let $\underline{IS}_{cr}$ denote the Zariski open subvariety of $\underline{IS}$ consisting of the completely reachable pairs. It follows that R induces a morphism

$$R: \underline{IS}_{cr} \to A^{reg}_{n,(n+1)m} \tag{3.1.2}$$

Note that R is a GL_n-invariant morphism. I.e.

$$R(SFS^{-1},SG) = SR(F,G) \tag{3.1.3}$$

3.2. Nice Selections and Successor Selections.

In section 2 we have seen that selections play on important role in the description of the quotient $\underline{A}_{n,s}/GL_n$. In view of (3.1.3) it is to be expected that they will also be important in the case of GL_n acting on $\underline{IS}$. Certain selections of the (n+1)m columns of the R(F,G) play a special role. To define them we number the (n+1)m columns by pairs of integers (lexicographically ordered) as follows

$$01, \dots, 0m;\ 11, \dots, 1m;\ \dots\ ;\ n1, \dots, nm$$

A selection α is called <u>nice</u> if $(i,j) \in \alpha \Rightarrow (i',j) \in \alpha$ for all $i' \leq i$.
Given a nice selection α its <u>successor selections</u> are obtained as follows: take any $(i,j) \in \{01,\dots,nm\}$ such that $(i,j) \notin \alpha$ but $(i',j) \in \alpha$ for all $i' < i$. Now take away from $\alpha \cup (i,j)$ any of the original elements of α. The result is a successor selection. Note that a successor selection may be nice but need not be.
Example, take $m = 4$, $n = 6$

$$
\begin{array}{cccc}
* & x & x & x \\
. & x & * & x \\
. & x & . & * \\
. & * & . & . \\
. & . & . & . \\
. & . & . & .
\end{array}
\qquad\qquad
\begin{array}{ccc}
0,1 & \ldots & 0,4 \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
6,1 & \ldots & 6,4
\end{array}
$$

The crosses constitute a nice selection. Its successor selections are obtained by adding one of the stars and deleting one of the crosses.

3.3. Lemma.

If (F,G) is a completely reachable pair then there is a nice selection α such that $\det(R(F,G)_\alpha) \neq 0$.

3.4. Successor Indices.

Let α be a nice selection. The successor indices of α are those elements $(i,j) \in \{01,\ldots,nm\}$ such that $(i',j) \in \alpha$ for all $i' < i$. I.e. in the example of subsection 3.2 the $*$'s mark the successor indices of the nice selection given by the x's.

We now define an algebraic morphism $\psi_\alpha : \underline{A}^{mn} \to \underline{IS}$ as follows. Let $\sigma(\alpha)$ be the set of successor indices of the nice selection α. The subset $\alpha \cup \sigma(\alpha)$ has precisely n + m elements. Give this subset the ordering induced by the (lexicographic) ordering of $\{01,\ldots,nm\}$. Write an element $x \in \underline{A}^{mn}$ as an array of m columns of length n; let x_i denote the i-th column in this array. We now assign to each element (i,j) of $\alpha \cup \sigma(\alpha)$ a column c(i,j) of length n as follows: if (i,j) is the ℓ-th element of α then $c_{(i,j)} = e_\ell$, the ℓ-th unit vector. If (i,j) is the ℓ-th element of $\sigma(\alpha)$ then $c_{(i,j)} = x_\ell$, the ℓ-th column of x. Writing G_i (resp. F_i) for the i-th column of G (resp. F) we now define ψ_α by

$$
\begin{aligned}
&\psi_\alpha(x) = (F,G), \text{ where} \\
&\quad G_i = \text{column assigned to i-th element of } \alpha \cup \sigma(\alpha) \\
&\quad F_i = \text{column assigned to (m+i)-th element of } \alpha \cup \sigma(\alpha).
\end{aligned}
$$

Thus in the example of subsection 3.2 we have

$$G_1 = x_1,\ G_2 = e_1,\ G_3 = e_2,\ G_4 = e_3$$

$$F_1 = e_4,\ F_2 = x_2,\ F_3 = e_5,\ F_4 = e_6,\ F_5 = x_3,\ F_6 = x_4$$

Note that if $\psi_\alpha(x) = (F,G)$, then $R(F,G)_\alpha$ = unit matrix, and if (i,j) is the ℓ-th element of $\sigma(\alpha)$ then $R(F,G)_{(i,j)}$, the (i,j)-th column of R(F,G), is equal to x_ℓ, the ℓ-th column of x. (This is easy to check; if $(i,j) \in \alpha \cup \sigma(\alpha)$ is the $(m+\ell)$-th element of $\alpha \cup \sigma(\alpha)$ then (i-1,j) is the ℓ-th element of α).

3.5. Lemma.

$R\psi_\alpha : \underline{A}^{mn} \to \underline{A}^{reg}_{n,(n+1)m}$ is an embedding which as image the subvariety of $\underline{A}^{reg}_{n,(n+1)m}$ consisting of the matrices of the form $R(F,G)$ for which $R(F,G)_\alpha = I_n$, the $n \times n$ unit matrix.

Proof. Follows from 3.4 above.

3.6. Lemma.

Let α be a nice selection. Denote with U_α the subvariety of $\underline{IS}_{cr}$ consisting of all completely reachable pairs (F,G) for which $\det(R(F,G)_\alpha) \neq 0$.
Then $U_\alpha \simeq GL_n \times \underline{A}^{nm}$.

Proof. Let $(F,G) \in U_\alpha$. There is a unique invertible matrix S such that $(S^{-1}R(F,G))_\alpha = I_n$, the $n \times n$ unit matrix. In fact $S = R(F,G)_\alpha$. Further $S^{-1}R(F,G) = R(S^{-1}FS, S^{-1}G)$. Now apply lemma 3.5.

3.7. The Coarse Moduli Space $\underline{M}_{m,n}$

It follows directly from lemma 3.6 that the quotients U_α/GL_n exist for all nice selections α. (Note also that U_α is GL_n-invariant). To construct the quotient $\underline{IS}_{cr}/GL_n$ it therefore suffices to patch the various affine pieces $V_\alpha = U_\alpha/GL_n \simeq \underline{A}^{mn}$ together. This is done as follows: let α,β be two nice selections. Let

$$V_{\alpha\beta} = \{x \in V_\alpha \mid \det((R\psi_\alpha(x))_\beta) \neq 0\}$$

$$V_{\beta\alpha} = \{x \in V_\beta \mid \det((R\psi_\beta(x))_\alpha) \neq 0\}$$

The open subvarieties $V_{\alpha\beta}$ of V_α and $V_{\beta\alpha}$ of V_β are now identified by means of the isomorphisms $\phi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ defined by

$$\phi_{\alpha\beta}(x) = x' \in V_{\beta\alpha},$$

where x' is the unique point of $V_{\beta\alpha}$ such that

$$R\psi_\beta(x') = (R\psi_\alpha(x))_\beta^{-1} R\psi_\alpha(x).$$

This is a well defined isomorphism in view of lemma 3.5. Patching together all the V_α for all nice selections α gives us, in view of lemma 3.3, a prescheme $\underline{M}_{m,n}$ of which the points correspond bijectively to the orbits of GL_n in $\underline{IS}_{cr}$. This does not yet show that $\underline{M}_{m,n}$ is a variety. However, using the same general techniques it is not difficult to write down equations for $\underline{M}_{m,n}$.

More precisely: the assignment: orbit(F,G) $\to$ $(\det(R(F,G)_\gamma)_\gamma \in \underline{G}_{n,(n+1)m} \subset \underline{P}^N$, where γ runs through all selections, embeds $\underline{M}_{m,n}$ in $G_{n,(n+1)m} \subset \underline{P}^N$. One now writes down a set of homogeneous equations.

$$q_{\alpha\beta}(\ldots,x_\gamma,\ldots) = 0 \tag{3.7.1}$$

(one equation for each pair: (nice selection α, successor selection of α)). The variety $\underline{M}_{m,n}$ as a subvariety of $\underline{P}^N$ (or $\underline{G}_{n,(n+1)m}$) then consists of those points $(x_\gamma)_\gamma$ satisfying the equations (3.6.1) such that moreover for at least one nice selection α, $x_\alpha \neq 0$. Thus $\underline{M}_{m,n}$ is a quasi projective variety. Cf. [2] for more details. (Note also that the affine pieces + patching data description of $\underline{M}_{m,n}$ given above is compatible with the affine pieces + patching data description of $\underline{G}_{n,(n+1)m}$ indicated in section 2.

3.8. Example.

$\underline{M}_{2,2}$ is obtained by patching together three affine pieces $V_\alpha, V_\beta, V_\gamma$, all isomorphic to $\underline{A}^4$. Let $V_\alpha, V_\beta, V_\gamma$ be the affine pieces corresponding respectively to the nice selections $\alpha = \{01,02\}$, $\beta = \{01,11\}$, $\gamma = \{02,12\}$.
Take coordinates (a_1,a_2,a_3,a_4) for V_α, (b_1,b_2,b_3,b_4) for V_β, (c_1,c_2,c_3,c_4) for V_γ arranged in columns (a_1,a_2) and (a_3,a_4), etc....
Then we see that

$$V_{\alpha\beta} = \{a \in V_\alpha \mid a_3 \neq 0\}$$
$$V_{\beta\alpha} = \{b \in V_\beta \mid b_2 \neq 0\}$$

and the identification isomorphism is given by

$$b_1 = -a_1a_3^{-1} \qquad b_2 = a_3^{-1}$$
$$b_3 = a_2a_3 - a_1a_4 \quad b_4 = a_1 + a_4$$

Further

$$V_{\alpha\gamma} = \{a \in V_\alpha \mid a_2 \neq 0\}$$
$$V_{\gamma\alpha} = \{c \in V_\gamma \mid c_2 \neq 0\}$$

with identifications

$$c_1 = -a_2^{-1}a_4 \qquad c_3 = a_2a_3 - a_1a_4$$
$$c_2 = a_2^{-1} \qquad c_4 = a_1 + a_4$$

And finally

$$V_{\beta\gamma} = \{b \in V_\beta \mid b_1^2 + b_1b_2b_4 - b_2^2b_3 \neq 0\}$$
$$V_{\gamma\beta} = \{c \in V_\gamma \mid c_1^2 + c_1c_2c_4 - c_2^2c_3 \neq 0\}$$

with identifications

$$c_1 = (b_1+b_2b_4)(b_1^2+b_1b_2b_4-b_2^2b_3)^{-1} \qquad c_3 = b_3$$
$$c_2 = (-b_2)(b_1^2+b_1b_2b_4-b_2^2b_3)^{-1} \qquad c_4 = b_4$$

3.9. Warning.

We have seen that $\underline{IS}_{cr}/GL_n = \underline{M}_{m,n}$. Now $\underline{DS}_{cr} = \underline{IS}_{cr} \times \underline{A}^{pn}$ and the action of GL_n on $\underline{DS}_{cr}$ is such its restriction to $\underline{IS}_{cr}$ is faithfull. It does not follow from this that $\underline{DS}_{cr}/GL_n \simeq \underline{M}_{m,n} \times \underline{A}^{pn}$ as is incorrectly claimed in subsection 4.6.A of [2]. (This would be the case if $\underline{IS}_{cr}$ were isomorphic to $\underline{M}_{m,n} \times GL_n$; this, however, is not true if $m \geq 2$). The following example may serve to illustrate the difficulty involved.
Let GL_1 act on $\underline{A}^2 \times \underline{A}^1$ as follows

$$\lambda(x_1,x_2,y) = (\lambda x_1,\lambda x_2,\lambda y)$$

Let $\underline{A}^2_{reg_1} = \{x \in \underline{A}^2 | x_1 \neq 0 \text{ or } x_2 \neq 0\}$. The quotients $\underline{A}^2_{reg_1}/GL_1$ and $(\underline{A}^2_{reg} \times \underline{A}^1)/GL_1$ both exist and are respectively equal to $\underline{P}^1$, the projective line, and $\underline{P}^2 \setminus \{pt\}$, the projective plane minus the point $(0,0,1)$. Thus we have

$$(\underline{A}^2_{reg}/GL_1) \times \underline{A}^1 = \underline{P}^1 \times \underline{A}^1$$
$$(\underline{A}^2_{reg} \times \underline{A}^1)/GL_1 = \underline{P}^2 \setminus \{pt\}$$

But the algebraic varieties $\underline{P}^2 \setminus \{pt\}$ and $\underline{P}^1 \times \underline{A}^1$ are not isomorphic.
Remark. It is true that the geometric quotient $\underline{DS}_{cr}/GL_n$ exists and it is a quasi-projective variety as we expect to show in a subsequent note.

4. FAMILIES OF DYNAMICAL SYSTEMS

The next topic we take up is that of a family of input pairs (F,G) parametrized by a variety S. The notion of a (locally trivial) vectorbundle is assumed to be known (cf. e.g. [1] Ch.2 for the algebraic case, or [3] for the topological version).

4.1. Families of Completely Reachable Pairs over a Variety.

As a first primitive approximation of a family of completely reachable pairs parametrized by a variety S we could define a family over S to be a morphism $S \to \underline{IS}_{cr}$. This turns out not to be suffiently general. Cf. 6.2 below. A more general concept is: a family Σ of pairs over a variety S consists of

(i) an n-vectorbundle E over S

(ii) a vectorbundle endomorphism $F: E \to E$

(iii) m sections $g_1, \ldots, g_m : S \to E$

Given a point $s \in S$ we have over s

$(i)_s$ the fibre $E(s)$ which is a vectorspace of dimension n

$(ii)_s$ a vectorspace endomorphism $F(s): E(s) \to E(s)$

$(iii)_s$ m vectors $g_1(s), \ldots, g_m(s)$ in $E(s)$

i.e. after choosing a basis in $E(s)$ we have a pair (F,G). The family Σ is said to be completely reachable if these induced pairs over the points of S are all completely reachable, i.e. if the vectors

$$F(s)^i g_j(s), \quad i = 1, \ldots, n; \; j = 1, \ldots, m$$

span all of $E(s)$ for all $s \in S$.

A family in the sense of a morphism $S \to \underline{IS}_{cr}$ corresponds to a family Σ over S for which the bundle E is isomorphic to $S \times \underline{A}^n$, the trivial n-vectorbundle over S.

Two families Σ, Σ' over S are said to be isomorphic if there exists a vectorbundle isomorphism $\phi: E \to E'$ such that $\phi F = F'\phi$ and such that $\phi g_i = g_i'$.

Remark. There is another possible definition of families of input pairs; however, this other definition is not "rigid" enough for "fine moduli scheme" purposes. Cf. [2] for details.

4.2. The Functor: Isomorphism Classes of Families of Input Pairs.

Let Σ be a family of input pairs over a variety S, and let $f: T \to S$ be a morphism of varieties. Let $\Sigma = (E,F,g_1,\ldots,g_m)$. We now define an induced family $f^!\Sigma$ over T by pulling everything back along f. I.e.
$f^!\Sigma = (f^!E,f^!F,f^!g_1,\ldots,f^!g_m)$, where $f^!E$ is the induced bundle over T, $f^!F$ the induced endomorphism over T and if we identify $(f^!E)(t)$ with $E(f(t))$ then $(f^!g_i)(t) = g_i(f(t))$. (The bundle $f^!E$ has as its fibre over t the fibre of E over $f(t)$; these fibres are fitted together in the obvious way). The family $f^!\Sigma$ is completely reachable if (and only if) the family Σ is completely reachable.

We now define a functor $\mathcal{F}_{m,n}: \underline{Sch}_k \to \underline{Sets}$ from varieties over k to the category of sets as follows.

$\mathcal{F}_{m,n}(S)$ = set of isomorphism classes of completely reachable families of pairs with m inputs and state space dimension n

$\mathcal{F}_{m,n}(f) : \mathcal{F}_{m,n}(S) \to \mathcal{F}_{m,n}(T)$ is the mapping induced by $\Sigma \mapsto f^!\Sigma$ if $f: T \to S$ is a morphism in $\underline{Sch}_k$.

5. THE FINE MODULI SCHEME $\underline{M}_{m,n}$

5.1. $\underline{M}_{m,n}$ is a Coarse Moduli Scheme.

Let Σ be a completely reachable family of pairs over a variety S. Then for every $s \in S$ we have (after choosing a basis in $E(s)$) a completely reachable pair $(F(s),G(s))$.

The pair $(F(s),G(s))$ is unique modulo a choice of basis in $E(s)$ and hence defines a unique point of $\underline{M}_{m,n}$. Thus we find a continuous algebraic map $f_\Sigma : S \to \underline{M}_{m,n}$. This map f_Σ only depends on the isomorphism class of Σ. It turns out that we have defined a morphism of functors $\Phi: \mathfrak{F}_{m,n} \to \underline{Sch}_k(\ ,\underline{M}_{m,n})$, $\Phi(S)(\Sigma) = (f_\Sigma: S \to \underline{M}_{m,n})$. Note also that $\Phi(\mathrm{Spec}(k))$ is an isomorphism. Finally one can prove that every functor morphism $\Psi: \mathfrak{F}_{m,n} \to \underline{Sch}_k(\ ,\underline{M})$ into a representable functor factors uniquely through Φ, via a morphism $h: \underline{M}_{m,n} \to \underline{M}$. I.e. $\underline{M}_{m,n}$ is a coarse moduli scheme. (Cf. [2], [5] or [6] for a definition of this notion).

In fact, more is true: $\underline{M}_{m,n}$ is a fine moduli scheme, which by definition means that the functor morphism Φ above is an isomorphism of functors. Or in other words: there exists a universal completely reachable family Σ^u over $\underline{M}_{m,n}$ such that for every family Σ over a variety S there is a unique morphism $f: S \to \underline{M}_{m,n}$ such that $f^!\Sigma^u = \Sigma$. The next thing to do is to construct this universal family Σ^u.

5.2. Construction of the Universal Family Σ^u.

Let $V_\alpha \simeq \underline{A}^{mn}$ be the affine piece of $\underline{M}_{m,n}$ corresponding to the nice selection α. Over V_α we take the trivial bundle $E_\alpha = V_\alpha \times \underline{A}^n$. Let $\psi_\alpha : V_\alpha \to \underline{IS}_{cr}$ be the morphism defined in subsection 3.4. Write $\psi_\alpha(x) = (F_\alpha(x),G_\alpha(x))$. We now define the bundle endomorphism $F_\alpha : E_\alpha \to E_\alpha$ by the formula $F_\alpha(x,v) = (x,F_\alpha(x)v)$ and the sections $g_{1\alpha}, \ldots, g_{m\alpha} : V_\alpha \to E_\alpha$ are defined by $g_{i\alpha}(x) = (x,$ i-th column of $G_\alpha(x))$.

We now construct the universal family Σ^u by patching together the partial families $(E_\alpha,F_\alpha,g_{1\alpha},\ldots,g_{m\alpha})$. This is done as follows. Let $E_{\alpha\beta} = E_\alpha|V_{\alpha\beta}$, $E_{\beta\alpha} = E_\beta|V_{\beta\alpha}$ and let $\phi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ be the isomorphism constructed in 3.7 above. We now define the isomorphism $\tilde{\phi}_{\alpha\beta} : E_{\alpha\beta} \to E_{\beta\alpha}$ by the formula

$$\tilde{\phi}_{\alpha\beta}(x,v) = (\phi_{\alpha\beta}(x),\ (R\psi_\alpha(x))_\beta^{-1}v) \tag{5.2.1}$$

It is easy to check that these isomorphisms are compatible with the endomorphisms F_α,F_β and the sections $g_{i\alpha},g_{i\beta}$, $i = 1, \ldots, m$, so that we find a family $\Sigma^u = (E^u,F^u,g_1^u,\ldots,g_m^u)$ such that $\Sigma^u|V_\alpha \simeq (E_\alpha,F_\alpha,g_{1\alpha},\ldots,g_{m\alpha})$ and hence such that the point of $\underline{M}_{m,n}$ corresponding to $\Sigma^u(s)$ is precisely s. I.e. $f_{\Sigma^u} : \underline{M}_{m,n} \to \underline{M}_{m,n}$, the morphism induced by the family Σ^u over $\underline{M}_{m,n}$ (cf. 5.1 above), is the identity morphism.

5.3. Theorem.

$\underline{M}_{m,n}$ is a fine moduli space with universal family Σ^u.

(For a proof cf. [2])

5.4. Remark.

Let E be the canonical n-bundle over the Grassmannian $\underline{G}_{n,(n+1)m}$, i.e. $E = \{(x,v) \in \underline{G}_{n,(n+1)m} \times \underline{A}^{(n+1)m} \mid v \in x\}$, where x is interpreted as an n-dimensional linear subspace of $\underline{A}^{(n+1)m}$. Let $\bar{R}$: $\underline{M}_{m,n} \to \underline{G}_{n,(n+1)m}$ be the embedding induced by the GL_n-invariant embedding R: $\underline{IS}_{cr} \to \underline{A}^{reg}_{n,(n+1)m}$. Then $\bar{R}^{!}E = E^u$ the underlying bundle of the universal family Σ^u over $\underline{M}_{m,n}$.

6. CANONICAL FORMS

In this section we discuss the existence and nonexistence of canonical forms.

6.1. Triviality of E^u and the Existence of Canonical Forms.

Suppose that E^u, the underlying bundle of the universal family Σ^u, were trivial; i.e. there is an isomorphism χ: $E^u \to \underline{M}_{m,n} \times \underline{A}^n$. Let e_i: $\underline{M}_{m,n} \to \underline{M}_{m,n} \times \underline{A}^n$ be the section $e_i(x) = (x,e_i)$ where e_i is the i-th unit (column) vector in $\underline{A}^n$. If there were such an isomorphism χ we would have a canonical basis, viz. $\{\chi^{-1}e_1(x),\ldots,\chi^{-1}e_n(x)\}$, in every fibre $E^u(x)$ of E^u which varies continuously with x. Let $(F_\chi(x),G_\chi(x))$ be the matrices corresponding to $\Sigma^u(x)$ with respect to this basis. Let π : $\underline{IS}_{cr} \to \underline{M}_{m,n}$ be the natural projection. Then

$$(F,G) \mapsto \pi(F,G) = x \mapsto (F_\chi(x),G_\chi(x))$$

would be a globally defined continuous algebraic canonical form on $\underline{IS}_{cr}$. Inversely, suppose there were a globally defined continuous algebraic canonical form on $\underline{IS}_{cr}$, say $(F,G) \mapsto (\bar{F},\bar{G})$. We can now define a family Σ^c over $\underline{M}_{m,n}$ as follows, $\Sigma^c = (E^c,F^c,g_1^c,\ldots,g_m^c)$, where $E^c = \underline{M}_{m,n} \times \underline{A}^n$, $F^c(x,v) = (x,\bar{F}_x v)$, $g_i^c(x) = (x,\text{i-th column of } \bar{G}_x)$ where (F_x,G_x) is any pair such that $\pi(F_x,G_x) = x$. Because Σ^u is universal there is a unique morphism f: $\underline{M}_{m,n} \to \underline{M}_{m,n}$ such that $f^{!}\Sigma^u = \Sigma^c$. But because $\pi(\bar{F}_x,\bar{G}_x) = x$, f is the identity morphism (cf. section 5.1) which would imply that $E^c \simeq E^u$, i.e. that E^u is trivial.
We have therefore proved

Theorem. The existence of a globally defined, continuous algebraic canonical form for $\underline{IS}_{cr}$ is equivalent to the triviality of E^u, the underlying bundle of the universal family Σ^u over $\underline{M}_{m,n}$.

6.2. Nonexistence of Canonical Forms for $\underline{IS}_{cr}$.

Let i: $\underline{G}_{n,(n+1)m} \to \underline{P}^N$ be the canonical embedding of the Grassmannian into projective space (cf. section 2). Let L be the canonical line bundle over $\underline{P}^N$, i.e. L(x) = the affine line which x represents. Let E be the canonical n-bundle over $\underline{G}_{n,(n+1)m}$. Then

(6.2.1) $$\overset{n}{\wedge} E \simeq i^{!}L$$

where $\overset{n}{\wedge}$ denotes the n-th exterior product.

Let $\bar{R}$: $\underline{M}_{m,n} \to \underline{G}_{n,(n+1)m}$ be the embedding induced by R. By 5.4 we have that $\bar{R}^{!}E = E^u$. Hence $\overset{n}{\wedge}E^u \simeq \bar{R}^{!}i^{!}L$, which is a very ample line bundle. Hence the sections of $\overset{n}{\wedge}E^u \to \underline{M}_{m,n}$ separate the points of $\underline{M}_{m,n}$ (cf. [1] Ch.II). Hence if E^u were trivial, then $\overset{n}{\wedge}E^u$ would be the trivial line bundle and sections of the trivial line bundle correspond bijectively to morphisms $\underline{M}_{m,n} \to \underline{A}^1$. I.e. if E^u were trivial then the morphisms $\underline{M}_{m,n} \to \underline{A}^1$ would separate points. It is easily seen (cf. [2] for details) from the affine pieces + patching data description of $\underline{M}_{m,n}$ that there are not enough morphisms $\underline{M}_{m,n} \to \underline{A}^1$ to do this when $m \geq 2$. Thus E^u is not trivial and there does not exist a continuous canonical form for $\underline{IS}_{cr}$ if $m \geq 2$. The nontriviality of E^u justifies the definition of family which we have used.

6.3. Nonexistence of Canonical Forms (continued)

There is an easier way to prove the nonexistence of canonical forms for $\underline{IS}_{cr}$. Suppose there existed a continuous canonical form fro $\underline{IS}_{cr}$, say $(F,G) \to (\bar{F},\bar{G})$ then we have n^2 morphisms a_{ij}: $\underline{M}_{m,n} \to A^1$ defined as follows $a_{ij}(x)$ = (i,j)-th entry of $(\bar{F}_x,\bar{G}_x)$, where (F_x,G_x) is any pair such that $\pi(F_x,G_x) = x$. These morphisms would separate the points of $\underline{M}_{m,n}$. But this cannot be done by morphisms to $\underline{A}^1$ if $m \geq 2$, hence a continuous canonical form does not exist for $\underline{IS}_{cr}$ if $m \geq 2$. A fortiori there does not exist a continuous canonical form for $\underline{IS}$ if $m \geq 2$.

There is a GL_n-invariant embedding $\underline{IS}_{cr} \to \underline{DS}_{cr}$, viz. $(F,G) \to (F,G,0)$ where 0 denotes an appropriate zero matrix. Hence there also does not exist a continuous canonical form for $\underline{DS}_{cr}$ and $\underline{DS}$ if $m \geq 2$. If $m = 1$ there does exist a global continuous canonical form for $\underline{IS}_{cr}$ and $\underline{DS}_{cr}$. Summing up, we have

6.4. Theorem.

If $m = 1$, there is a globally defined continuous algebraic canonical form for $\underline{IS}_{cr}$ and $\underline{DS}_{cr}$.

If $m \geq 2$, there is no globally defined canonical form for $\underline{IS}$, $\underline{IS}_{cr}$, $\underline{DS}$, $\underline{DS}_{cr}$.

7. CONCLUDING REMARKS AND OPEN QUESTIONS.

7.1. The moduli space $\underline{M}_{m,n}$ is not complete (for all m,n); i.e. it is not a closed subvariety of $\underline{P}^N$ (or $\underline{G}_{n,(n+1)m}$). Let $\bar{\underline{M}}_{m,n}$ be its closure. E.g. $F_1 = t^{-1}e_1$

$F_i = ie_i$, $i = 2, \ldots, n$; $G_1 = t^n e_1 + e_2 + \ldots + e_n$, $G_i = 0$, $i = 2, \ldots, m$. Then

$$R(F,G) = \left(\begin{array}{cccc|cccc|c|cccc} t^n & 0 & \ldots & 0 & t^{n-1} & 0 & \ldots & 0 & & 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & 2 & 0 & \ldots & 0 & & 2^n & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ldots & \vdots & \vdots & & \vdots \\ 1 & 0 & \ldots & 0 & n & 0 & \ldots & 0 & & n^n & 0 & \ldots & 0 \end{array}\right)$$

which as t goes to 0 specializes to an element of $\underline{A}^{reg}_{n,(n+1)m}$ which is not of the form $R(F',G')$ for any $(F',G') \in \underline{IS}_{cr}$ (in view of lemma 3.3) and which hence gives rise to a point in $\bar{\underline{M}}_{m,n}$ which is not in $\underline{M}_{m,n}$.
The question arises whether it is possible to interpret the missing points, i.e. the points of $\bar{\underline{M}}_{m,n} \setminus \underline{M}_{m,n}$, as (generalized?) dynamical systems?

7.2. The group GL_m of basis changes in input space acts on $\underline{M}_{m,n}$. If $m < n$, then there is an open dense subset U of $\underline{M}_{m,n}$ such that the stabilizer of this action is GL_1 (diagonally embedded in GL_m) for all $x \in U$. (So what we really have is an action of PGL_{m-1} on $\underline{M}_{m,n}$). By general theorems (cf. [5]) we then know that a geometric quotient V/GL_m exists for a suitable dense open subset V of $\underline{M}_{m,n}$.
Problem: calculate the maximal V and describe the quotient V/GL_m. In particular (in view of canonical forms) one would like to know whether the points of V/GL_m can be separated by morphisms to $\underline{A}^1$.

7.3. Let V_α be the subvariety of $\underline{M}_{m,n}$ corresponding to the nice selection α. There is a global continuous algebraic canonical form for $\pi^{-1}V_\alpha$, where $\pi : \underline{IS}_{cr} \to \underline{M}_{m,n}$ is the natural projection, viz $(F,G) \to \psi_\alpha\pi(F,G)$, where ψ_α is the morphism defined in 3.4 above. The V_α are also maximal subvarieties for which a canonical form exists for $\pi^{-1}V_\alpha$. However, not every subvariety V of $\underline{M}_{m,n}$ for which a canonical form exists for $\pi^{-1}V$, is contained in one of the V_α, α a nice selection. E.g. let β be a not nice selection and $W_\beta = \{(F,G) \in \underline{IS}_{cr} \mid \det R(F,G)_\beta \neq 0\}$. Then there is a canonical form on W_β. (NB. W_β can be nonempty as the family)

$$F_t = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad G_t = \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$$

shows. This family also shows that W_β need not be contained in any of the V_α, α a nice selection. The following could be true, let λ be a linear form in the expressions $\det(R(F,G)_\beta)$, where β runs through all selections. Let W_λ be the subvariety of $\underline{IS}_{cr}$ where λ is $\neq 0$. If $V \subset \underline{IS}_{cr}$ is a subvariety for which a canonical form exists, then V is contained in one of the W_λ.

7.4. What kind of morphisms between the various $\underline{M}_{m,n}$ does the partial realization algorithm induce ? This could be interesting also because of 7.1.

7.5. We have seen in theorem 6.4 that there is no canonical form on $\underline{DS}_{cr}$ or $\underline{DS}$ if $m \geq 2$. Let $\underline{DS}_{cr,co}$ be the subspace of $\underline{DS}$ consisting of completely reachable and completely observable linear dynamical systems. The nonexistence of a canonical form for $\underline{IS}_{cr}$ does not imply the nonexistence of a canonical form for $\underline{DS}_{cr,co}$, and, a priori, a canonical form for $\underline{DS}_{cr,co}$ could exist also for $m \geq 2$. Indeed, such a canonical form does exist if $p = 1$ (p is the number of outputs), n and m arbitrary. The geometric quotient $\underline{DS}_{cr,co}/GL_n$ does exist, cf. also section 3.9 above, but in this case there also exists an embedding of $\underline{DS}_{cr,co}/GL_n$ in an affine space, so that the argument of 6.3 above cannot be used to prove nonexistence of canonical forms. Possibly one shall have to use results like 6.1 to decide whether $\underline{DS}_{cr,co}$ admits a global continuous algebraic canonical form or not.

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