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NORM MAPS FOR FORMAL GROUPS III

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1. INTRODUCTION.

Let K be discretely valued complete field of characteristic zero with algebraically closed residue field k of characteristic $p > 0$. Let A be the ring of integers of K and let F be a one dimensional commutative formal group over A. Let K_∞/K be a Γ-extension (associated to the prime p), i.e. K_∞/K is galois and $Gal(K_\infty/K) \simeq \mathbb{Z}_p$, the p-adic integers, and let K_n be the invariant field of $p^n Gal(K_\infty/K)$. There are natural norm maps

$$F\text{-}Norm_{n/o}\colon F(K_n) \to F(K)$$

Let v be the normalized exponential valuation on K, i.e. $v(\pi) = 1$, where π is a uniformizing element of K. Let $F^s(K)$, $s \in \mathbb{R}$, $s \geq 1$ denote the filtration subgroup of F(K) consisting of all elements x of A such that $v(x) \geq S$. Let h be the height of the formal group F and let e_K be the (absolute) ramification index of K, i.e. $v(p) = e_K$. In [3] we proved.

There exist constants c_1 and c_2 such that for all $n \in \mathbb{N}$

$$F^{\beta_n}(K) \subset Im(F\text{-}Norm_{n/o}) \subset F^{\alpha_n}(K)$$

where $\alpha_n = h^{-1}(h-1)ne_K - c_1$, $\beta_n = h^{-1}(h-1)ne_K + c_2$

The proof in [3] that there exists a constant c_1 such that the second inclusion holds is relatively easy, but the proof in [3] that there is a c_2 such that the first inclusion holds is very long and laborious. It is the purpose of the present note to give a much shorter and more conceptual proof of this part of the theorem by using some results on the logarithm of F. This proof is similar in spirit to the proof sketched in section 12 of [3] for the main theorem of [2].

For more complete definitions of the notions mentioned above cf. [2] and [3]. Some motivation as to why one would want to study the images of norm maps of formal groups can be found in the introduction of [2] and especially in [6].

So the theorem we are going to prove in this paper is

Theorem A. Let K_∞/K be a Γ-extension of a mixed characteristic local field K with algebraically closed residue field. Let F be a one dimensional commutative formal group over A of height h over A. Then there exists a constant c such that

$$F^{\beta_n}(K) \subset \mathrm{Im}(F\text{-Norm}_{n/o})$$

for all n, where $\beta_n = h^{-1}(h-1)ne_K + c$.

(If $h = \infty$, $h^{-1}(h-1)$ is taken to be equal to 1).
All formal groups in this paper will be one dimensional commutative. The notation introduced above will remain in force throughout this paper. In addition we use A_n for the ring of integers of K_n; π_n for a uniformizing element of K_n; v_n for the normalized exponential valuation of K_n, i.e. $v_n(\pi_n) = 1$; and $\mathrm{Tr}_{n/o}$ is the trace map from K_n to K. The natural numbers are denoted by $\mathbb{N}$.

2. RECAPITULATION OF SOME RESULTS AND DEFINITIONS.

2.1. Let L/K be a cyclic extension of degree p. There is a unique integer $m(L/K) \geq 1$ such that for all n, $\mathrm{Tr}_{L/K}(\pi_L^n A_L) = \pi_K^r A_K$, where $r = [p^{-1}((m(L/K)+1)(p-1)+n)]$ where [y] denotes the entier of y. We shall use m_n to denote the number $m(K_n/K_{n-1})$.

2.2. Lemma. (Tate [7]). There is a constant m_o such that $m_n = (1+p+\ldots+p^{n-1})e_K + m_o$ for all sufficiently large n.

2.3. Let L/K be any totally ramified extension. We define the function $\lambda_{L/K}$ as follows $\lambda_{L/K}(n) = r$ iff $\mathrm{Tr}_{L/K}(\pi_L^n A_L) = \pi_L^r A_K$. The function $\lambda_{L/K}$ can of course be described in terms of the various $m(L_i/L_{i-1})$ where $K = L_1 \subset L_2 \subset \ldots \subset L_s = L$ is a tower of cyclic extensions of prime degree. It follows immediately from this that

2.4. Lemma. $\lambda_{L/K}(t) = e_L^{-1} e_K t + e_t$, where the numbers e_t are bounded indepently of t.

2.5. Lemma. ([3] lemma 3.4). Let L/K be a totally ramified extension, then there is a $t_o \in \mathbb{N}$ such that for all $t \geq t_o$

$$F\text{-Norm}_{L/K}(F^t(L)) = F^{\lambda_{L/K}(t)}(K)$$

2.6. Reduction of the Proof of Theorem A.

If K_∞/K is a Γ-extension, then so is K_∞/K_r for all $r \in \mathbb{N}$. In view of 2.2 and 2.5 this reduces the proof of theorem 1.1 to the case where K_∞/K is Γ-extension such that $m_n = (1+\ldots+p^{n-1})e_K + m_o$ for all $n \in \mathbb{N}$. Indeed if K_∞/K is any Γ-extension, then by 2.2 there is an $r \in \mathbb{N}$ such that $m_n = m(K_n/K_{n-1}) =$ $(1+\ldots+p^{n-r-1})e_K p^r + m_o + (1+p+\ldots+p^{r-1})e_K$ for all $n > r$. Now apply lemma 2.5 with $L = K_r$ (using that $F\text{-Norm}_{n/o} = F\text{-Norm}_{r/o}(F\text{-Norm}_{n/r})$).

2.7. Lemma. Let F be a formal group over A and f(X) its logarithm. Then for t large enough f is an isomorphism

$$F^t(K) \xrightarrow{f} G_a^t(K)$$

where G_a is the additive formal group, i.e. $G_a(X,Y) = X + Y$

Proof. We have $f(F(X,Y)) = f(X) + f(Y)$ and $nb_n \in A$ if $f(X) = \Sigma b_n X^n$. The lemma follows easily from this.

2.8. Idea of the Proof of Theorem A.

We consider the diagram

$$\begin{array}{ccc} F(K_n) & \xrightarrow{f} & K_n \\ \downarrow{\scriptstyle F\text{-Norm}} & & \downarrow{\scriptstyle Tr_{n/o}} \\ F(K) & \xrightarrow{f} & K \end{array}$$

which is commutative. It now suffices to prove that there is a constant c such that

$$\pi^{\beta_n} A \subset Tr_{n/o} f(\pi_n A_n)$$

This follows directly from lemma 2.7.

3. LEMMAS ON f(X).

3.1. Let $h = \text{height}(F) < \infty$. Let F^* be the reduction of the formal group F to a formal group over k, the residue field of K. Because k is algebraically closed F^* is classified by its height h. Let F_T be the p-typically universal formal group of [4]. Substituting 1 for T_h and 0 for all T_i with $i \neq h$ we obtain a formal group G over A such that G^* is of height h and hence isomorphic to F^* by a theorem of Lazard, because k is algebraically closed, cf. e.g. [1]. It now follows from [4] part I section 5.3 and [5] that F is isomorphic to a formal group F_t obtained from F_T by substituting t_i for T_i, $i = 1, 2, \ldots$ where $t_i \in \pi A$, $i = 1, \ldots, h-1$, $t_h = 1$, $t_j = 0$, $j = h+1, h+2, \ldots$ We can therefore assume that F is equal to such an F_t. It follows that if

$$(3.1.1) \qquad F(X,Y) = f^{-1}(f(X) + f(Y) \qquad f(X) = X + a_1X^p + a_2X^{p^2} + \ldots$$

then the coefficients of $f(X)$ satisfy relations

$$(3.1.2) \qquad \begin{array}{c} pa_n = a_{n-1}t_1^{p^{n-1}} + a_{n-2}t_2^{p^{n-2}} + \ldots + a_{n-h}t_h^{p^{n-h}}, \quad n \geq h \\ t_1, \ldots, t_{h-1} \in \pi A, \ t_h = 1 \end{array}$$

3.2. Lemma. If $\text{height}(F) < \infty$ then there is no $n_o \in \mathbb{N}$ such that $v(a_n) \geq 0$ for all $n \geq n_o$.

Proof. If $v(a_n) \geq 0$ for all $n \geq n_o$. Then $v(a_n) \geq 0$ for all $n \geq n_o-1$ by 3.1.2 (because $t_h = 1$), and thus with induction $v(a_n) \geq 0$ for all $n \geq 1$, which means that $f(X)$ is an isomorphism of F with the additive group and hence implies $\text{height}(F) = \infty$.

3.3. Lemma. If $h < \infty$ then there is an $n_o \in \mathbb{N}$ such that

$$v(a_{n_o+rh}) = v(a_{n_o}) - re_K, \ v(a_{n_o}) < 0$$

for all $r \in \mathbb{N}$.

Proof. Let $n_1 \in \mathbb{N}$ be such that $p^n \geq ne_K$ for $n \geq n_1$. Then for $n \geq n_1+h$ we have that $v(a_{n-i}t_i^{p^{n-1}}) \geq 0$, $i = 1, \ldots, h-1$. Now let $n_o \geq n_1$ be such that $v(a_{n_o}) < 0$. Such an n_o exists by lemma 3.2. Then by (3.1.2) we have that $v(a_{n_o+h}) = v(a_{n_o}) - e_K$ and with induction $v(a_{n_o+rh}) = v(a_{n_o}) - re_K$, $r \in \mathbb{N}$.

3.4. Lemma. Let $h < \infty$. There is a constant c such that

$$v(a_n) \geq -h^{-1}ne_K - c$$

for all $n \in \mathbb{N}$.

Proof. We have that (cf.[4])

$$a_n = \sum_{(i_1,\ldots,i_r)} \frac{t_{i_1} t_{i_2}^{p^{i_1}} \cdots t_{i_2}^{p^{i_1 \cdots i_{r-1}}}}{p^r} \tag{3.4.1}$$

where the sum is over all sequences $(i_1,\ldots,i_r)$ such that $i_1 + \ldots + i_r = n$, $i_j \in \{1,\ldots,h\}$. Let $s(i_1,\ldots,i_r)$ be the number of indices j such $i_j = h$. Let $\ell_1, \ldots, \ell_{r-s}$ be the indices in $(i_1, \ldots, i_r)$ which are different from h in their original order. Then

$$\begin{aligned} v(a_n) &\geq 1 + p^{\ell_1} + \ldots + p^{\ell_1+\ldots+\ell_{r-s}} - re_K \\ &\geq (1+p+\ldots+p^{r-s}) - re_K \end{aligned} \tag{3.4.2}$$

Choose c' such that $1+p+ \ldots + p^{c'+1} \geq e_K$ and $c = e_K c'$. If $r \leq \frac{n}{h} + c'$ the term $p^{-r} t_{i_1} t_{i_2}^{p^{i_1}} \cdots t_{i_r}^{p^{i_1 \cdots i_{r-1}}}$ has valuation $\geq -h^{-1}ne_K - e_K c_1$. Suppose that $r = \frac{n}{h} + c' + d, d \in \mathbb{N}$. Because $\ell_1 + \ldots + \ell_{r-s} + hs = n$ we have that $r-s + hs \leq n$, hence $(h-1)s \leq n-r = (h-1)(\frac{n}{h}) - (c'+d)$ hence $s \leq \frac{n}{n} - \frac{c'+d}{h-1}$ and $r-s \geq c'+d$ and therefore

$$(3.4.3) \qquad 1 + p + \ldots + p^{r-s} - re_K \geq 1 + p + \ldots + p^{c'+d} - (\tfrac{n}{n}+c'+d)e_K$$

$$\geq p^d(1+p+\ldots+p^{c'}) - (\tfrac{n}{n}+c')e_K - de_K$$

$$\geq -(\tfrac{n}{n}+c')e_K$$

which proves the lemma.

3.5. Remark. The estimate of 3.4 is (up to a constant) best possible. Because we see from lemma 3.3 that for n of the form $n_o + rh$ we have for a certain constant d that

$$v(a_n) = -h^{-1}ne_K + d$$

4. VARIOUS FUNCTIONS AND ESTIMATES.

From now on K_∞/K is a Γ-extension such that $m_n = (1+p+\ldots+p^{n-1})e_K + m_o$ for all $n \in \mathbb{N}$; F is a formal group over A of height $h < \infty$ of the form $F(X,Y) = f^{-1}(f(X) + f(Y))$, where f(X) is as in (3.1.1) and (3.1.2).

4.1. The functions μ_n, σ_n, j_n, ℓ_n.
We define for all $n \in \mathbb{N}$, $t \in \mathbb{N}$, $i \in \mathbb{N}$

$$(4.1.1) \quad \mu_n(p^i,t) = ie_K + \lambda_{n-i/o}(t) \text{ if } i \leq n, \quad \mu_n(p^i,t) = ne_K + p^{i-n}t \text{ if } i \geq n$$

$$(4.1.2) \qquad \sigma_n(t) = \min_i\{v\,(a_i) + \mu_n(p^i,t)\}$$

$$(4.1.3) \qquad j_n(t) = \text{smallest integer } i \text{ such that } \sigma_n(t) = v\,(a_i) + \mu_n(p^i,t)$$

$$(4.1.4) \qquad \ell_n(t) = n - j_n(t)$$

4.2. Lemma. For every n and t there are only finitely many i such that $\sigma_n(t) = v\,(a_i) + \mu_n(p^i,t)$.

This follows immediately from (4.1.1) and lemma 3.4.

4.3. We define

$$(4.3.1) \qquad r_n = \left[\frac{(1+m_n)(p-1)+1}{p}\right]$$

Lemma. Suppose that $m_o \geq 2$ and $e_K \geq p$. Then for all $n \geq r \geq 0$

$$\lambda_{n/n-r}(2r_{n+1}-1) \geq (r+1)e_K p^{n-r} + p^{n-r}$$

Proof. One easily sees that $\lambda_{n/n-r}(m_n) = m_{n-r} + re_K p^{n-r} =$

$(1+p+\ldots+p^{n-r-1})e_K + m_o + re_K p^{n-r}$.

Hence it suffices to prove that

$$2r_{n+1}-1 \geq m_n + e_K p^n + p^n - (p^r+\ldots+p^{n-1})e_K - m_o p^r$$

$$= e_K p^n + p^n + (1+p+\ldots+p^{r-1})e_K + m_o - m_o p^r$$

If $r = n$, then $m_o p^r \geq p^n$ because $m_o \geq 2$ and we also have $p^n \leq p^{n-1}e_K$ because $e_K \geq p$.

It follows that it suffices to prove that

$$2r_{n+1} - 1 \geq e_K p^n + (1+p+\ldots+p^{n-1})e_K + m_o$$

We have $2r_{n+1}-1 \geq 2p^{-1}(p^{n+1}-1)e_K + 2p^{-1}(p-1)m_o + 2p^{-1} -1$.

Now if $p > 2$ then $2p^{-1}(p-1)m_o \geq m_o+1$ because $m_o \geq 2$ and if $p = 2$ then $2p^{-1} = 1$. Hence $2r_{n+1}-1 \geq 2p^n e_K - 2p^{-1}e_K + m_o \geq (1+\ldots+p^{n-1})e_K+m_o+p^n e_K$.

4.4. Trace Lemma ([3] Proposition 4.1)

Let $\pi_{n-1} = (-1)^{p-1}N_{n/n-1}(\pi_n)$, where $N_{n/n-1}$ is the norm map $K_n \to K_{n-1}$

Then we have

$$(4.4.1) \qquad Tr_{n/n-1}(\pi_n^{p^t}) \equiv p\pi_{n-1}^t \mod \pi_{n-1}^{2r_n+t+1}$$

4.5. Lemma. If $m_o \geq 2$ and $e_K \geq p$ then

$$v(Tr_{n/o}(\pi_n^{p^r t})) \geq \mu_n(p^r,t)$$

Proof. First let $r \leq n$. Then we have because of 4.4 that

$$Tr_{n/n-1}(\pi_n^{p^r t}) \equiv p\pi_{n-1}^{p^{r-1}t} \quad \mathrm{mod}(v_{n-1} \text{ valuation } \quad 2r_n+tp^{r-1}-1)$$

$$Tr_{n-1/n-2}(p\pi_{n-1}^{p^{r-1}t}) \equiv p^2\pi_{n-2}^{p^{r-2}t} \quad \mathrm{mod}(v_{n-2} \text{ valuation } 2r_{n-1}+tp^{r-2}-1+p^{n-1}e_K)$$

$$\vdots$$

$$Tr_{n-r+1/n-r}(p^{r-1}\pi_{n-r+1}^{p^t}) \equiv p^r\pi_{n-r}^t \quad \mathrm{mod}(v_{n-r} \text{ valuation } 2r_{n-r+1}+t-1+(r-1)p^{n-r}e_K)$$

Now $\lambda_{n-i/n-r}(2r_{n-i+1}+p^{r-i}t-1+p^{n-i}(i-1)e_K) = t + p^{n-r}(i-1)e_K + \lambda_{n-i/n-r}(2r_{n-i+1}-1) \geq t + re_K p^{n-r} + p^{n-r}$ by lemma 4.3. It follows that

(4.5.1) $$Tr_{n/n-r}(\pi_n^{p^r t}) \equiv p^r\pi_{n-r}^t \quad \mathrm{mod}(v_{n-r}\text{-valuation } t+re_K p^{n-r}+p^{n-r})$$

Now $v_{n-r}(p^r\pi_{n-r}^t) = re_K p^{n-r} + t$. So that

(4.5.2) $$v(Tr_{n/o}(\pi_n^{p^r t})) \geq re_K + \lambda_{n-r/o}(t) = \mu_n(p^r,t)$$

Now suppose that $r > n$, then replacing t with $p^{r-n}t$ and r with n we obtain from (4.5.1)

(4.5.3) $$Tr_{n/o}(\pi_n^{p^r t}) \equiv p^n\pi^{p^{r-n}t} \quad \mathrm{mod}(v\text{-valuation } p^{r-n}t+ne_K+1)$$

which proves the lemma also in this case.

4.6. Lemma.

If $m_o \geq 2$ and $e_K \geq p$ and if t is such that $\lambda_{n-r/o}(t+1) = \lambda_{n-r/o}(t) + 1$ and $r \leq n$ then $v(Tr_{n/o}(\pi_n^{p^r t})) = \mu_n(p^r,t)$; if $r > n$ then $v(Tr_{n/o}(\pi_n^{p^r t})) = \mu_n(p^r,t)$ for all t.

Proof. If $x \in A_{n-r}$ and $v_{n-r}(x) = s$ and $\lambda_{n-r/o}(s+1) = \lambda_{n-r/o}(s) + 1$ then always $v(Tr_{n-r/o}(x)) = \lambda_{n-r/o}(s)$. Lemma 4.6 now follows immediately from (4.5.1). The second statement of the lemma follows from (4.5.3).

4.7. Lemma.

For every $t \in \mathbb{N}$ there is a constant c such that

$$\sigma_n(t) \leq h^{-1}(h-1)ne_K + c$$

Proof. Let i_o be such that $v(a_{i_o}) < 0$, $v(a_{i_o+rh}) = v(a_{i_o}) - re_K$ for $r \in \mathbb{Z}$, $r \geq -1$. For $n \leq i_o$ take $i = i_o$, $t = t_o$. Then we have

$$\sigma_n(t) \leq v(a_{i_o}) + \mu_n(p^{i_o},t) \leq p^{i_o-n} \leq p^{i_o}t$$

If $n > i_o$, let i be the largest number of the form $i_o + rh$, which is smaller then n. Then $n-i \leq h$ and we have

$$\sigma_n(t) \leq v(a_i) + \mu_n(p^i,t) = v(a_{i_o}) - re_K + \lambda_{n-i/o}(t) + ie_K$$

$$\leq ne_K - re_K + \lambda_{n-i/o}(t)$$

Now $\lambda_{n-i/o}(t)$ is bounded because $n-i \leq h$. Let $d = \max\{\lambda_{1/o}(t),\ldots,\lambda_{h/o}(t)\}$. As $i_o + rh + h \geq n$ we have that $r \geq h^{-1}n - 1 - h^{-1}i_o$ so that indeed for all $n \in \mathbb{N}$

$$\sigma_n(T) \leq h^{-1}(h-1)ne_K + c$$

with $c = \max(p^{i_o}t,\ (1+h^{-1}i_o)e_K+d)$

5. PROOF OF THEOREM A.

By lemma 2.2 and 2.6 we can assume that the Γ-extension K_∞/K is such that $m_n = (1+p+\ldots+p^{n-1})e_K + m$, for all $n \in \mathbb{N}$ and that $e_K \geq p$, $m_o \geq 2$.

5.1. Lemma.

Let L/K be an extension. Then there is a $t \in \mathbb{N}$ such that $F\text{-Norm}_{L/K}(F(L)) \supset F^t(K)$.

Proof. If we have $F(X,Y) = X + Y + \sum_{i,j\geq 1} a_{ij}X^iY^i$. It follows that

if s is such that $\lambda_{L/K}(s) < [L:K]^{-1}2s$, and $v_L(x) = s$ then $F\text{-Norm}(x) \equiv Tr_{L/K}(s)$ mod (v -valuation $\lambda_{L/K}(s)+1$). Up to a constant we have $\lambda_{L/K}(s) = [L:K]^{-1}s$ and the lemma follows.

5.2. Proof of theorem A in the case $h = \infty$. This follows from lemma 5.1. Cf. also [3].

5.3. In view of 5.2. We can assume that $h < \infty$. Hence we can assume that $F(X,Y)$ is a formal group with logarithm $f(X)$ such that (3.1.1) and (3.1.2) hold. Given all this we have available the various functions defined in section 3 and 4 and the various lemma's of sections 3 and 4.

Choose n_o such that $v(a_{n_o}) < 0$ and $v(a_{n_o+rh}) = v(a_{n_o}) - re_K$, $r \geq 0$

and such that $p^n \geq ne_K$ for $n \geq n_o$. Note that if $n \geq n_o+h$ and $v(a_n) < -1$ then $v(a_{n-h}) = v(a_n) + 1$ by (3.1.2).

Let $t_o \in \mathbb{N}$ be such that $t_o \geq (1+p+\ldots+p^{h-1})e_K + m_o$ and choose a constant c_o as in lemma 4.7. Now let $n_1 \in \mathbb{N}$ be such that $n_1 \geq n_o+h$, and such that $\sigma_n(t_o) < ne_K$.

5.4. Lemma.

If $n \geq n_1$, $j_n(t_o) \leq n$.

Proof. Suppose $n'=j_n(t_o) > n$. Then

$$v(a_{n'}) + \mu_n(p^{n'},t_o) = \sigma_n(t_o) < ne_K$$

But $\mu_n(p^{n'},t_o) = ne_K + p^{n'-n}t$. Hence $v(a_{n'}) < -1$ and $v(a_{n'-h}) = v(a_{n'}) + 1$. Then if $n' \geq n+h$ we have

$$v(a_{n'-h}) + \mu_n(p^{n'-h},t_o) = v(a_{n'}) + 1 + ne_K + p^{n'-h-n}t_o \leq$$

$\leq$ $v(a_{n'}) + \mu_n(p^{n'},t_o)$ which is a contradiction. And if $n'-h < n$.

We have

$$v(a_{n'-h}) + \mu_n(p^{n'-h},t_o) = v(a_{n'}) + 1 + (n'-h)e_K + \lambda_{n-n'+h/o}(t)$$

which is also $\leq v(a_{n'}) + \mu_n(p^{n'},t_o)$ because $\lambda_{i/o}(t_o) \leq t_o$ for $i = 1, \ldots, h$ if $t_o \geq (1+p+\ldots+p^{h-1})e_K + m_o$.

q.e.d.

5.5. Proof of Theorem A.

We assume all the conditions mentioned above. Let n_1 be as in 5.3 above. By lemma 5.1 it suffices to prove theorem A for $n \geq n_1$. To do this it suffices according to 2.8 to prove that

$Tr_{n/o}f(\pi_n A_n) \supset \pi^{\beta_n}A$ for $n \geq n_1$. Note that because $f(F(X,Y)) = f(X) + f(Y)$ we have

$$(5.5.1) \qquad x,y \in Tr_{n/o}f(\pi_n A_n) \Rightarrow x + y \in Tr_{n/o}f(\pi_n A_n)$$

Now let $t_o \in \mathbb{N}$ be larger than $(1+p+\ldots+p^{h-1})e_K + m_o$, and let $j = j_n(t_o)$. Then $j \leq n$ by lemma 5.4. Let $\ell = \ell_n(t_o) = n - j_n(t_o) = n - j$, and let t be the largest integer such that $t \geq t_o$ and $\lambda_{\ell/o}(t) = \lambda_{\ell/o}(t_o)$ Then we have (cf. 4.1)

$$(5.5.2) \qquad \mu_n(p^i,t) \geq \mu_n(p^i,t_o) \quad \text{for all } i = 1, 2,\ldots$$

$$\mu_n(p^j,t) = \mu_n(p^j,t_o)$$

And hence (cf. 4.1)

(5.5.3) $$\sigma_n(t) = \sigma_n(t_o),\ j_n(t) = j_n(t_o) = j$$

Now we also know by lemma's 4.6 and 4.5 that

(5.5.4) $$v(Tr_{n/o}(a_j\pi_n^{p^j t})) = v(a_j) + \mu_n(p^j,t)$$

$$v(Tr_{n/o}(a_i\pi_n^{p^i t})) \geq v(a_i) + \mu_n(p^i,t) . \qquad i \neq j$$

Let $x \in A$. Then it follows from (5.5.4) and lemma 4.2 that

(5.5.5) $$Tr_{n/o}f(x\pi_n^t) \equiv b_o x^{p^j} + b_1 x^{p^{j+1}} + \ldots + b_r x^{p^{j+r}} \quad \mathrm{mod}\,\pi^{\sigma_n(t)+1}$$

where

(5.5.6) $$v(b_o) = \sigma_n(t) = \sigma_n(t_o),\ v(b_i) \geq \sigma_n(t),\ \ i = 1, \ldots, r$$

Because k is algebraically closed this implies that

(5.5.7) $$Tr_{n/o}f(\pi_n A_n)/\pi^{\sigma_n(t_o)+1}A \supset \pi^{\sigma_n(t_o)}A/\pi^{\sigma_n(t_o)+1}A$$

We obtain an inclusion (5.5.7) for every $t_o \in \mathbb{N}$, $t_o \geq (1+p+\ldots+p^{h-1})e_K + m_o$
Now also $\sigma_n(1+p+\ldots+^{h-1})e_K+m_o) = h^{-1}(h-1)ne_K + c$ for a certain constant c.

Hence in view of (5.5.1) and completeness of A (or lemma 5.3) theorem A will be proved if we can show for every $n \geq n_1$ that all $s \in \mathbb{N}$, $s \geq\ \ s_o = \sigma_n((1+p+\ldots+p^{h-1})e_K+m_o))$ occur as a $\sigma_n(t)$ for some t.
This is done by induction on $s - s_o$.
The induction hypothesis is: there is a $t_o \geq (1+p+\ldots+p^{h-1})e_k + m_o$ such that $\sigma_n(t_o) = s \geq 0$.
Let $j_o = j_n(t_o)$, then $j_o \leq n$. Let $\ell_o = n - j_o$. Let $t_1 = t_o + p^{\ell_o}$.
Then

$$v(a_i) + \mu_n(p^i,t_1) \geq v(a_i) + \mu_n(p^i,t_o) + 1 \text{ if } i \geq j_o$$

$$(5.5.8) \quad v(a_{j_o}) + \mu_n(p^{j_o},t_1) = v(a_{j_o}) + \mu_n(p^{j_o},t_o) + 1$$

$$v(a_i) + \mu_n(p^i,t_1) \geq v(a_i) + \mu_n(p^i,t_o) \quad \text{if } i < j_o$$

It follows that

$$(5.5.9) \qquad \sigma_n(t_1) \leq \sigma_n(t_o) + 1$$

If $\sigma_n(t_1) = \sigma_n(t_o) + 1$ we are done. If $\sigma_n(t_1) = \sigma_n(t_o)$, then because of (5.5.8) we must have $j_n(t_1) = j_1 < j_o$. Let $\ell_1 = n - j_1$ and $t_2 = t_1 + p^{\ell_1}$. Then $\sigma_n(t_2) \leq \sigma_n(t_1) + 1$; if .... This process must stop and finally give a t such that $\sigma_n(t) = s + 1$, because $j_o > j_1 > . . \geq 0$. This concludes the proof of the theorem.

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SYMBOLS USED.

Latin lower case   k,p,v,s,x,h,e,c,n,m,y,r,t,f,b,a,j,ℓ

Latin upper case   K,A,F,L,X,G,Y,T,N,

Latin lower case as sub-or superscript   p,n,s,t,r,a,h,i,j,

Latin upper case as sub or superscript   K,L,T

Latin lower case bold face

Latin upper case bold face $\mathbb{Z}$ (integers), $\mathbb{R}$(reals), $\mathbb{N}$(natural numbers)

Greek lower case $\pi,\lambda,\alpha,\beta,\mu,\sigma$,

Greek upper case Γ

Greek lower case as sub-or superscript β,α,σ,

Numerals   0,1,2,3,6,

Numerals as sub-or superscript 0,1,2,

Special symbols $>, /, \infty, \simeq, (\,), \in, \geq, -, =, [\,], +, \subset, \div, \Sigma, <, \neq, \leq$

Special symbols as sub-or superscript $\infty, /, -, *, +$,

Frequently occurring groups of letters in formula s Gal, Norm, Im,Tr

Conventions:

Greek: underline in red