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A UNIVERSAL FORMAL GROUP AND COMPLEX COBORDISM

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The purpose of this note is to 'announce' some of the results of [5], [6], [7] pertaining to formal groups and complex cobordism. These should have been written up a number of years ago. The phrase "formal group" is used as an abbreviation for commutative one-dimensional formal group (law).

1. **Introduction.** Below we give an explicit recursion formula for the logarithm of a universal commutative formal group and a p-typically universal commutative formal group. These give us a universal formal group F_U defined over $\mathbf{Z}[U] = \mathbf{Z}[U_2, U_3, U_4, \ldots]$ and a p-typically universal formal group F_T over $\mathbf{Z}[T_1, T_2, \ldots]$. Possibly the best way to look at these formal groups is as follows. To fix ideas let p be a fixed prime number and let A be a commutative ring with unit such that every prime number $\neq p$ is invertible in A. Let F_T be the one-dimensional p-typically universal formal group and G a one-dimensional formal group over A. Cartier [4] associates to G a module of curves $C(G)$ over a certain ring $\text{Cart}_p(A)$. The ring $\text{Cart}_p(A)$ has as its elements expressions $\Sigma\, V^i[a_{ij}]\,\mathbf{f}^j$, $a_{ij} \in A$, which are added and multiplied according to certain rules, cf. [4] and [9]; V stands for the 'Verschiebung' associated to the prime number p and $\mathbf{f}$ stands for the 'Frobenius' associated to the prime number p. The left modules C over $\text{Cart}_p(A)$ which arise as modules of curves of some one-dimensional commutative formal group are of the form

$$C \simeq \text{Cart}_p(A)\Big/\text{Cart}_p(A)\left(\mathbf{f} - \sum_{i=1}^{\infty} V^i[t_i]\right), \qquad t_i \in A.$$

Now let F_t be the formal group over A obtained by substituting t_i for T_i. Then $C(F_t) = C$.

2. **The formulae.** Choose a prime number p and let

$$l_n(T) = \sum T_{i_1} T_{i_2}^{p^{i_1}} \cdots T_{i_s}^{p^{i_1+\cdots+i_{s-1}}}/p^s \tag{2.1}$$

where the sum is over all sequences $(i_1, i_2, \ldots, i_s)$, $i_j \in \mathbf{N} = \{1, 2, 3, \ldots\}$ such that $i_1 + \cdots + i_s = n$.

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Further, let

$$a_n(U) = \sum \frac{k(q_1, \ldots, q_s, d) \ldots k(q_s, d)}{p_1 p_2 \cdots p_s} \tag{2.2}$$
$$\cdot\, U_{q_1} U_{q_2}^{q_1} \cdots U_{q_s}^{q_1 \cdots q_{s-1}} U_d^{q_1 \cdots q_s}$$

where we take $U_d = 1$ if $d = 1$; the sum is over all sequences $(q_1, \ldots, q_s, d)$ with $q_i = p_i^{r_i}$, p_i a prime number, $r_i \in \mathbf{N}$ and $d = 1$ or $d > 1$ and not a power of a prime number; the integers $k(q_1, \ldots, q_s, d)$ can be chosen arbitrarily subject to the following congruences:

(2.3)

$$k(q_1, \ldots, q_s, d) \equiv \begin{cases} 1 \bmod p_1 \\ 0 \bmod p_2^{j-1} \end{cases} \quad \text{if } q_1 = p_1^{r_1}, q_2 = p_2^{r_2}, \ldots, q_j = p_2^{r_j},$$
$$p_1 \neq p_2,\ q_{j+1} \text{ not a power of } p_2,$$
$$k(q_1, \ldots, q_s, d) \equiv 1 \bmod p_1^j \quad \text{if } q_1 = p_1^{r_1}, \ldots, q_j = p_1^{r_j},\ q_{j+1} \text{ not a power of } p_1.$$

We now define

$$f_T(X) = \sum_{n \geq 0} l_n(T) X^{p^n}, \qquad f_U(X) = \sum_{n \geq 1} a_n(U) X^n, \tag{2.4}$$

where we take $l_0(T) = 1$ and $a_1(U) = 1$.

One has the following recursion formula for the T_i in terms of the l_i

$$p l_n(T) = l_{n-1}(T) T_1^{p^{n-1}} + l_{n-2}(T) T_2^{p^{n-2}} + \cdots + l_1(T) T_{n-1}^p + T_n. \tag{2.5}$$

The situation for the a_i and U_i is slightly more complicated. We have

$$\nu(n) a_n(U) = U_n + \sum_{i=1}^{\infty} (-1)^{i+1} \sum^{(i)} \rho(n, d_1) a_d(U) U_{d_i}^d U_{d_{i-1}}^{dd_i} \cdots U_{d_1}^{dd_i \cdots d_2} \tag{2.6}$$

if we choose the $k(q_1, \ldots, q_s, d)$ in a certain special way (cf. [5, part II]). Here $\Sigma^{(i)}$ is the sum over all sequences $(d, d_i, d_{i-1}, \ldots, d_1)$ such that $d, d_i, \ldots, d_1 \in \mathbf{N}$, $d_1 \neq 1, s, d_j > 1$ and not a power of a prime number for $j = 2, \ldots, i$ and $dd_i \cdots d_1 = s$. (Note that there are contributions with $d = 1$ in $\Sigma^{(i)}$ if $i \geq 2$ but no contributions with $d = 1$ in $\Sigma^{(1)}$.) The numbers $\upsilon(n)$ and $\rho(n, d_1)$ which occur in (2.6) are obtained as follows. For every pair of prime numbers let $c(p, p')$ be an integer such that $c(p, p) = 1$, $c(p, p') \equiv 1 \bmod p$ and $c(p, p') \equiv 0 \bmod p'$ if $p \neq p'$. Now for all (s, d) such that $d \mid s$ we define: $r(s, d) = 1$ if $d = 1$ or $d > 1$ and not a power of a prime number, $r(s, p^r) = \Pi c(p', p)$ where the product is over the set prime numbers p' which divide s. We define $\nu(n) = 1$ if $n = 1$ or $n > 1$ and not a power of a prime number and $\nu(p^r) = p$ if $r \in \mathbf{N}$. $\rho(s, d)$ is now defined as $\nu(s)\nu(d)^{-1} r(s, d)$.

3. **Universality theorems.** We define

(3.1) $F_U(X, Y) = f_U^{-1}(f_U(X) + f_U(Y)), \quad F_T(X, Y) = f_T^{-1}(f_T(X) + f_T(Y))$

where f_U^{-1} and f_T^{-1} are the inverse power series to f_U and f_T; i.e. $f_U^{-1}(f_U(X)) = X$ and similarly for f_T. One now has

3.2. THEOREM. *$F_T(X, Y)$ is a formal power series with coefficients in $\mathbf{Z}[T_1, T_2, \ldots]$. $F_U(X, Y)$ is a formal power series with coefficients in $\mathbf{Z}[U_2, U_3, \ldots]$.*

The two power series hence define commutative formal groups over $\mathbf{Z}[T]$ and $\mathbf{Z}[U]$.

3.3. THEOREM. *F_U is a universal formal group. F_T is a p-typically universal formal group.*

I.e. if $G(X, Y)$ is any formal group (resp. p-typical formal group) over a commutative ring with unit A, then there is a unique homomorphism ϕ: $\mathbf{Z}[U] \longrightarrow A$ (resp. ϕ: $\mathbf{Z}[T] \longrightarrow A$) such that $G(X, Y)$ is equal to the formal group obtained from F_U (resp. F_T) by applying ϕ to its coefficients.

There are more dimensional analogues for the F_U and F_T and corresponding more dimensional analogues of Theorems 3.2 and 3.3. Cf. [**5**].

4. **Application to complex cobordism and Brown-Peterson cohomology.** Let MU denote the unitary (co)bordism spectrum and BP the Brown-Peterson spectrum. The associated cohomology theories are complex oriented and hence define groups over $MU(pt)$ and $BP(pt)$. The logarithms of these formal groups are by [**11**], [**12**], cf. also [**1**, part II], equal to

$$\log \mu_{MU}(X) = \sum_{n\geqslant 0} m_n X^{n+1}, \tag{4.1}$$
$$\log \mu_{BP}(X) = \sum_{n\geqslant 0} m_{p^n-1} X^{p^n}$$

with $m_n = (n+1)^{-1}[\mathbf{CP}^n]$, where $\mathbf{CP}^n$ is the complex projective space of (complex) dimension n, and $m_0 = 1$. By [**12**], cf. also [**1**], we have that the formal group μ_{MU} is universal and that μ_{BP} is p-typically universal.

Hence there are uniquely determined isomorphisms ϕ: $\mathbf{Z}[U] \longrightarrow MU(pt)$ and Ψ: $\mathbf{Z}[T] \longrightarrow BP(pt)$ taking (2.2) and (2.1) into (4.1). It follows that the $\phi(U_2), \phi(U_3), \ldots$ are a free polynomial basis for $MU(pt)$ and that the $\Psi(T_1), \Psi(T_2), \ldots$ are a free polynomial basis for $BP(pt)$. Knowing $\log \mu_{MU}$ and $\log \mu_{BP}$ we can calculate these $\varphi(U_n)$ and $\Psi(T_n)$ by means of formulae (2.6) and (2.5). We find $BP(pt) \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$, $MU(pt) = \mathbf{Z}[u_2, u_3, \ldots]$ with the v_i and u_i related to the m_i by the formulae:

$$pm_{p^n-1} = m_{p^{n-1}-1}v_1^{p^{n-1}} + m_{p^{n-2}-1}v_2^{p^{n-2}} + \cdots + m_{p-1}v_{n-1}^p + v_n, \tag{4.2}$$

$$(4.3)\quad \nu(n)m_{n-1} = u_n + \sum_{i=1}^{\infty} (-1)^i \sum^{(i)} \rho(n, d_1)m_{d-1}u_{d_i}^{d}u_{d_{i-1}}^{dd_i} \cdots u_{d_1}^{dd_i\cdots d_2}.$$

BP is a direct summand of $MU\mathbf{Z}_{(p)}$, where $\mathbf{Z}_{(p)}$ denotes the integers localized at p. Because formula (4.3) reduces to (4.2) if $n = p^s$ under the identification $v_i = u_{p^i}$, we see that the v_i are integral i.e. they live in $MU(pt)$ not just in $MU\mathbf{Z}_{(p)}(pt)$. Cf. also [2].

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