ERASMUS UNIVERSITEIT ROTTERDAM

ECONOMETRISCH INSTITUUT

Report 7504/M

MODULI AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS

by Michiel Hazewinkel and Rudolf E. Kalman

Writtenin April 1974, while the first named author
stayed at the University of Florida in Gainesville

MODULI AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS

1. Introduction

In this paper we are concerned with linear systems (F, G, H), where F is an $n \times n$ matrix, G an $n \times m$ matrix, and H an $r \times n$ matrix (i.e. there are m inputs, r ouputs, the dimension of the system is n) and the equivalence relation induced by basis change in the state space. I.e., (F, G, H) is isomorphic to (TFT^{-1}, TG, HT^{-1}), $T \in GL_n(k)$ where k is the base field we are working over. For convenience we shall assume that k is algebraically closed. (Cf., however, (4.6)).

Let $\underline{L}_{cr}$ denote the space of completely reachable linear systems. It turns out that the orbit space $\underline{L}_{cr}/GL_n$ exists and it has a nice natural geometric structure. It is, in fact, a quasi-projective variety.

Moreover, this space turns out to be a fine moduli scheme for continuous (algebraic) families of completely reachable systems. I.e., its points correspond bijectively with equivalence classes of linear systems and over the moduli space there exists a universal family of linear systems from which every family can be obtained by pullback.

Unfortunately (or fortunately, depending on one's point of view), the underlying n-vector bundle of this universal family is not trivial if $m \geq 2$, (i.e. if there are 2 or more inputs; the bundle is trivial if $m = 1$) and this ruins all chances of finding global continuous algebraic canonical forms (c.f. (4.5)).

It should be remarked, however, that the local coordinates of the moduli variety are very closely related to certain (currently very popular) local canonical forms for (F, G, H).

Most of the time we shall be concerned with the input aspect only. I.e. we study pairs (F, G) under the equivalence relation (F, G) ~ ~ $(T^{-1}FT, TG)$. It is trivial to extend the theory to triples (F, G, H) (cf. 4.6). Dually we could have elected to study pairs (F, H) under (F, H) ~ (TFT^{-1}, HT^{-1}) and completely observable systems instead of completely reachable ones.

The contents of the paper are:

Some notation. The field we work over is denoted k; for convenience we take k algebraically closed. All schemes, varieties are over k; we consider only reduced algebraic separated schemes (varieties). The category of schemes over k is denoted $\underline{Sch}_k$, and $\underline{Sets}$ denotes the category of sets. If S, T $\in \underline{Sch}_k$ then $\underline{Sch}_k(T, S)$ is the set of morphisms from T to S. If S $\in \underline{Sch}_k$ then $\underline{O}_S$ is the sheaf of germs of functions on S. Projective space of dimension n and affine space of dimension n are respectively denoted by $\underline{\underline{P}}_k^n$ and $\underline{\underline{A}}_k^n$; GL_n is the group scheme of invertible $n \times m$ matrices, $GL_n(k)$ is its group of k-points; $G_{n,r}$ is the Grassmann variety of subspaces of dimension n in r-space.

If X is a finite set then $\# X$ denotes the number of elements of X.

2. The space $\underline{N}_{n,m}$.

2.1. Completely reachable systems and $n \times m(n+1)$ matrices.

Let (F, G) be a linear input system, where G is an $n \times m$ matrix and F an $n \times n$ matrix (i.e. there are m inputs and the dimension of the state space is n). Then (F, G) is completely reachable if and only if the rank of the matrix $R(F, G)$ is n, where

2.1.1. $$R(F, G) = (G, FG, \ldots, F^nG).$$

Cf. e.g. [4].

In this section we describe the image of the algebraic morphism

2.1.2. $$R: \underline{FG}^{cr}_{m,n} \to \underline{A}^{reg}_{n,\, m(n+1)}$$

where $\underline{FG}^{cr}_{m,n}$ is the space of all completely reachable pairs (F, G) and $\underline{A}_{n,m(n+1)}$ is the space of all $n \times m(n+1)$ matrices, and $\underline{A}^{reg}_{n,m(n+1)}$ is the open subset of $\underline{A}_{n,m(n+1)}$ consisting of the matrices of maximal rank (i.e. rank n).

The group $GL_n(k)$ of invertible $n \times n$ matrices with coefficients in k acts on (F, G) as

2.1.3. $$(F, G) \mapsto (SFS^{-1}, SG), \quad S \in GL_n(k).$$

Thus if we let $GL_n(k)$ act on $n \times m(n+1)$ matrices A as

2.1.4. $$A \mapsto SA, \quad S \in GL_n(k)$$

then R is a $GL_n(k)$-morphism.

2.2. Nice selections and successor selections.

The invariants of the action described above of $GL_n(k)$ on $\underline{A}_{n,m(n+1)}$ are ratios of expressions of the form $\det(A_\alpha)$, where α is a subset of size n of $\{1, 2, \ldots, m(n+1)\}$ which is given the natural order and A_α is the matrix consisting of the columns of A with column index in α. We shall call such subsets of size n of $\{1, 2, \ldots, m(n+1)\}$ selections. It is natural to expect that the expressions $\det(A_\alpha)$, α a selection, will be important in the description of the image of R.

Certain of these selections play a special role. To define them we number the $m(n+1)$ columns by pairs of integers as follows:

$$01, \ldots, 0m; 11, \ldots, 1m: \ldots; n1, \ldots, nm.$$

2.2.1. Definitions. A selection of α is called nice if $(i, j) \in \alpha \Rightarrow (i', j) \in \alpha$ for all $i' \leq i$.

Given a nice selection α, its successor selections are obtained as follows. Take any $(i, j) \in \{01, \ldots, nm\}$ such that $(i, j) \notin \alpha$ and $(i', j) \in \alpha$ for all $i' < i$. Now take away from $\alpha \cup \{i, j\}$ any of the original elements of α and reorder (if necessary) the resulting subset of $\{01, \ldots, nm\}$.

Note that a successor selection of a nice selection may be nice but need not be.

2.3. Equations for $\overline{\underline{N}}_{m,n}$.

We denote by $\underline{N}_{m,n}$ the image $R(\underline{FG}^{cr}_{m,n}) \subset \underline{A}^{reg}_{n,m(n+1)}$ and by $\overline{\underline{N}}_{m,n}$ its closure in the algebraic variety $\underline{A}^{reg}_{n,m(n+1)}$.

It is easily seen that $\underline{N}_{m,n}$ is neither open nor closed in $\underline{A}^{reg}_{n,m(n+1)}$. E.g. the matrix

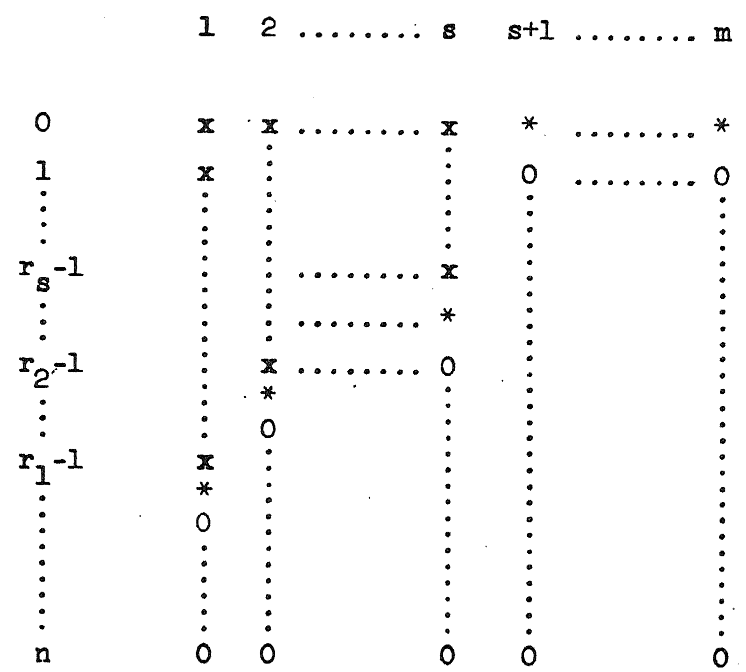
2.3.1. $$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is in $\overline{\underline{N}}_{m,n}$ as the specialization of

2.3.2. $$\begin{pmatrix} t^2 & 0 & t & 0 & 1 & 0 \\ 0 & t^2 & 0 & t & 0 & 1 \end{pmatrix}$$

as $t \to 0$. Cf. however (2.4).

Let α be a nice selection and suppose that $\det(R(F, G)_\alpha) \neq 0$, for a certain couple $(F, G) \in \underline{FG}^{cr}_{m,n}$. Pictorially we can represent α as shown below (where the columns of G have been permuted for convenience (if necessary)).

2.3.3 

The first row consists of the column indices 01, ..., 0m of R(F, G), the second row of the column indices 11, ..., 1m, etc.; the crosses x indicate the column indices in α the successors of α are obtained by adding one of the stars * and then deleting any of the crosses x.

Multiplying $R(F, G)$ with $(R(F, G)_\alpha)^{-1} = S$ we obtain a $n \times m(n+1)$ matrix R' (which is also in $R(\underline{FG}^{cr}_{m,n})$) such that $R'_\alpha = I_n$, the $n \times n$ unit matrix.

It is now obvious that the elements of the columns of R' indexed by stars in the array above are given by the numbers $\pm \det(R'_\beta)$ where β is a successor of α. (Indeed the successors obtained by adding the index of the column in question and then deleting any of the elements of α.)

<u>There are now precisely one</u> F' <u>and</u> G' <u>such that</u> $R' = R(F', G')$. Indeed if $g_1, \ldots, g_m$ are the columns of G' and $f_1, \ldots, f_n$ the columns of F' then

$$g_1 = e_1,\ g_2 = e_{r_1+1},\ \ldots,\ g_s = e_{r_1+\ldots+r_{s-1}+1},\ g_{s+1} = R'_{o,s+1},\ \ldots,\ g_m = R'_{o,m}$$

$$f_1 = e_2,\ f_2 = e_3,\ \ldots,\ f_{r_1-1} = e_{r_1},\ f_{r_1} = R'_{r_1-1,1}$$

$$f_{r_1+1} = e_{r_1+2},\ \ldots,\ f_{r_1+r_2-1} = e_{r_1+r_2},\ f_{r_1+r_2} = R'_{r_2-1,2}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$f_{r_1+\ldots+r_{s-1}+1} = e_{r_1+\ldots+r_{s-1}+2},\ \ldots,\ f_n = f_{r_1+\ldots+r_s} = R'_{r_s-1,s}$$

where R'_{ij} denotes the (ij)-column of R'. As F' and G' are now known we can calculate the other elements of $R' = R(F', G')$.

Let now γ be any selection, then we find in this way expressions

2.3.4. $\det(R'_\gamma) = P'_{\alpha\gamma}(\det(R'_\beta))$

where β runs through the successors of α, and $P'_{\alpha\gamma}$ is some polynomial.

Now for any selection δ

2.3.5. $\det(R(F, G)_\delta) = \det(R'_\delta)\det(R(F, G)_\alpha)$.

So putting in appropriate powers of $\det(R'_\alpha) = 1$ in 2.3.4 and using 2.3.5 we obtain for every selection γ homogeneous relations

2.3.6. $P_{\alpha\gamma}(\det(R(F, G)_\alpha; \det(R(F, G))_\gamma; \ldots, \det(R(F, G)_\beta, \ldots) = 0$

where β runs through the successors of α.

By Weyl's irrelevancy of algebraic inequalities principle (cf. e.g. [1]) these relations hold for all $R(F, G)$.

2.4. Description of $\underline{N}_{m,n}$ and $\overline{\underline{N}}_{m,n}$.

In this section we show that $\overline{\underline{N}}_{m,n}$ is the subvariety of $\underline{A}^{reg}_{n,m(n+1)}$ defined by the equations 2.3.6 (one equation for each pair (α, γ), α a nice selection, γ any selection), and describe $\underline{N}_{m,n}$ as the open subvariety of $\overline{\underline{N}}_{m,n}$ given by the condition: there is a nice selection α such that $\det(R_\alpha) \neq 0$.

To do this we need a lemma.

2.4.1. Lemma. If (F, G) is a completely reachable pair, then there is a nice selection such that $\det(R(F, G)_\alpha \neq 0$.

PROOF. We define a nice subselection of $\{01, \ldots, nm\}$ as an ordered subset σ of $\{01, \ldots, nm\}$ of size $r \leq n$, such that $(i, j) \in \sigma$ implies $(i', j) \in \sigma$ for all $i' \leq i$.

Now let σ be a nice subselection of maximal size such that the columns $R(F, G)_{ij}$ for $(i,j) \in \sigma$ are independent. (Note that $\#\sigma \geq 1$ because otherwise we would have $G = 0$ contradicting that $R(F, G)$ has maximal rank). Let V be the space spanned by the columns of $R(F, G)$ with index in σ. Rearranging the columns of G if necessary we can assume that B is spanned by the vectors

$$\begin{array}{lll} g_1 & g_2 \cdots\cdots\cdots & g_s \\ Fg_1 & \vdots & \vdots \\ \vdots & \vdots & F^{r_s}g_s \\ \vdots & F^{r_1}g_2 & \\ F^{r_1}g_1 & & \end{array}$$

Maximality of σ then gives that

$$F^{r_1+1}g_1 \in V, \quad F^{r_2+1}g_2 \in V, \quad \ldots, \quad F^{r_s+1}g_s \in V, \quad g_{s+1} \in V, \quad \ldots, \quad g_m \in V$$

and a very easy induction then gives

$$F^{r_1+k}g_1 \in V, \quad \ldots, \quad F^{r_s+k}g_s \in V, \quad F^k g_{s+1} \in V, \quad \ldots, \quad F^k g_m \in V$$

for all $k \in \underline{\underline{N}}$. Thus V contains all the vectors $F^i g_j$, i.e. $\dim V = \text{rank}(G, FG, \ldots, F^n G) = n$ and σ is therefore a nice subselection of size n, i.e. a nice selection.

2.4.2. Corollary

(i) $\underline{\bar{N}}_{m,n}$ is the closed subset of $\underline{A}^{reg}_{n,m(n+1)}$ given by equations 2.3.6.

(ii) $\underline{N}_{m,n}$ is the open subset of $\underline{\bar{N}}_{m,n}$ consisting of matrices $A \in \underline{\bar{N}}_{m,n}$ for which there is at least one nice selection α such that $\det(A_\alpha) \neq 0$.

(iii) The morphism $R: \underline{FG}^{cr}_{m,n} \to \underline{A}_{n,m(n+1)}$ is injective.

Proof. (iii) follows immediately from 2.4.1 because we can recover (F, G) from R(F, G) by means of the method described in subsection 2.3. To prove (i) and (ii) take any $A \in \underline{A}^{reg}_{n,m(n+1)}$ such that equations 2.3.3 hold and such that there is a nice selection α with $\det(A_\alpha) \neq 0$. Now calculate an (F, G) from A using the method of subsection 2.3. Then R(F, G) = A because equations 2.3.6 (for that α, any γ) hold for A.

3. The Grassman variety and moduli schemes for linear systems.

In this section we describe the Grassman variety and the moduli scheme for linear input systems (F, G).

3.1. The Grassman variety.

Consider the space $\underline{A}_{n,r}^{reg}$ (where $r > n$) of maximal rank $n \times r$ matrices. The group $GL_n(k)$ acts on this space as $(S, A) \mapsto SA$. The geometric quotient for this action exists; it is called the Grassman variety $G_{n,r}$. (Cf. [5] for a discussion of $G_{n,r}$ and [6], [7] for the definition of "geometric quotient".)

It can be described as follows.

For each selection α (i.e. a subset of size n of $\{1, \ldots, r\}$) and $A \in \underline{A}_{n,r}^{reg}$ we let A_α denote the matrix consisting of the columns in A with column index in α. We define a function x_α: $\underline{A}_{n,r}^{reg} \to \underline{\underline{A}}^1$ by $x_\alpha(A) = \det(A_\alpha)$.

Let α run through all selections (there are $\binom{r}{n}$ selections). Then, because at least one of the $x_\alpha(A)$ is non-zero (because A has maximal rank), we obtain a morphism

3.1.1. φ: $\underline{A}_{n,r}^{reg} \to \underline{\underline{P}}_k^N$, $A \to (x_\alpha)_\alpha \in \underline{\underline{P}}_k^N$, $N = \binom{r}{n} - 1$

where α runs through all selections. Note that $\varphi(SA) = \varphi(A)$, for all $S \in GL_n(k)$, so φ is constant on the orbits. We are going to describe the image of φ.

For each selection α let $U_\alpha = \{A \in \underline{A}_{n,r}^{reg} | x_\alpha(A) \neq 0\}$, and for

$A \in U_\alpha$ we define $c_\alpha(A) = A_\alpha^{-1}A$.

It is clear that for each $A \in U_\alpha$, and each selection β

3.1.2. $$\det(A_\alpha) \cdot \det((A_\alpha^{-1}A)_\beta) = \det(A_\beta).$$

Now $(A_\alpha^{-1}A)_\alpha = I_n$ and the elements of $A_\alpha^{-1}A$ which are not in a column with index in α are of the form $\det((A_\alpha^{-1}A)_\gamma)$ for certain γ and therefore can be written $x_\gamma(A)(x_\alpha(A))^{-1}$. Substituting this in 3.1.2 and multiplying with an appropiate power of $x_\alpha(A)$ we obtain a set of homogeneous relations

3.1.3. $$q_{\alpha\beta}(\ldots, x_\gamma(A), \ldots) = 0$$

which are satisfied (using Weyl's irrelevancy principle again) by all $A \in \underline{A}_{n,r}^{reg}$. (If $\#(\alpha \cap \beta) = n - 1$, $q_{\alpha,\beta}$ is the trivial relation $x_\alpha(A)x_\beta(A) - x_\alpha(A)x_\beta(A)$).

3.1.4. Proposition.

Im φ is the subset of $\underline{\underline{P}}_k^N$ described by equations 3.1.3.

<u>Proof</u>. Let $x = (\ldots, x_\gamma, \ldots)$ be an element of $\underline{\underline{P}}_k^N$ satisfying 3.1.3. There is an α such that $x_\alpha \neq 0$. We can assume $x_\alpha = 1$. Now let the matrix $B_\alpha(x)$ be constructed as follows.

$(B_\alpha(x))_\alpha = I_n$, $b_{ij} = x_\beta$ for $j \notin \alpha$, where β is the selection obtained by adding j to α and deleting the index of the column in $(B_\alpha(x))_\alpha$ which is equal to the i-th unit vector.

Then $B_\alpha(x) \in \underline{A}_{n,r}^{reg}$ and $\varphi(B_\alpha(x)) = x$ because x satisfies the relations 3.1.3 (for this particular α and all β). This follows immediately from the way in which the relations were obtained.

We shall denote the subvariety of $\underline{\underline{P}}_k^N$ defined by equations 3.1.3 by $G_{n,r}$ and call it the <u>Grassmann variety</u>. Note that $G_{1,r} = \underline{\underline{P}}_k^r$.

3.1.5. <u>Corollary</u> (of the proof).

$\{x \in G_{n,r} \mid x_\alpha \neq 0\} \simeq \underline{\underline{A}}^s$ with $s = n(r - n)$; $\dim G_{n,r} = n(r-n)$. The isomorphism is given by the morphisms $x \mapsto B_\alpha(x)$.

3.2. <u>Families of linear systems</u>.

Let S be an algebraic variety over k. In the definition of what a <u>family of linear input systems over</u> S is (or a <u>family parametrized</u> by S) we have some choice. We could e.g. define a family over S as consisting of an n-vector bundle $\underline{E}$ over S, an m-vector bundle $\underline{E}'$ over S, an endomorphism of vector bundles $F: \underline{E} \to \underline{E}$ and a homomorphism of vector bundles $G: \underline{E}' \to \underline{E}$. If we take this as a definition there is certainly not going to be a fine moduli scheme for this functor, because tensoring everything with a nontrivial line bundle then gives a locally isomorphic but not globally isomorphic family. It is therefore more natural to take a rigidified version of the previous tentative definition.

3.2.1. <u>Definitions</u>.

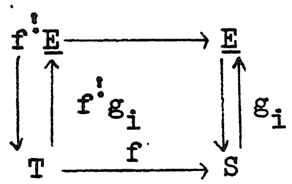
A <u>family</u> $\mathcal{F}$ <u>of linear input systems of dimension</u> n <u>with</u> m <u>inputs</u>, over an algebraic variety S, consists of an n-dimensional vector bundle $\underline{E}$ over S, a vector bundle endomorphism $F: \underline{E} \to \underline{E}$, and m sections $g_1, \ldots, g_m: S \to \underline{E}$.

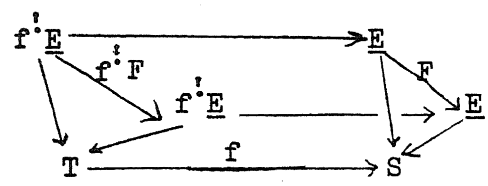
The family $(\underline{E}, F, g_1, \ldots, g_m)$ is <u>completely reachable</u> if for every $s \in S$ we have that the fibre $\underline{E}(s)$ at $s \in S$ is spanned by the vectors $F(s)^i g_j(s)$, $i = 0, 1, \ldots, n$; $j = 1, \ldots, m$.

Two families $\mathcal{T} = (\underline{E}, F, g_1, \ldots, g_m)$ and $\mathcal{T}' = (\underline{E}', F', g_1', \ldots, g_m')$ over S are said to be isomorphic if there exists a vector bundle isomorphism $\sigma\colon \underline{E} \to \underline{E}'$ such that $F'\sigma = \sigma F$, $\sigma g_i = g_i'$, $i = 1, \ldots, m$.

We let $\underline{F}_{m,n}(S)$ be the set of isomorphism classes of completely reachable families of linear input systems $\mathcal{T}$ of dimension n with m inputs over S.

Let $f\colon T \to S$ be a morphism of algebraic varieties and let $\mathcal{T} = (\underline{E}, F, g_1, \ldots, g_m)$ be a family of linear systems over S. Then the induced family $f^!\mathcal{T} = (f^!\underline{E}, f^!F, f^!g_1, \ldots, f^!g_m)$ over T is obtained by "pulling everything back along f". I.e. $f^!\underline{E}$ is the induced vector bundle over T, $f^!F$ is its induced endomorphism and if we identify $(f^!\underline{E})(t)$ with $\underline{E}(f(t))$ then $(f^!g_i)(t) = g_i(f(t))$. The following diagrams are therefore commutative.





The family $(f^!\underline{E}, f^!F, f^!g_1, \ldots, f^!g_m)$ is completely reachable if (and only if) the family $(\underline{E}, F, g_1, \ldots, g_m)$ is completely reachable.

Thus we have defined a contravariant functor.

3.2.2. $\underline{F}_{m,n}\colon \underline{Sch_k} \to \underline{Sets}$

For convenience we recall what a fine and coarse moduli space for $\underline{F}_{m,n}$ would be.

3.2.3. Definitions

A fine moduli space for $\underline{F}_{m,n}$ is a scheme $M_{m,n}$ and an isomorphism of functors Φ: $\underline{F}_{m,n} \Rightarrow \underline{Sch}_k(, M_{m,n})$. A coarse moduli space for $\underline{F}_{m,n}$ is a scheme $M_{m,n}$ and a morphism of functors Φ: $\underline{F}_{m,n} \to \underline{Sch}_k(, M_{m,n})$ such that $\Phi(\mathrm{Spec}(k))$ is an isomorphism and such that for every scheme N and functor morphism Φ: $\underline{F}_{m,n} \to \underline{Sch}(, N)$ there is a unique morphism h: $M_{m,n} \to N$ such that $\Psi = \underline{Sch}_k(, h) \circ \Phi$.

If $M_{m,n}$ is a moduli space then the k-points of $M_{m,n}$ are in 1-1 correspondence with isomorphism classes of linear input systems (dim n, m inputs). The map $S \to M_{m,n}$ associated to a family $(\underline{E}, F, g_1, \ldots, g_m)$ over S associates to $s \in S$ the point in $M_{m,n}$ corresponding to the isomorphism class of $(F(s), g_1(s), \ldots, g_m(s))$.

3.3. A coarse moduli space for families of linear input systems.

Let $\underline{\overline{M}}_{m,n}$ be the subvariety of $G_{n,m(n+1)}$ defined by the equations 2.3.6. More precisely, $\underline{\overline{M}}_{m,n}$ is the subvariety of $G_{n,m(n+1)}$ defined by the equations

3.3.1. $$P_{\alpha\gamma}^{\prime}(x_\alpha, x_\gamma, \ldots, x_\beta, \ldots) = 0$$

where the $P_{\alpha\gamma}$ are the polynomials of 2.3.6. Then $\underline{\overline{M}}_{m,n}$ is a projective variety, since it is a closed subvariety of $G_{n,m(n+1)} \subset \underline{\underline{P}}^N$. As a subvariety of $\underline{\underline{P}}^N$ it is also given by equations 3.3.1 (the equations $q_{\alpha\beta}(\ldots) = 0$ of 3.1.3 are consequences of 3.3.1 as is clear from the way we obtained these equations).

Let $\underline{M}_{m,n}$ be the open subspace of $\underline{\overline{M}}_{m,n}$ defined by

$$\underline{M}_{m,n} = \{x \in \underline{\overline{M}}_{m,n} \mid \exists \text{ nice selection } \alpha \text{ such that } x_\alpha \neq 0\}.$$

3.3.2. Theorem.

$\underline{M}_{m,n}$ is a coarse moduli space for $\underline{F}_{m,n}$.

Proof. We know that $G_{n,m(n+1)}$ is the geometric quotient of $\underline{A}^{reg}_{n,m(n+1)}$ by GL_n. Now $\underline{N}_{m,n} \subset \underline{A}^{reg}_{n,m(n+1)}$ is invariant under GL_n and hence so is the closed subvariety $\overline{\underline{N}}_{m,n} \subset \underline{A}^{reg}_{n,m(n+1)}$ and the equations defining $\overline{\underline{N}}_{m,n}$ are polynomials in the x_α. Hence $\overline{\underline{M}}_{m,n}$ is the geometric quotient of $\overline{\underline{N}}_{m,n}$ by GL_n and the geometric quotient of $\underline{N}_{m,n}$ is $\underline{M}_{m,n}$. The fact that $\underline{M}_{m,n}$ is a coarse moduli space for $\underline{F}_{m,n}$ now follows from the general theory connecting quotient spaces and coarse moduli spaces. (Cf. [G] and [M]). Given $(\underline{E}, F, g_1, \ldots, g_m)$ over S, then $\Phi(S)$ assigns to s the point of $\underline{M}_{m,n}$ obtained as follows. Choose any basis in $E(s)$. Let $\overline{F}(s)$, $\overline{g}_1(s), \ldots, \overline{g}_n(s)$ be the matrix and coordinate vectors of F and $g_1, \ldots, g_m$ with respect to this basis. Then $\Phi(S)(s) = \varphi(\overline{g}_1(s)\ldots\overline{g}_m(s);\ \overline{F}(s)\overline{g}_1(s)\ldots\overline{F}(s)\overline{g}_m(s);\ \ldots;\ \overline{F}^n(s)\overline{g}_1(s), \ldots, \overline{F}^n(s)\overline{g}_m(s))$ where φ is the morphism 3.1.1.

3.4. The canonical bundle over $G_{n,r}$

Every matrix $A \in \underline{A}^{reg}_{n,r}$ defines an n-dimensional subspace of $\underline{\underline{A}}^r$, viz. the subspace spanned by the rows of the matrix A. Two matrices A, B span the same subspace iff there is an $S \in GL_n(k)$ such that $A = SB$. Thus $G_{n,r}$ "is" the space of n-dimensional subspaces of $\underline{\underline{A}}^r$.

The canonical n-bundle ξ_n over $G_{n,r}$ can now informally be described as the bundle over $G_{n,r}$ whose fiber over $x \in G_{n,r}$ is the n-space represented by x. More precisely ξ_n is the subbundle of $G_{n,r} \times k^r$ defined by (cf. e.g. [3])

3.4.1 $$\xi_n = \{(x, y) \in G_{n,r} \times \underline{A}^n \mid y \in x\}.$$

In terms of trivial local pieces and patching data we have the following description. For each selection α from $\{1, \ldots, r\}$ let $V_\alpha = \{x \in G_{n,r} \mid x_\alpha \neq 0\}$. Then $V_\alpha \simeq \underline{\underline{A}}^{n(r-n)}$. Let $B_\alpha(x)$ be the unique element of $\underline{A}_{n,r}$ such that $\varphi(B_\alpha(x)) = x$ and $(B_\alpha(x))_\alpha = I_n$. (Cf. the proof of 3.1.4 for the construction of $B_\alpha(x)$.) The bundle ξ_n is trivial over each V_α. The trivialization being given by

$$V_\alpha \times \underline{\underline{A}}^n \to G_{n,r} \times \underline{\underline{A}}^r$$

3.4.2. $(x, (t_1, \ldots, t_n)) \mapsto (x, (t_1, \ldots, t_n)B_\alpha(x))$

In order to describe the bundle ξ_n over all of $G_{n,r}$ it therefore suffices to give the identification isomorphisms

$$\rho_{\alpha\beta}: (V_\alpha \cap V_\beta) \times \underline{\underline{A}}^n \to (V_\beta \cap V_\alpha) \times \underline{\underline{A}}^n$$

and these are obtained as follows. Let $x \in V_\alpha \cap V_\beta$ and let $T_{\alpha\beta}(x)$ be the unique element of $GL_n(k)$ such that

3.4.3. $T_{\alpha\beta}(x)B_\alpha(x) = B_\beta(x)$.

Then

3.4.4. $\rho_{\alpha\beta}(x, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}) = (x, T_{\alpha\beta}(x)\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix})$

3.5. <u>A canonical family of linear systems over</u> $\underline{M}_{m,n}$.

Let α be a nice selection. For each $x \in W_\alpha = V_\alpha \cap \underline{M}_{m,n} \subset G_{n,m(n+1)} \subset \underline{\underline{P}}^N$ let $B_\alpha(x)$ be the matrix such that $(B_\alpha(x))_\alpha = I_n$ and $\varphi(B_\alpha(x)) = x$. Because $x \in \underline{M}_{m,n}$ there exist a unique $n \times n$ matrix $F_\alpha(x)$ and $n \times m$ matrix $G_\alpha(x)$ such that $R(F_\alpha(x), G_\alpha(x)) = B_\alpha(x)$.

We now define a canonical family $(\underline{E}^c, F^c, g_1^c, \ldots, g_m^c)$ of linear input systems over $\underline{M}_{m,n}$ as follows.

The bundle $\underline{E}^c$ is trivial over each $W_\alpha = V_\alpha \cap \underline{M}_{m,n}$, and the identification maps $\rho_{\alpha\beta}$: $W_\alpha \cap W_\beta \times \underline{\underline{A}}^n \to W_\alpha \cap W_\beta \times \underline{\underline{A}}^n$ are given by

$$\rho_{\alpha\beta}(x, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}) = (x, T_{\alpha\beta}(x)\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix})$$

where $T_{\alpha\beta}(x)$ is as above. I.e. the bundle over $\underline{M}_{m,n}$ is the restriction to $\underline{M}_{m,n}$ of the canonical bundle over $G_{n,m(n+1)}$.

The endomorphism F^c and the sections $g_1^c, \ldots, g_m^c$ are given over W_α with respect to the canonical basis of $\underline{\underline{A}}^n$ by the matrices $F_\alpha(x)$ and $G_\alpha(x)$. Then, because $T_{\alpha\beta}(x)B_\alpha(x) = B_\beta(x)$ and $R(F_\alpha(x), G_\alpha(x)) = B_\alpha(x)$, $R(F_\beta(x), G_\beta(x)) = B_\beta(x)$ and maximality of rank we have,

$$T_{\alpha\beta}(x)G_\alpha(x) = G_\beta(x)$$

$$T_{\alpha\beta}(x)F_\alpha(x) = F_\beta(x)T_{\alpha\beta}(x)$$

so that the identification isomorphisms of the bundle transform the local endomorphisms and sections into each other. I.e. we have defined a family of linear systems over $\underline{M}_{m,n} = \bigcup_{\alpha \text{ is nice}} W_\alpha$. (Cf. 2.2).

3.6. $\underline{M}_{m,n}$ <u>is a fine moduli scheme</u>.

3.6.1. <u>Theorem</u>. $\underline{M}_{m,n}$ is a fine moduli scheme for families of completely reachable linear m-input systems of dimension n. The canonical family $\mathcal{F}^c = (\underline{E}^c, F^c, g_1^c, \ldots, g_m^c)$ of (3.5) is the universal family over $\underline{M}_{m,n}$.

Proof. We have to show two things.

(i) Let $f: S \to \underline{M}_{m,n}$ be any morphism. Then $\Phi(S)(f^{!}\mathcal{T}^{c}) = f$.

(ii) Let $\mathcal{T}$ be a family of systems over S and let $f = \Phi(S)(\mathcal{T})$ then $f^{!}\mathcal{T}^{c}$ is isomorphic to $\mathcal{T}$.

To prove (i), note that the linear system over a point $s \in S$ defined by $f^{!}\mathcal{T}^{c}$ is given (up to equivalence) by the matrices $F_\alpha(f(s))$, $G_\alpha(f(s))$ where α is any nice selection such that $f(s) \in W_\alpha \subset \underline{M}_{m,n}$. It now follows immediately from the definition of $\Phi(S)$ (cf. 3.2) that $\Phi(S)(f^{!}\mathcal{T}^{c})(s) = f(s)$.

As to (ii), let $\mathcal{T} = (\underline{E}, F, g_1, \ldots, g_m)$. For every nice selection α let $S_\alpha = f^{-1}W_\alpha$. Then $S = \bigcup S_\alpha$. Let $s \in S_\alpha$ and let $A(s)$ be the matrix $R(F(s), G(s))$ where $F(s)$ and $G(s)$ are the matrices of the linear system defined by $\mathcal{T}$ over s relative some basis of $E(s)$. Then because $F(s) \in W_\alpha$ $\det(A(s)_\alpha) \neq 0$ in view of the definition of $\Phi(S)$. So locally (= over each S_α) $f^{!}\mathcal{T}^{c}$ and $\mathcal{T}$ are "the same". This means that the bundle $\underline{E}$ of $\mathcal{T}$ is trivial over any S_α. And the isomorphism a_α: $\underline{E}|S_\alpha \to S_\alpha \times \underline{\underline{A}}^n$ transforms $F|S_\alpha$ and $g_1|S_\alpha, \ldots, g_m|S_\alpha$ into $(F_\alpha(f(s)), G_\alpha(f(s)))$.

To see how these trivial pieces fit together, let $s \in S_\alpha \cap S_\beta$. Choose any basis in $E(s)$ and let $A(s)$ be the matrix of $R(F(s), G(s))$ where $F(s)$ and $G(s)$ are the matrices of the system over s relative to the chosen basis. Then, relative to this basis and the canonical basis in $\underline{\underline{A}}^n$, the isomorphism a_α at s takes $A(s)$ into $A(s)_\alpha^{-1}A(s) = R(F_\alpha(f(s)), G_\alpha(f(s))$. Similarly for β. It follows that the identification isomorphism

$$S_\alpha \cap S_\beta \times \underline{\underline{A}}^n \to S_\beta \cap S_\alpha \times \underline{\underline{A}}^n$$

is given by the matrix $A(s)_\beta^{-1}A(s)_\alpha$ and

$$R(F_\beta(f(s), G_\beta(f(s)) = A(s)_\beta^{-1}A(s)_\alpha R(F_\alpha(f(s), G_\alpha(f(s))$$

which shows that

$$A(s)_\beta^{-1}A(s)_\alpha = T_{\alpha\beta}(f(s))$$

because $R(F_\alpha(f(s), G_\alpha(f(s))$ is of maximal rank. So that the pieces over S_α, α nice, of $f^!\tau^c$ and τ also fit together in the same way. (This, by the way, is also "immediately clear" because the isomorphism must transform the F and G in the right way, and rank maximality then determines the isomorphism uniquely). This concludes the proof of the theorem.

3.7. <u>A lower codimensional projective embedding for</u> $\underline{M}_{m,n}$.

Let η be the set of selections from $\{01, \ldots, 0m; 11, \ldots, 1m; \ldots; n1, \ldots, nm\}$ and let ζ be the subset of η of nice selections and successors of nice selections. Let $\underline{\underline{P}}^N$ be the projective space with coordinate functions labelled by η and $\underline{\underline{P}}^{N'}$ the projective space with coordinate functions labelled by ζ. Let $\pi\colon \underline{\underline{P}}^N \to \underline{\underline{P}}^{N'}$ be the natural projection (not everywhere defined).

Now consider the composed map

$$i\colon\ \underline{M}_{m,n} \to \underline{\underline{P}}^N \overset{\pi}{\to} \underline{\underline{P}}^{N'}$$

where the first arrow is the embedding of $\underline{M}_{m,n}$ is $\underline{\underline{P}}^N$ given by equations 3.3.1.

Then first of all $i\colon \underline{M}_{m,n} \to \underline{\underline{P}}^{N'}$ is everywhere defined on $\underline{M}_{m,n}$. (because if $x \in \underline{M}_{m,n}$ there is a nice selection α such that $x_\alpha \neq 0$), and secondly i is injective because once we know the values of the x_β, β a successor of α (and $x_\alpha \neq 0$) we can calculate $B_\alpha(x)$ and hence all the other x_γ. (Cf. 2.3).

This gives us an embedding $\underline{M}_{m,n} \to \underline{\underline{P}}^{N'}$ of considerable lower codimension than the original embedding $\underline{M}_{m,n} \to \underline{\underline{P}}^{N}$. This can be convenient for calculations.

To obtain equations for the closure of $\underline{M}_{m,n}$ in $\underline{\underline{P}}^{N'}$ proceed exactly in the same way as for $\overline{\underline{M}}_{m,n} \subset \underline{\underline{P}}^{N}$.

E.g. for $\underline{M}_{2,3}$ we have $N = 55$ and $N' = 15$.

4. Local description of $\underline{M}_{m,n}$. Canonical forms.

In this section we give a local pieces and patching data description of $\underline{M}_{m,n}$, we calculate a specific example $(\underline{M}_{2,2})$, we prove that the canonical bundle over $\underline{M}_{m,n}$ (cf. 3.4 and 3.5) is nontrivial if $m \geq 2$ and conclude that no globally defined continuous canonical forms exist for m-input linear systems of dimension n if $m \geq 2$.

4.1. Local description of $\underline{M}_{m,n}$.

Let α be a nice selection from $\{01, \ldots, 0m; 11, \ldots, 1m; \ldots; n1, \ldots, nm\}$ and let σ_α be the set of selections which are successors to α. Note that $\#\sigma_\alpha = mn$. For each $\beta \in \sigma_a$ let $y_\beta \in k$ be arbitrary then there exists precisely one pair of matrices (F, G) such that

$$R(F, G)_\alpha = I_n, \quad \det (R(F, G)_\beta) = y_\beta$$

(cf. (2.3)).

It follows that

$$W_\alpha = \{x \in M_{m,n} \mid x_\alpha \neq 0\} \rightleftarrows \underline{\underline{A}}^{mn}$$

Let this isomorphism be ψ_α. An immediate consequence is

4.1.1. Proposition. $\dim \underline{M}_{m,n} = mn$; $\underline{M}_{m,n}$ is connected and irreducible.

Let $y = (y_\beta) \in \underline{\underline{A}}^{mn}$, and let γ be a nice selection. The condition $\det (B_\alpha(\psi_\alpha(y))_\gamma) \neq 0$ defines an open subset of $\underline{\underline{A}}^{mn}$ which gets identified with the open subset of $\underline{\underline{A}}^{mn} \simeq W_\gamma$ defined by $\det (B_\gamma(\psi_\gamma(z))_\alpha) \neq 0$ according to the obvious formulas. (Calculate $\det (B_\alpha(\psi_\alpha(y))_\delta$ for all successors δ of γ; cf. 4.3 for a specific example.)

<u>Example. Equations for</u> $\underline{M}_{2,2}$.

We consider the Grassmann variety $G_{2,6} \subset \underline{\underline{P}}^{14}$. We number the coordinates $\underline{\underline{P}}^{14}$ as $x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{23}, x_{24}, x_{25}, x_{26}, x_{34}, x_{35}, x_{36}, x_{46}, x_{56}$ (x_{ab} corresponds to the selection ab from $\{1, 2, \ldots, 6\}$.

The equations for $G_{2,6}$ in $\underline{\underline{P}}^{14}$ are

$$x_{12}x_{34} = x_{13}x_{24} - x_{23}x_{14} \qquad *$$
$$x_{12}x_{35} = x_{13}x_{25} - x_{23}x_{15} \qquad *$$
$$x_{12}x_{36} = x_{13}x_{26} - x_{23}x_{16} \qquad *$$
$$x_{12}x_{45} = x_{14}x_{25} - x_{24}x_{15} \qquad *$$
$$x_{12}x_{46} = x_{14}x_{26} - x_{24}x_{16} \qquad *$$
$$x_{12}x_{56} = x_{15}x_{26} - x_{25}x_{16} \qquad *$$
$$x_{13}x_{45} = x_{14}x_{35} - x_{15}x_{34} \qquad *$$
$$x_{13}x_{46} = x_{14}x_{36} - x_{16}x_{34} \qquad *$$
$$x_{13}x_{56} = x_{15}x_{36} - x_{35}x_{16} \qquad *$$
$$x_{14}x_{56} = x_{15}x_{46} - x_{45}x_{16}$$
$$x_{23}x_{45} = x_{24}x_{35} - x_{34}x_{25} \qquad *$$
$$x_{23}x_{46} = x_{24}x_{36} - x_{34}x_{26} \qquad *$$
$$x_{23}x_{56} = x_{25}x_{36} - x_{35}x_{26}$$
$$x_{24}x_{56} = x_{25}x_{46} - x_{45}x_{26} \qquad *$$
$$x_{34}x_{56} = x_{35}x_{46} - x_{45}x_{36}$$

The equations for $\underline{M}_{2,2}$ as a subvariety of $G_{2,6}$ are

$$x_{12}x_{25} = x_{13}x_{24} - x_{23}^2$$
$$x_{12}x_{26} = x_{14}x_{23} - x_{23}x_{24}$$
$$x_{12}x_{15} = x_{13}x_{14} - x_{13}x_{23}$$
$$x_{12}x_{16} = x_{14}^2 - x_{13}x_{24}$$

$$x_{13}x_{34} = x_{12}x_{35}$$

4.2.2 $$x_{13}x_{14} = x_{12}x_{15} - x_{13}x_{23}$$
$$x_{13}^2x_{36} = x_{12}x_{15}x_{35} - x_{13}x_{35}x_{23}$$
$$x_{13}^2x_{16} = x_{12}x_{13}x_{35} - x_{13}x_{15}x_{23} + x_{12}x_{15}^2$$

$$x_{24}x_{34} = x_{12}x_{46}$$
$$x_{24}x_{23} = x_{24}x_{14} - x_{12}x_{26}$$
$$x_{24}^2x_{45} = x_{14}x_{24}x_{46} - x_{12}x_{26}x_{46}$$
$$x_{24}^2x_{25} = x_{12}x_{46}x_{24} + x_{14}x_{12}x_{26} - x_{12}x_{26}^2$$

To obtain equations for $\underline{M}_{2,2}$ as a subspace of $\underline{\underline{P}}^{14}$ one adds the equations marked with a (*) of 4.2.1 to the equations of 4.2.2. Equations for the closure of $\underline{M}_{2,2}$ in $\underline{\underline{P}}^{N'}$ ($N' = 8$ in this case) are obtained by disregarding all equations involving x_{16}, x_{25}, x_{34}, x_{36}, x_{45}, or x_{56}.

4.3. Local pieces and patching data description of $\underline{M}_{2,2}$.

There are three nice selections from {01, 02; 11, 12; 21, 22}, so $\underline{M}_{2,2}$ is obtained by patching three pieces V_1, V_2, V_3 together. All three pieces are isomorphic to $\underline{\underline{A}}^4$. Let V_1 be the piece corresponding to the nice selection $\alpha = \{01, 02\}$, V_2 the one corresponding to $\beta = \{01, 11\}$ and V_3 the one corresponding to $\gamma = \{02, 12\}$.

The canonical matrix $B_\alpha(a)$ for $a \in V_1$ is of the form $(\alpha = \{01, 11\})$

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 & a_1^2 + a_2a_3 & a_1a_2 + a_2a_4 \\ 0 & 1 & a_3 & a_4 & a_1a_3 + a_3a_4 & a_2a_3 + a_4^2 \end{pmatrix}$$

The canonical matrix $B_\beta(b)$ for $b \in V_2$ is of the form $(B = \{01, 11\})$

$$\begin{pmatrix} 1 & b_1 & 0 & b_2b_3 & b_3 & b_1b_3 + b_2b_3b_4 \\ 0 & b_2 & 1 & b_1 + b_2b_4 & b_4 & b_2b_3 + b_1b_4 + b_2b_4^2 \end{pmatrix}$$

From this we see that V_{12} and V_{21}, the parts of V_1 and V_2 that must be identified, are given by

$$V_{12} = \{a \in V_1: \ a_3 \neq 0\}$$
$$V_{21} = \{b \in V_2: \ b_2 \neq 0\}$$

and the identification is given by

4.3.1. $$b_1 = -a_1a_3^{-1} \qquad b_2 = a_3^{-1}$$
$$b_3 = a_2a_3 - a_1a_4 \qquad b_4 = a_1 + a_4$$

The canonical matrix $B_\gamma(c)$ for $c \in V_3$ is of the form $(\gamma = \{02, 12\}$

$$\begin{pmatrix} c_1 & 1 & c_2c_3 & 0 & c_1c_3 + c_2c_3c_4 & c_3 \\ c_2 & 0 & c_1 + c_2c_4 & 1 & c_2c_3 + c_1c_4 + c_2c_4^2 & c_4 \end{pmatrix}$$

And one finds

$$V_{13} = \{a \in V_1 \mid a_2 \neq 0\}$$
$$V_{31} = \{c \in V_3 \mid c_2 \neq 0\}$$

and the identification is given by

4.3.2. $c_1 = -a_2^{-1}a_4 \qquad c_3 = a_2a_3 - a_1a_4$

$c_2 = a_2^{-1} \qquad c_4 = a_1 + a_4$

Finally

$$V_{23} = \{b \in V_2 \mid b_1^2 + b_1b_2b_4 - b_2^2b_3 \neq 0\}$$
$$V_{32} = \{c \in V_3 \mid c_1^2 + c_1c_2c_4 - c_2^2c_3 \neq 0\}$$

and the identification is given by

4.3.3. $c_1 = \dfrac{b_1 + b_2b_4}{b_1^2 + b_1b_2b_4 - b_2^2b_3} \qquad c_2 = \dfrac{-b_2}{b_1^2 + b_1b_2b_4 - b_2^2b_3}$

$c_3 = b_3 \qquad c_4 = b_4$

4.4. <u>Nontriviality of the canonical bundle over $\underline{M}_{m,n}$ if $m \geq 2$</u>

Let $\mathcal{T}^c = (E^c, F^c, g_1^c, \ldots, g_m^c)$ be the universal family defined over $\underline{M}_{m,n}$. In this section we shall show that $\underline{E}^c$ is not the trivial bundle. More precisely we shall do this for $\underline{M}_{2,2}$ but the argument generalizes immediately.

Let ξ_n be the canonical n-plane bundle on bundle on $G_{n,r}$ and similarly let ξ_1 be the canonical line bundle on $G_{1,N+1} = \underline{\underline{P}}_k^N$ where $N = \binom{n}{r} - 1$ and let $G_{n,r} \to \underline{\underline{P}}^N$ be the canonical embedding (cf. 3.1 and 3.4). Then

$$\overset{n}{\wedge}\xi_n \simeq \xi_1 \mid G_{n,r}$$

where $\overset{n}{\wedge}\xi_n$ denotes the n-th exterior product of ξ_n.

Now, $\underline{E}^c$, the underlying bundle of $\mathcal{T}^c$, is $\xi_n \mid M_{m,n}$. Suppose that $\underline{E}^c$ were trivial over $\underline{M}_{m,n}$, then $\overset{n}{\wedge}\underline{E}^c$ would be the trivial line bundle, but $\overset{n}{\wedge}\underline{E}^c$ is the restriction of ξ_1 to $M_{m,n}$ and the sheaf of sections of ξ_1 is $\underline{O}_{\underline{\underline{P}}^N}(1)$ which is very ample.

Therefore if $\underline{E}^c$ were trivial we would have that the sheaf $\underline{O}_{\underline{M}_{m,n}}$ is very ample. But this in turn (cf. [2], Ch. II) implies that the open sets

$$D(f) = \{ x \in \underline{M}_{m,n} \mid f(x) \neq 0 \}, \quad f \in \Gamma(\underline{M}_{m,n}, \underline{O}_{\underline{M}_{m,n}})$$

form a basis for $\underline{M}_{m,n}$.

Now let f be a global section of $\underline{O}_{\underline{M}_{2,2}}$, i.e. a function $\underline{M}_{2,2} \to \underline{\underline{A}}^1$. Restricted to V_1, f is a polynomial in a_1, a_2, a_3, a_4. But from equations 4.3.1 we see that the specialization of $f(a_1, a_2, a_3, a_4)$ as $a_3^{-1} \to 0$ but $a_2 a_3 \to$ some finite limit, must exist. This shows that $f(a_1, a_2, a_3, a_4)$ can involve no pure powers of a_3. Therefore

4.4.2. $\forall f\colon \underline{M}_{2,2} \to \underline{\underline{A}}^1$, either $(0, 0, a_3, 0) \in D(f) \cap V_1$ for all $a_3 \in k$
or $(0, 0, a_3, 0) \notin D(f) \cap V_1$ for all $a_3 \in k$

so that the $D(f)$ do not form a basis for the open sets of $\underline{M}_{2,2}$. It is easy to generalize this argument. Therefore

4.4.3. Proposition.

The underlying bundle $\underline{E}^c$ of the universal family of linear systems γ^c over $\underline{M}_{m,n}$ is nontrivial if $m \geq 2$.

Remark. $\underline{M}_{1,n} \simeq \underline{\underline{A}}^n$ and the underlying bundle $\underline{E}^c$ of the universal family over $\underline{M}_{1,n}$ is trivial.

4.5. Canonical forms.

The nontriviality of $\underline{E}^c$, or more precisely the nonexistence of sufficiently many functions on $\underline{M}_{m,n}$ for $m \geq 2$ ruins the chances for the existence of a global continuous algebraic canonical form for linear input systems with 2 or more inputs. Indeed suppose such a form existed and let $\overline{F}(s)$, $\overline{G}(s)$ be the canonical form for the system $\Sigma^c(s)$, $s \in \underline{M}_{m,n}$. Then because $\Sigma^c(s)$ is not equivalent to $\Sigma^c(s')$ for $s \neq s'$ this would give us an embedding

$$\underline{M}_{m,n} \to \underline{\underline{A}}^{n(n+m)}, \quad s \mapsto (\overline{F}(s), \overline{G}(s))$$

which would imply the existence of sufficiently many functions $f: \underline{M}_{m,n} \to \underline{A}^1$ to separate points. As this is not the case (cf. 4.4) we see that there does not exist a continuous algebraic canonical form.

Of course there exist many discontinuous algebraic canonical forms, e.g. order the nice selections from $\{01, \ldots, 0m; 11, \ldots, 1m; \ldots; n1, \ldots, nm\}$. Let these be $\alpha_1, \ldots, \alpha_r$, and define the canonical form by

$$(F, G) \to (\overline{F}, \overline{G})$$

where $\overline{F}$, $\overline{G}$ are such that

$$R(\overline{F}, \overline{G}) = R(F, G)_{\alpha_i}^{-1} R(F, G) \quad \text{if} \quad \det(R(F, G)_{\alpha_j}) = 0 \quad j = 1, \ldots, i-1$$
$$\text{and} \quad \det(R(F, G)_{\alpha_i}) \neq 0$$

There does exist of course a global continuous algebraic canonical form for n-dimensional fully reachable systems with one input. Every fully reachable pair (F, g) being isomorphic to precisely one of the form

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & a_1 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & a_n \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

4.6. Concluding remarks.

A. Let (F, G, H) be a completely reachable system with inputs G and outputs H. A basis change in the state space changes the triple of matrices (F, G, H) into (SFS^{-1}, SG, HS^{-1}). A fine moduli space for completely reachable systems of dimension n with m-inputs and r-outputs is then $\underline{M}_{m,n} \times \underline{\underline{A}}^{nr}$.

B. In the preceding we assumed that k was algebraically closed. This is unnecessary. All the algebraic varieties discussed and constructions performed exist (resp. can be carried out) over any base field.

References

1. J. Dieudonné and A. Carrell, "Invariant theory, old and new", Advances in Math., 4 (1970) 1-80.

2. A. Grothendieck and J. Dieudonné, "Éléments de la géométric algébrique. Ch. I, II, ...", Publ. Math. de l'I. H. E. S., 4, 8, ..., 1960, 1961, ...

3. D. Husemoller, THEORY OF FIBRE BUNDLES, McGraw-Hill, 1966.

4. R. E. Kalman, P. L. Falb, and M. A. Arbib, TOPICS IN MATHEMATICAL SYSTEM THEORY, McGraw-Hill, 1969.

5. S. Kleiman and D. Laksov, "Schubert calculus", Am. Math. Monthly (1972).

6. D. Mumford, GEOMETRIC INVARIANT THEORY, Springer, 1965.

7. D. Mumford and K. Suominen, "Introduction to the theory of moduli", in Proceedings of the Fifth Nordic Summer School in Mathematics, ALGEBRAIC GEOMETRY (F. Oort, ed.), Noordhoff, 1972.